TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Một áp dụng của Định lý tuyến tính hóa đối với bài toán hội tụ Tauber trong không gian có trọng của các hàm chỉnh hình Gâteaux

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TÓM TẮT

Mục đích của bài báo là nghiên cứu Định lý tuyến tính hóa để giải quyết một bài toán về hội tụ Tauber trên không gian Banach có trọng các hàm chỉnh hình Gâteaux giữa các không gian lồi địa phương.

Từ khóa: *Tuyến tính hóa, không gian lồi địa phương, các bất biến tôpô tuyến tính, các hàm chỉnh hình, các hàm đa điều hòa dưới.*

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An application of Linearization Theorem to the problem on Tauberian convergence in weighted space of Gâteaux holomorphic functions

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ABSTRACT

The aim of this paper is to study a linearization theorem for weighted spaces of Gâteaux holomorphic functions between locally convex spaces and its applications to the problem on Tauberian convergence.

Keywords: *Linearization, locally convex spaces, topological linear invariants, holomorphic functions, plurisubharmonic functions.*

1. INTRODUCTION

It is known that linearization technique is very important because it can help us in simplifying calculations. Indeed, if E and F are Banach spaces and U is an open subset of E, then the linearization results help in identifying a given class of holomorphic functions defined on U, $F-$valued with the space of continuous linear mappings from a certain Banach space G to F, i.e., a holomorphic mapping is being identified with a linear operator through linearization results. The first linearization result for such spaces was obtained by Mazet in 1984. The problem has received much attention in the past few years. However, most of the results are directed to spaces of functions between Banach spaces. Six years after the announcement of Mazet, Mujica obtained a linearization theorem for the space $H^\infty(D, F)$ of Banach-valued bounded holomorphic mappings defined on an open subset of a Banach space.

Weighted spaces of holomorphic functions defined on an open subset of a finite or infinite dimensional Banach space have been studied widely in the literature by several mathematicians. Whereas the results in the finite dimensional case, we attribute to the contributions of Bierstedt, Bonet, Galbis, Summers, Meise, Rubel, Shields, etc., the infinite dimensional case was introduced by Garcia, Maestre, and Rueda, and further investigated by Beltran, Jordá, Rueda, Gupta, Baweja, etc.

In this paper, we are concerned with proving a linearization theorem for weighted Gâteaux holomorphic functions between locally convex spaces and applying to solve some related problems in weighted spaces of holomorphic functions.

Let E, F be locally convex spaces and v be a weight on a domain D in E. Denote $H_v(D, F)$ (resp. $H_{G,v}(D, F)$) the space of all F-valued (resp. Gâteaux) holomorphic functions on D such that $(v.f)(D)$ is bounded in F equipped with the topology induced by the family $\{\|\cdot\|_{v,p}\}_{p\in cs(F)}$ of semi-norms where

$$\|f\|_{v,p} := \sup_{x\in D} v(x)p(f(x)) \quad \forall p \in cs(F).$$

Let $A_v(D)$ (resp. $A_{G,v}(D)$) be a subspace of $H_v(D)$ (resp. $H_{G,v}(D)$) such that the closed unit ball

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is compact for the compact open topology τ_0. Denote

$$A_v(D,F) := \{f : D \to F :\ u \circ f \in A_v(D)\ \forall u \in F'\}.$$

We wish to investigate the linearization theorem in the weighted spaces of F-valued functions in a weak sense

$$A_{G,v}(D,F) := \{f : D \to F :\ u \circ f \in A_{G,v}(D)\ \forall u \in F'\}.$$

We will use these result as one of main tools to study the problem of Tauberian convergences which is to look for additional properties to ensure that every sequence/net of F-valued functions defined and converging on a subset of D is convergent entirely on D. The classical theorem of Vitali is an important example on the Tauberian convergence for holomorphic functions. For a sequense $\{f_m\}_{m\geq 1}$ of scalar-valued holomorphic functions which is uniformly bounded on compact subsets of a domain D in $\mathbb{C}^n$ the theorem says that if it is pointwise convergent to a function f on a subset X of D then it converges uniformly on compact subsets of D whenever X is not contained in any complex hypersurface.

In 2000, Arendt and Nikolski proved Vitali's theorem for nets of Banach-valued, one-complex variable holomorphic functions in the case where the small subset has an accumulation point (see [1] [Theorem 3.1]). They gave an easy direct proof based on the theorem on very weakly holomorphic and the uniqueness theorem. After that, more general, in 2013 Quang, Lam and Dai [2] have introduced theorems of Vitali-type for sequences, which are locally bounded as well as are bounded on bounded subsets, of Fréchet-valued holomorphic functions on a domain in a Fréchet space (see [2] [Theorems 6.1, 6.2, 6.3 and Corollaries]). The tools of linear topological invariants, introduced and investigated by Vogt (see [3,4,5]), were used in their proofs.

Very recently, Dieu, Manh, Bang and Hung [6] are concerned with finding analogues theorem of Vitali in which the uniform boundedness of the sequence under consideration is omitted. A possible approach is to impose stronger mode of convergence and/or the size of the small subset. The versions of Vitali theorem for bounded holomorphic functions and rational functions that are rapidly pointwise convergent on a non-pluripolar subset of a domain in $\mathbb{C}^n$ have been considered in their work. Here the speed of approximation is measured in terms of the growth of the sup-norms of functions. Motivated by the problem of finding local conditions for single-valuedness of holomorphic continuation, Gonchar [7] proved that a sequence of rational functions $\{r_m\}_{m\geq 1}$ in $\mathbb{C}^n$ ($\deg r_m \leq m$) that converges rapidly in measure on an open set X to a holomorphic function f defined on a bounded domain D ($X \subset D$) must converge rapidly in measure to f on the whole domain D. Much later, by using techniques of pluripotential theory, in [8] [Theorem 2.1] Bloom was able to prove an analogous result in which rapidly convergence in measure is replaced by rapidly convergence in capacity and the small subset X is only required to be compact and non-pluripolar.

