# VỀ MỘT PHƯƠNG TRÌNH MA TRẬN

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**TÓM TẮT.** Trong bài báo này, chúng tôi chỉ ra rằng với $A, B$ là các ma trận xác định dương và $M_1, M_2, \ldots, M_m$ là các ma trận không suy biến thì phương trình ma trận

$$X^p = A + \sum_{i=1}^{m} M_i^T \left( B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}} \right)^t M_i$$

có duy nhất nghiệm xác định dương $X^*$. Ngoài ra, bằng cách sử dụng phương pháp lặp, bài báo cũng chỉ ra dãy các ma trận hội tụ về nghiệm $X^*$ của phương trình trên.

**Từ khóa:** *Ma trận xác định dương, phương trình ma trận, định lý điểm bất động, phương pháp lặp.*

# ON A MATRIX EQUATION

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**ABSTRACT.** In this paper, we consider one matrix equation that involves a matrix generalization of the the weighted geometric mean. More precisely, for positive definite matrices $A$ and $B$, for nonsingular matrices $M_1, M_2, \ldots, M_m$, we show that the following equation

$$X^p = A + \sum_{i=1}^{m} M_i^T \left( B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}} \right)^t M_i$$

has a unique positive definite solution. We also study the multi-step stationary iterative method for this equation and prove the corresponding convergence.

**Keywords:** *Positive definite matrice, matrix equation, fixed point theorem, multi-step stationary iterative method.*

## 1. Introduction

Let $\mathbb{M}_n$ be the algebra of $n \times n$ matrices over $\mathbb{C}$ and let $\mathcal{P}_n$ denote the cone of positive definite matrices in $\mathbb{M}_n$. For a real-valued function $f$ and a Hermitian matrix $A \in \mathbb{M}_n$, the matrix $f(A)$ is understood by means of the functional calculus.

Let $A$, $B$ be positive definite matrices, it is well-known that the matrix geometric mean $A \sharp B = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^{1/2} A^{1/2}$ was firstly defined by Pusz and Woronowicz [1]. They showed that the geometric mean is the unique positive definite solution of the Riccati equation

$$XA^{-1}X = B. \tag{1}$$

In 2005, Lim [2] studied the inverse means problem for the geometric mean and the contraharmonic mean. Using the Riccati equation (1) as a lemma, he studied the following equation

$$X = A + 2BX^{-1}B,$$

where $A \leq B$ are positive definite matrices. He showed that this equation has a unique solution of the form $X = \frac{1}{2}\left(A + A \sharp \left(A + 4BA^{-1}B\right)\right)$. Lim and co-authors [3] studied the nonlinear equation

$$X = B \sharp (A + X).$$

They proved that this equation has a unique positive definite solution $X = \frac{1}{2}(B + B\sharp(B + 4A))$. Interestingly, both results were based on elementary approach by solving the corresponding quadratic equations. Recently, Lee and co-authors [4] studied the following matrix equation

$$X^p = A + M^T(X\sharp B)M.$$

Similar to the approach of Lim and Palfia [5], they used the Thompson metric and Banach fixed point theorem to show that the equation has a unique positive definite solution. Recently, Zhai and Jin [6] generalized the last equation for $m$ non-singular matrices. More precisely, they studied two non-linear matrix equations as follows

$$X^p = A + \sum_{i=1}^{m} M_i^T(X\sharp B)M_i$$

and

$$X^p = A + \sum_{i