Most recently, the problem on rapidly Tauberian convergence for sequences of polynomials have been studied by Quang, Vy, Hung and Bang [9] based on their researchs on Zorn spaces. They establish some results on the holomorphic extension of a Fréchet-valued continuous function f to an entire function from some non-polar balanced convex compact subset B of a Fréchet space whenever f is approximated fast enough on B by a sequence of polynomials.

In this paper, we use the linearization technique of weighted holomorphic functions to investigate the Tauberian convergence of sequences/nets in $H_v(E,F)$ and $A_v(D,F)$ as well as the holomorphic extension of weak-type holomorphic functions from a small subset of D in the weighted space $A_v(D,F)$, where E, F are Fréchet spaces.

The organization of the article is as follows.

We set up review in Section 2 some notations and terminologies of functional analysis, of holomorphic functions on locally convex spaces and of weighted spaces of holomorphic functions pertaining to our work. Some linear topological invariants of Fréchet spaces introduced and investigated by Vogt and some results on pluripolar/non-pluripolar sets are also recalled in this section.

In Section 3, we modify the method of Mujica in [10] to obtain Theorems 3.2 which permit to identify $A_{G,v}(D,F)$ with F-valued continuous linear mappings from a dense subspace $P^0_{A_v(D)}$ of the predual space $P_{A_v(D)}$. Recent works of Carando and Zalduendo [11], and of Mujica [12] are devoted to get linearization results for (unweighted) spaces of holomorphic functions between locally convex spaces; and of Beltrán [13] for weighted (LB)-spaces of entire functions on Banach spaces.

Based on the results from the previous work, in Section 4 we will investigate the problem on Tauberian convergences for sequences/nets in weighted spaces $H_{G,v}(E, F)$ and $A_{G,v}(D, F)$.

The first part of the section, we present some notations and state (without proofs) theorems on Zorn's property for the space (E_B, τ_E), where E_B the linear hull of some compact, absolutely convex subset B of a Fréchet space E and the topology τ_E on E_B is induced by the topology of E. The proofs of these results are analogous to the those encountered in the recent work of Quang and his colleagues [9]. We devoted to the study of the Tauberian convergence of sequences of Fréchet-valued Gâteaux holomorphic functions on a dense subset of a domain in a Fréchet space and holomorphic extension of the limit function in weighted space. Combining the results on (BB)-Zorn spaces with the linearization theorems, we give the conditions under which every bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ of holomorphic functions in $H_{G,v}((E_K, \tau_E), F)$ is convergent to a function $f \in H_{G,v}((E_K, \tau_E), F)$ uniformly on the compact subsets of (E_K, τ_E) whenever $\{f_m\}_{m\in\mathbb{N}}$ converges at every point of K where τ_E is the topology of E_K induced by the topology of E. Moreover, the function f admits a holomorphic extension in $H_v(E, F)$ if it is continuous at a simple point in K (Theorems 4.3, 4.4).

2. PRELIMINARIES

2.1. General notations

Standard notations of the theory of locally convex spaces as presented in the book of Schaefer [14] will be used in the paper. A locally convex space is always a complex vector space with a locally convex Hausdorff topology.

We always assume that the locally convex structure of a Fréchet space E is generated by an increasing system $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\ker\|\cdot\|_k$, by $\omega_k : E \to E_k$ the canonical map and by U_k the set $\{x \in E : \|x\|_k < 1\}$. Sometimes it is convenient to assume that $\{U_k\}_{k\in\mathbb{N}}$ is a neighbourhood basis of zero (shortly $\mathcal{U}(E)$).

If B is an absolutely convex subset of E we define a norm $\|\cdot\|_B^*$ on E', the strongly dual space of E with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)|, x \in B\}.$$

Obviously $\|\cdot\|_B^*$ is the gauge functional of B°. Instead of $\|\cdot\|_{U_k}^*$ we write $\|\cdot\|_k^*$. By E_B we denote the linear hull of B which becomes a normed space in a canonical way if B is bounded.

2.2. Some linear topological invariants

We say that a Fréchet space E has the property $(\widetilde{\Omega})$, and write $E \in (\widetilde{\Omega})$, if

$$\forall p\ \exists q\ d > 0\ \forall k\ \exists C > 0 \quad \text{such that}$$

$$\|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^*\|\cdot\|_p^{*d}.$$

For $B \in \mathcal{B}(E)$, the family of closed, bounded, absolutely convex sets in E, we say that E has the property $(\widetilde{\Omega}_B)$, and write $E \in (\widetilde{\Omega}_B)$, if

$$\forall\ p\ \exists\ q, d, C > 0 \quad \text{such that}$$

$$\|\cdot\|_q^{*1+d} \le C\|\cdot\|_B^*\|\cdot\|_p^{*d}.$$

The above properties have been introduced and investigated by Vogt [3,4,5].

Note that in [15] Dineen, Meise and Vogt proved that $E \in (\widetilde{\Omega})$ if and only if there exists $B \in \mathcal{K}(E)$ such that $E \in (\widetilde{\Omega}_B)$ where $\mathcal{K}(E)$ is the family of compact, absolutely convex subsets of E.

2.3. Weighted spaces of holomorphic functions

Let E and F be locally convex spaces and D be a domain in E. A function $f : D \to F$ is called *Gâteaux holomorphic* if for every $a \in D, b \in E$ and $\varphi \in F'$, the $\mathbb{C}$-valued function of one complex variable

$$\lambda \mapsto (\varphi \circ f)(a + \lambda b)$$

is holomorphic on a neighborhood of $0 \in \mathbb{C}$.

The function f is said to be *holomorphic* if it is Gâteaux holomorphic and continuous.

By $H(D, F)$ (resp. $H_G(D, F)$) we denote the vector space of all holomorphic (resp. Gâteaux holomorphic) functions on D with values in F. The space $H(D, F)$ equipped with the compact-open topology τ_0. We use $H_b(E, F)$ to denote the space of holomorphic functions from E into F which are bounded on every bounded set in E, and $H_{ub}(E, F)$ to indicate the set of holomorphic functions which are bounded on rU for some neighbourhood U of $0 \in E$ and for all $r > 0$. Note that $H_{ub}(E, F) \subseteq H_b(E, F)$.

Instead of $H(D, \mathbb{C})$, $H_G(D, \mathbb{C})$, $H_b(E, \mathbb{C})$, $H_{ub}(E, \mathbb{C})$ we write $H(D)$, $H_G(D)$, $H_b(E)$, $H_{ub}(E)$ respectively.

For details concerning holomorphic functions on locally convex spaces, we refer to the book of Dineen.[16]

For a domain D in E, a *weight* $v : D \to (0, \infty)$ is a continuous function which is strictly positive. Denote $H_v(D, F)$

$$:= \{f \in H(D, F) : \ (v.f)(D) \text{ is bounded on } D\};$$

$$H_v(D) := H_v(D, \mathbb{C})$$

$$= \{f \in H(D) : \ \|f\|_v := \sup_{x \in D} v(x)|f(x)| < \infty\};$$

$$H_{G,v}(D, F)$$

$$:= \{f \in H_G(D, F) : \ (v.f)(D) \text{ is bounded on } D\};$$

$$H_{G,v}(D) = H_{G,v}(D, \mathbb{C})$$

$$= \{f \in H_G(D) : \ \|f\|_v := \sup_{x \in D} v(x)|f(x)| < \infty\}.$$

The space $H_v(D, F)$ equipped with the topology generated by the family $\{\| \cdot \|_{v,p}\}_{p \in cs(F)}$ of seminorms. Then $H_v(D, F)$ is complete whenever F is complete, in particular, it is Banach if F is Banach. It is easy to check that

$$H_v(D, F) = \varprojlim_{p \in cs(F)} H_v(D, F_p)$$

where F_p is the completion of the canonically normed space $F/\ker p$.

2.4. Pluripolar sets

Let D be an open subset of a topological vector space E. An upper-semicontinuous function $\varphi : D \to [-\infty, +\infty)$ is called *plurisubharmonic*, and write $\varphi \in PSH(D)$, if for every $a \in D$ and $b \in E$ the function

$$\lambda \mapsto \varphi(a + \lambda b)$$

is subharmonic as a function of one complex variable on a neighborhood of $0 \in \mathbb{C}$. For example, if $f \in H(D, F)$, where D is an open subset of a locally convex space and F is a vector space with a seminorm $\| \cdot \|$ then the function $z \mapsto \log \|f(z)\|$ is plurisubharmonic on D.

Definition 2.1. A subset $B \subset D$ is said to be *pluripolar* in D if there exists a $\varphi \in PSH(D)$ such that $\varphi \not\equiv -\infty$ on any connected component of D and $\varphi\big|_B = -\infty$.

It is clear that the union of a finite number of pluripolar sets is pluripolar. We should say that pluripolar sets in infinite dimension space can be complicated. For example, there exist Fréchet spaces in which *every* bounded sets is pluripolar.

In the case where $\dim E < \infty$, a fundamental result of Josefson (see [17], [Theorem 4.7.4]) says that the function φ above can be chosen to be *globally* defined on E. That means a subset X of $\mathbb{C}^p$ is pluripolar if and only if there exists $\varphi \in PSH(\mathbb{C}^p)$, $\varphi \neq -\infty$ such that

$$X \subset \{z \in \mathbb{C}^p : \ \varphi(z) = -\infty\}.$$

Polarity of subsets in infinite dimensional spaces has been dealt with in detail in many places, and the reader is refered to [15,18,19] for further details.

3. LINEARIZATION OF WEIGHTED SPACES OF GÂTEAUX HOLOMORPHIC FUNCTIONS

Let $A_v(D)$ be a subspace of $H_v(D)$ such that the closed unit ball $B_{A_v(D)}$ is compact for the topology τ_0. Note that this condition implies that $A_v(D)$ is norm-closed and hence Banach because $H_v(D)$ is Banach. First, we present two illustrative examples for this assumption in the next section.

In [21] [Theorem 7] Jordá showed that if, for $m \in \mathbb{N}$, the space $\mathscr{P}(^m E)$ of continuous m-homogeneous polynomials on a Fréchet E, endowed with its norm topology, is contained $H_v(D)$ then the closed unit ball of $\mathscr{P}(^m E)$ is compact for the topology τ_0.

Now, we present another illustrative example. For each $f \in H(D)$ we consider the Taylor series representation at zero

$$f(z) = \sum_{k=0}^{\infty} (P_k f)(z), \quad z \in D,$$

where $P_k f$ is a k-hemogeneous polynomial, $k = 0, 1, \ldots$ The series converges to f uniformly on each compact subset of D.

Now given a sequence $\alpha := \{\alpha_n\}_{n \geq 0} \subset \mathbb{C} \setminus \{0\}$. For each $k \geq 1$, let us denote $|\alpha|_k = \sum_{m=0}^{k} |\alpha_m|$. For each $n \geq 0$, consider the linear operator $C_n : H_v(D) \to H_v(D)$ given by

$$(C_n f)(z) = \frac{1}{|\alpha|_n} \sum_{k=0}^{n} \Big(\alpha_k \sum_{j=0}^{k} (P_j f)(z) \Big), \quad z \in D. \tag{1}$$

Put

$$\mathcal{C}_{n,v}(D) = C_n(H_v(D)), \mathcal{C}_v(D) = \bigcup_{n \geq 0} \mathcal{C}_{n,v}(D)$$

and

$$A_v(D) := \overline{\mathcal{C}_v(D)}^{\tau_v}. \tag{2}$$

In [20] [Proposition 4] Quang showed that if, for D is a balanced, bounded open set in a Fréchet-Montel space E having the $(BB)_\infty$-property, v is a weight on D which is radial and vanishes at infinity outside compact sets of D, and $A_v(D)$ is defined by (2). Then, the closed unit ball of $(A_v(D), \tau_v)$ is τ_0-compact.

In [22] [Theorem 2.2.1] Vy have given linearization theorem for weighted holomorphic functions between locally convex spaces as follows.

Proposition 3.1. *Let v be a weight on a domain D in a metrizable locally convex space E and $A_v(D)$ be a subspace of $H_v(D)$ such that the closed unit ball is τ_0-compact. Then there exist a Banach space $P_{A_v(D)}$ and a mapping $\delta_D \in H(D, P_{A_v(D)})$ with the following universal property: For each complete locally convex space F, a function $f \in A_v(D, F)$ if and only if there is a unique mapping $T_f \in L(P_{A_v(D)}, F)$ such that $T_f \circ \delta_D = f$. This property characterize $P_{A_v(D)}$ uniquely up to an isometric isomorphism.*

Moreover, the mapping

$$\Phi : f \in A_v(D, F) \mapsto T_f \in (L(P_{A_v(D)}, F), \tau_\ell)$$

$$:= \varprojlim_{p \in cs(F)} (L(P_{A_v(D)}, F_p)$$

is a topological isomorphism.

By the Ng Theorem [23] [Theorem 1] the evaluation mapping

$$J : (A_v(D), \|\cdot\|_v) \to (P_{A_v(D)})'$$

given by

$$(Jf)(u) = u(f) \quad \forall u \in P_{A_v(D)},$$

is a topological isomorphism. Hence, the space $P_{A_v(D)}$ is called *the predual* of $A_v(D)$.

Now we consider the above result for weighted Gâteaux holomorphic functions.

Let D be an open subset of metrizable locally convex space E. Denote $\mathscr{F}(E)$ the family of all finite dimensional subspaces of E. By Proposition 3.1, for each $Y \in \mathscr{F}(E)$ there exists a unique map $p_Y \in L(P_{A_v(D \cap Y)}, P_{A_v(D)})$ such that the following diagram is commutative

$$\begin{array}{ccc} D \cap Y & \xrightarrow{\ id\ } & D \\ \big\downarrow{\scriptstyle \delta_{D\cap Y}} & & \big\downarrow{\scriptstyle \delta_D} \\ P_{A_v(D\cap Y)} & \xrightarrow{\ p_Y\ } & P_{A_v(D)} \end{array} \tag{3}$$

where id is the identity mapping and $P_{A_v(D \cap Y)}$ denotes the predual of $A_v(D \cap Y)$.

If $Y, Z \in \mathscr{F}(E)$ such that $Y \subset Z$, then by Proposition 3.1 again, there exists a unique map $p_{ZY} \in L(P_{A_v(D \cap Y)}, P_{A_v(D \cap Z)})$ such that the following diagram is commutative

$$\begin{array}{ccc} D \cap Y & \xrightarrow{\ id\ } & D \cap Z \\ \big\downarrow{\scriptstyle \delta_{D\cap Y}} & & \big\downarrow{\scriptstyle \delta_{D\cap Z}} \\ P_{A_v(D\cap Y)} & \xrightarrow{\ p_{ZY}\ } & P_{A_v(D\cap Z)} \end{array}$$

It follows that $p_Z \circ p_{ZY} = p_Y$ whenever $Y \subset Z$. Denote

$$P^0_{A_v(D)} := \bigcup_{Y \in \mathscr{F}(E)} p_Y(P_{A_v(D \cap Y)})$$

and equip $P^0_{A_v(D)}$ with the topology induced by $P_{A_v(D)}$.

Let $A_{G,v}(D)$ be a subspace of $H_{G,v}(D)$ such that the closed unit ball is compact for the topology τ_0. For each complete locally convex space F, we put

$$A_{G,v}(D, F) := \{f : D \to F : \ u \circ f \in A_{G,v}(D) \ \forall u \in F'\}.$$

Theorem 3.2. *Under the assumptions of Proposition 3.1,*

(a) $P^0_{A_v(D)}$ *is a dense subspace of* $P_{A_v(D)}$;

(b) $\delta_D \in H(D, P^0_{A_v(D)})$;

(c) For each complete locally convex space F, *the function* $f \in A_{G,v}(D, F)$ *if and only if there exists a unique linear mapping* $T_f : P^0_{A_v(D)} \to F$ *such that* $T_f \circ \delta_D = f$. *Moreover,* T_f *is continuous if and only if* f *is continuous.*

Proof. (a) By [22] [Theorem 3.1] we have $\delta_D : D \to P_{A_v(D)}$ be the evaluation mapping given by

$$\delta_D(x) = \delta_x$$

with δ_x is the evaluation, that means $\delta_x(g) := g(x)$ for all $g \in A_v(D)$; then, we have

$$\text{span}\{\delta_x : \ x \in D\} \text{ is a dense subspace of } P_{A_v(D)}. \tag{4}$$

By the commutative diagram (3), $\delta_x \in P^0_{A_v(D)}$ for every $x \in D$. Thus, by (4), $P^0_{A_v(D)}$ is dense in $P_{A_v(D)}$.

(b) It is known that $\delta_D \in H(D, P_{A_v(D)})$ (see Proposition 3.1) and $\delta_D(x) \in P^0_{A_v(D)}$ for all $x \in D$. For each $z \in D$ we write

$$\delta_D(z) = \sum_{n=0}^{\infty} P_n \delta_D(z).$$

Thus, in order to prove $\delta_D \in H(D, P^0_{A_v(D)})$ it suffices to check that

$$P_n\delta_D(a) \in \mathcal{P}(^nE, P^0_{A_v(D)})$$

for every $a \in D$ and $n \in \mathbb{N}$, where $\mathcal{P}(^nE, P^0_{A_v(D)})$ denotes the vector space of all continuous n-homogeneous polynomials from E into $P^0_{A_v(D)}$.

Fix $a \in D, n \in \mathbb{N}$ and $t \in E$. Let $r > 0$ such that $a + \eta t \in D$ for all $\eta \in \overline{\Delta}_r = \{\lambda \in \mathbb{C} : \ |\lambda| \leq r\}$ and let $Y \in \mathscr{F}(E)$ which is generated by a and t. Then, it follows from the commutative diagram (3) that

$$\begin{aligned} P_n\delta_D(a)(t) &= \frac{1}{2\pi i}\int_{|\eta|=r} \frac{\delta_D(a+\eta t)}{\eta^{n+1}} d\eta \\ &= \frac{1}{2\pi i}\int_{|\eta|=r} \frac{p_Y \circ \delta_{D\cap Y}(a+\eta t)}{\eta^{n+1}} d\eta \\ &= p_Y(P_n\delta_{D\cap Y})(a)(t), \end{aligned}$$

and hence $P_n\delta_D(a)(t) \in p_Y(P_{A_v(D\cap Y)}) \subset P^0_{A_v(D)}$.

(c) First, given a function $f : D \to F$. Assume that there exists a unique linear mapping $T_f : P^0_{A_v(D)} \to F$ such that $T_f \circ \delta_D = f$. Then, the commutative diagram (3) implies that $T_{Y,f} \circ \delta_{D\cap Y} = f\big|_{D\cap Y}$ for every $Y \in \mathscr{F}(E)$, where $T_{Y,f} := T_f\big|_{p_Y(P_{A_v(D\cap Y)})} \in L(P_{A_v(D\cap Y)}, F)$. By Theorem 3.1, $f\big|_{D\cap Y} \in A_v(D\cap Y, F)$. Hence $f \in A_{G,v}(D, F)$.

Conversely, for each $f \in A_{G,v}(D, F)$ and $Y \in \mathscr{F}(E)$, the function $f_Y := f\big|_{D\cap Y} \in A_v(D\cap Y)$. By Theorem 3.1, there is a unique map $T_Y \in L(P_{A_v(D\cap Y)}, F)$ such that $T_Y \circ \delta_{D\cap Y} = f_Y$. If $Y, Z \in \mathscr{F}(E)$ with $Y \subset Z$, the following diagram is commutative

$$\begin{array}{ccccc} D\cap Y & \xrightarrow{id} & D\cap Z & \xrightarrow{id} & D \\ \downarrow{\scriptstyle \delta_{D\cap Y}} & & \downarrow{\scriptstyle \delta_{D\cap Z}} & & \downarrow{\scriptstyle f} \\ P_{A_v(D\cap Y)} & \xrightarrow{p_{ZY}} & P_{A_v(D\cap Z)} & \xrightarrow{T_Z} & F \end{array}$$

This implies that $T_Z \circ p_{ZY} = T_Y$ whenever $Y \subset Z$. Thus, there is a linear map $T : P^0_{A_v(D)} \to F$ such that $T \circ p_Y = T_Y$ for each $Y \in \mathscr{F}(X)$. Since $T_Y \circ \delta_{D\cap Y} = f_Y$ we obtain $T \circ \delta_D = f$.

If we can find $S : P^0_{A_v(D)} \to F$ such that $T \circ \delta_D = f$ then $S \circ p_Y = T_Y = T \circ p_Y$ for every $Y \in \mathscr{F}(E)$. This yields $S = T$.

Finally, since $f = T \circ \delta_D$, it is obvious that f is continuous if T is continuous. The converse inclusion follows from Theorem 3.1.

Now, note that a family $\mathcal{F} \subset A_v(D, F)$ is bounded if and only if it is amply bounded, that means the family $\{\omega_p \circ f : \ f \in \mathcal{F}\}$ is bounded in $A_v(D, F_p)$ for all $p \in cs(F)$, where $\omega_p : F \to F_p$ is the canonical map.

Proposition 3.3. *Under the assumptions of Proposition 3.1, a family* $\{f_j\}_{j\in I} \subset A_v(D, F)$ *is bounded if and only if the corresponding family* $T_{f_j} \subset L(P_{A_v(D)}, F)$ *is equicontinuous.*

Proof. Let $\{T_{f_j}\}_{j\in I}$ be the corresponding family of $\{f_j\}_{j\in I}$, that is $f_j(x) = T_{f_j}(\delta_x)$ for each $x \in D$ (see Proposition 3.1). Let $D^*_v := \{v(x)\delta_x : \ x \in D\} \subset P_{A_v(D)}$.

Assume that $\{T_{f_j}\}_{j\in I} \subset L(P_{A_v(D)}, F)$ is equicontinuous. Then for each $p \in cs(F)$ there exists $C_p > 0$ such that

$$T_{\omega_p \circ f_j}(D^*_v) = \{v(x)(\omega_p \circ f_j)(x) : x \in D\}$$

$$\subset C_p B_{F_p} \quad \forall j \in I$$

where B_{F_p} is the unit ball of F_p.

We now note that

$$\overline{\mathrm{acx}}(D_v^*) = (D_v^*)^{\circ\circ} = B^{\circ}_{A_v(D)} = B_{P_{A_v(D)}} \tag{5}$$

where $\overline{\mathrm{acx}}(D_v^*)$ denotes the closed balanced convex hull of $D_v^* := \{v(x)\delta_x : x \in D\}$ in $P_{A_v(D)}$. Thus, we have

$$\begin{aligned} T_{\omega_p \circ f_j}(B_{P_{A_v(D)}}) &= T_{\omega_p \circ f_j}(\overline{\mathrm{acx}}(D_v^*)) \\ &\subset \overline{\mathrm{acx}}(T_{\omega_p \circ f_j}(D_v^*)) \\ &\subset C_p \overline{B}_{F_p} \quad \forall j \in I. \end{aligned}$$

This implies that the family $\{\omega_p \circ f_j\}_{j\in I}$ is bounded for all $p \in cs(F)$, hence the family $\{f_j\}_{j\in I}$ is bounded.

Conversely, take a 0-neighbourhood V in F. We can assume that $V := \varepsilon \bigcap_{i=1}^m \omega_{p_i}^{-1}(B_{F_{p_i}})$. By the hypothesis, for each p_i, $1 \le i \le m$, there exists $C_{p_i} > 0$ such that

$$\{v(x)(\omega_{p_i} \circ f_j)(x) : x \in D\} \subset \varepsilon C_{p_i}(B_{F_{p_i}}) \quad \forall j \in I$$

and hence

$$\{v(x) f_j(x) : x \in D\} \subset V \quad \forall j \in I.$$

Then we have

$$T_{f_j}(D_v^*) = \{v(x) T_{f_j}(\delta_x) : x \in D\}$$

$$= \{v(x) f_j(x) : x \in D\} \subset V \quad \forall j \in I.$$

Consequently,

$$T_{f_j}(B_{P_{A_v(D)}}) = T_{f_j}(\overline{\mathrm{acx}}(D_v^*)) \subset \overline{V} \quad \forall j \in I,$$

and the proposition is proved.

The following is a consequence of Theorem 3.2 and Proposition 3.3.

Corollary 3.4. *Under the assumptions of Theorem 3.1, a family $\{f_j\}_{j\in I} \subset A_{G,v}(D,F)$ is bounded if and only if the corresponding family $T_{f_j} \subset L(P^0_{A_v(D)}, F)$ is equicontinuous.*

4. TAUBERIAN CONVERGENCES IN WEIGHTED SPACES OF GÂTEAUX HOLOMORPHIC FUNCTIONS

We will apply the results of the previous section to investigate the problem on Tauberian convergences for sequences/nets as well as the extension of limit functions in weighted spaces $H_v(E,F)$ and $A_v(D,F)$.

First, we present some notions which will be needed in subsequent sections.

A subset $M \subseteq D$ is said to be a *set of uniqueness for* $A_v(D)$ if each $f \in A_v(D)$ such that $f\big|_M = 0$ then $f \equiv 0$.

A subset $M \subset D$ is said to be *sampling for* $A_v(D)$ if there exists a constant $C \ge 1$ such that for every $f \in A_v(D)$ we have

$$\sup_{z\in D} v(z)|f(z)| \le C \sup_{z\in M} v(z)|f(z)|. \tag{6}$$

Remark 1. For $M \subset D$, denote $M_v^* := \{v(x)\delta_x : x \in M\} \subset B_{P_{A_v(D)}}$ where $B_{P_{A_v(D)}}$ denotes the unit ball of $P_{A_v(D)}$.

1. Since the closed unit ball $B_{A_v(D)}$ of the space $A_v(D)$ is τ_0-compact, by the Hahn-Banach theorem, it is easy to check that the following are equivalent:
   (i) M is a set of uniqueness for $A_v(D)$;
   (ii) M_v^* is separating in $A_v(D)$;
   (iii) $\langle M_v^* \rangle := \mathrm{span} M_v^*$ is $\sigma(P_{A_v(D)}, A_v(D))$-dense.
2. For the norm given by $\|f\|_{M,v} := \sup_{z\in M} v(z)|f(z)|$ on $A_v(D)$, it is obvious that the following are equivalent:
   (i) M is sampling for $A_v(D)$;
   (ii) $\|\cdot\|_v \simeq \|\cdot\|_{M,v}$ on $A_v(D)$.
3. Obviously, if M is sampling for $A_v(D)$ then M_v^* is separating in $A_v(D)$, hence, M is a set of uniqueness for $A_v(D)$.

Now, we state Theorems of Zorn type which will be needed in subsequent sections.

Definition 4.1. A locally convex space E is said to be *(BB)-Zorn space* (or, to have *(BB)-Zorn Property*) if for every open subset D of E, every $f \in H_G(D)$ which is bounded on bounded sets in D and continuous at a single point of D is holomorphic on D.

In [9], Quang and his colleagues have investigated the Zorn property of the linear hull of some convex balanced compact subset B of a Fréchet space. In the same way as in [9] we get the following.

Theorem 4.1. *Let E be a nuclear Fréchet space with the topology τ_E and $E \in (\widetilde{\Omega})$. Then there exists $K \in \mathcal{K}(E)$ such that (E_K, τ_E) is a (BB)-Zorn space. Moreover, for every domain D in E we have*

$$H_b(D(K), \tau_E) = H_b(D)$$

where $D(K) := D \cap E_K$.

In particular, $H(E_K, \tau_E) = H(E) = H_{ub}(E)$.

Theorem 4.2. *Let E be a Fréchet-Schwartz space with an absolute Schauder basis and the topology τ_E and $E \in (\widetilde{\Omega})$. Then there exists a non-pluripolar set $K \in \mathcal{K}(E)$ such that (E_K, τ_E) is a (BB)-Zorn space. Moreover, for every domain D in E we have*

$$H_b(D(K), \tau_E) = H_b(D)$$

where $D(K) := D \cap E_K$.

In particular, $H_b(E_K, \tau_E) = H(E) = H_b(E)$.

Note that, under the assumptions of E the set K is non-pluripolar (see [15] [Theorem 9]).

The first results of this section are concerned with the weighted Tauberian convergence of sequences of Gâteaux holomorphic functions in a space (E_K, τ_E), where E_K the linear hull of some $K \in \mathcal{K}(E)$ where the topology τ_E on E_K is induced by the topology of a Fréchet space E.

Theorem 4.3. *Let E, F be Fréchet spaces and v be a weight on E. Assume that E is nuclear with the topology τ_E and $E \in (\widetilde{\Omega})$. Then there exists a non-pluripolar set $K \in \mathcal{K}(E)$ satisfying the following: if $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence in $H_{G,v}((E_K, \tau_E), F)$ such that $\{f_m\}_{m\in\mathbb{N}}$ is convergent at each $x \in K$ to a function f which is continuous at some $x_0 \in K$ then f has an extension $\widetilde{f} \in H_v(E, F)$ and $\{f_m\}_{m\in\mathbb{N}}$ is convergent to $\widetilde{f}$ uniformly on the compact subsets of (E_K, τ_E).*

Proof. By Theorem 4.1 there exists a non-pluripolar $K \in \mathcal{K}(E)$ such that (E_K, τ_E) is a (BB)-Zorn space.

First, without loss of generality we may assume that $\mathscr{F}((E_K, \tau_E)) = \{Q_1, \dots, Q_n, \dots\}$ where $\dim Q_n = n$ and $Q_n \subset Q_{n+1} \subset (E_K, \tau_E)$ for all $n \in \mathbb{N}$. Then

$$(E_K, \tau_E) = \bigcup_{n\in\mathbb{N}} Q_n.$$

Since $\dim Q_n = n < \infty$, by Montel's theorem, the closed unit ball $B_{n,v}$ of $H_v(Q_n)$ is τ_0-compact. As in Theorem 3.2 we denote

$$P^0_{H_v(E_K,\tau_E)} := \bigcup_{n\in\mathbb{N}} p_{Q_n}(P_{H_v(Q_n)})$$

and equip $P^0_{H_v(E_K,\tau_E)}$ with the topology induced by the predual $P_{H_v(E_K,\tau_E)}$ of $H_v(E_K, \tau_E)$.

Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence as in the theorem. From Theorem 3.2 there exists a sequence $\{T_{f_m}\}_{m\in\mathbb{N}} \subset L(P^0_{H_v(E_K,\tau_E)}, F)$ such that $T_{f_m} \circ \delta_{E_K} = f_m$ for all $m \in \mathbb{N}$.

Then, since $\{f_m\}_{m\in\mathbb{N}}$ is bounded, it follows from Corollary 3.4 that $\{T_{f_m}\}_{m\in\mathbb{N}}$ is equicontinuous.

Then the topology of pointwise convergence on $P_{H_v(E_K,\tau_E)}$ coincides with the topology of pointwise convergence on $(E_K, \tau_E)^*_v$ by [24] [39.4(1)]. Thus $\{T_j\}_{j\in I}$ is pointwise convergent to $T \in L(P^0_{H_v(E_K,\tau_E)}, F)$. The convergence is uniform on the compact subsets of $P^0_{H_v(E_K,\tau_E)}$, by [24] [39.4(2)].

By Proposition 3.1 there exists $f^* \in H_{G,v}((E_K, \tau_E), F)$ such that $f^*(x) = T(\delta_x)$ for all $x \in E_K$. It is obvious that $\{f_j\}_{j\in J}$ is convergent to f^*. On the other hand, the function $\delta_{E_K} : x \mapsto \delta_x$ is holomorphic (see Proposition 3.1) and then continuous. Thus, if $L \subset E_B$ is compact then $\{\delta_x : x \in L\}$ is compact in $P_{H_v(E_K,\tau_E)}$. It follows that $\{f_j\}_{j\in J}$ is convergent to f^* uniformly on the compact subsets of (E_K, τ_E). It is obvious that $f^* = f$ on K.

On the other hand, since K is not pluripolar, by [15] [Theorem 2] $E \in (\widetilde{\Omega}_K)$. It implies that E_K is dense in E.

Then, because f is continuous at some $x_0 \in K$, by Theorem 4.1, $u \circ f^* \in H_v(E_K, \tau_E)$ for all $u \in F'$. Consequently, it follows from Theorem 4.1 that $u \circ f^* \in H(E_K, \tau_E)$ admits an extension $f^*_u \in H_v(E)$ for all $u \in F'$.

Now, because $E \in (\widetilde{\Omega}_K)$, by [25] [Lemma 2.2], K is a set of uniqueness for $H(E)$, hence, it is also of uniqueness for $H_v(E)$.

From Remark 1, $\langle K^*_v \rangle := \text{span} K^*_v$ is $\sigma(P_{H_v(E)}, H_v(E))$-dense, and hence it is norm-dense.

Put $\widehat{T} : \langle K_v^* \rangle \to F$, $\widehat{T}(\delta_x) := f^*(x)$.

Since F' is separating, $\widehat{T}$ is well defined.

Let $x = \sum_{k=1}^n \alpha_k v(x_k)\delta_{x_k} \in B_{\langle K_v^* \rangle}$, the unit ball of $\langle K_v^* \rangle$. For each $u \in F'$ we have the estimate

$$\begin{aligned}|\langle \widehat{T}x, u\rangle| &= \Big|\Big\langle \sum_{k=1}^n \alpha_k v(x_k) f^*(x_k), u \circ f^* \Big\rangle\Big| \\ &= \Big|\Big\langle \sum_{k=1}^n \alpha_k v(x_k)\delta_{x_k}, u \circ f \Big\rangle\Big| \\ &\le \|u \circ f^*\|_v.\end{aligned}$$

This means that $\widehat{T}(B_{\langle K_v^* \rangle})$ is $\sigma(F, F')$-bounded and then it is bounded. Thus, $\widehat{T}$ is a bounded linear mapping.

Since $\langle K_v^* \rangle$ is norm-dense in $P_{H_v(E)}$ we can extend $\widehat{T}$ to $\widetilde{T} : P_{H_v(E)} \to F$.

Finally, by applying Proposition 3.1 there exists unique $\widetilde{f} \in H_v(E, F)$ such that $\widetilde{T} \circ \delta_E = \widetilde{f}$. Obviously, $\widetilde{f} = f^*$ on E_K.

From Theorem 4.2, as Theorem 4.3, we have

Theorem 4.4. *Let E, F be Fréchet spaces and v be a weight on E. Assume that E is Schwartz with an absolute Schauder basis and the topology τ_E and $E \in (\widetilde{\Omega})$. Then there exists a non-pluripolar set $K \in \mathcal{K}(E)$ satisfying the following: if $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence in $H_{G,v}((E_K, \tau_E), F)$ such that $\{f_m\}_{m\in\mathbb{N}}$ is convergent at each $x \in K$ to a function f which is continuous at some $x_0 \in K$ then f has an extension $\widetilde{f} \in H_v(E, F)$ and $\{f_m\}_{m\in\mathbb{N}}$ is convergent to a function $\widetilde{f}$ uniformly on the compact subsets of (E_K, τ_E).*

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REFERENCES

1. W. Arendt and N. Nikolski. Vector-valued holomorphic functions revisited, *Math. Z.*, **2000**, *234*, 777-805.
2. T. T. Quang, L. V. Lam and N. V. Dai. On $\sigma(\cdot, W)$-holomorphic functions and theorems of Vitali-type, *Complex Anal. Oper. Theory*, **2013**, *7*(1), 237-259.
3. D. Vogt. Charakterisierung der Unterräume von s,*Math. Z.*, **1977**, *155*, 109-117.
4. D. Vogt. Subspaces and quotient spaces of s,*in Functional Analysis: Surveys and Recent Results III (ed. K. D. Bierstedt, B. Fuchssteiner), North-Holland. Math. Studies*, **1977**, *27*, 167-187.
5. D. Vogt. Eine Charakterisierung der Potenzeihenräume von endlichem typ und ihre Folgerungen,*Manuscripta Math.*, **1982**, *37*, 269-301.
6. N. Q. Dieu, P. V. Manh, P. H. Bang and L. T. Hung. Vitali's theorem without uniform boundedness, *Publ. Mat.*, **2016**, *60*, 311-334.
7. A. A. Gonchar. A local condition for the single-valuedness of analytic functions of several variables,*(Russian), Mat. Sb. (N.S.)*, **1974**, *93*(135), 296-313.
8. T. Bloom. On the convergence in capacity of rational approximants,*Constr. Approx.*, **2001**, *17*(1), 91-102.
9. T. T. Quang, D. T. Vy, L. T. Hung and P. H. Bang. The Zorn property for holomorphic functions, *Ann. Polon. Math.*, **2017**, *120*(2), 115-133.
10. J. Mujica. A completeness criterion for indutive limits of Banach spaces, *In: Functional Analysis, Holomorphy ans Approximation Theory II, North-Holland Math. Studies*, North-Holland Amsterdam, **1984**, *86*, 319-329.
11. D. Carando and I. Zalduendo. Linearization of functions, *Math. Ann.*, **2004**, *328(4)*, 683-700.
12. J. Mujica. Linearization of bounded holomorphic mappings on Banach spaces, *Trans. Amer. Math. Soc.*, **1991**, *324*(2), 867-887.
13. M. J. Beltrán. Linearization of weighted (LB)-spaces of entire functions on Banach spaces, *Revis. Real Acad. Ciencias Exactas, Físicas y Naturales A*, **2012**, *106*(2), 275-286.

14. H. Schaefer. *Topological vector spaces*, Springer Verlag, 1971.

15. S. Dineen, R. Meise and D. Vogt. Characterization of nuclear Fréchet spaces in which every bounded set is polar, *Bull. Soc. Math. France.*, **1984**, *112*, 41-68.

16. S. Dineen. *Complex Analysis on Infinite Dimensional Spaces*, Springer, New York, 1999.

17. M. Klimek. *Pluripotential Theory*, Oxford Press, 1991.

18. S. Dineen, R. Meise, D. Vogt. Polar Subsets of Locally Convex Spaces, *in: Aspects of Mathematics and its Applications, Ed. J. A. Barroso, Elsevier Science Publisher, B.V.*, **1986**, *34*, 295-319.

19. S. Dineen and Ph. Noverraz. Gaussian measures and polar sets in locally convex spaces, *Arkiv. Mat.*, **1979**, *17*, 217-223.

20. T. T. Quang. On some classes of weighted spaces of weakly holomorphic functions, *Acta Mathematica Vietnamica*, (accepted), DOI: 10.1007/s40306-020-00394-y.

21. E. Jordá. Weighted Vector-Valued Holomorphic Functions on Banach Spaces, *Abstract and Applied Analysis Volume 2013*, **2013**, 9 pages.

22. Dương Thanh Vỹ. *Hội tụ kiểu Tauber cho hàm và ánh xạ chỉnh hình,* luận án tiến sĩ, Trường Đại học Quy Nhơn, 2019.

23. K. F. Ng. On a theorem of Dixmier, *Math. Scand.*, **1971**, *29*, 279-280.

24. G. Köthe. *Topological Vector Spaces II,* Springer, Berlin, Germany, 1979.

25. N. D. Lan. LB^{∞}- structure of space of germs of holomorphic functions, *Publ. Mat.*, **2000**, *44*, 177-192.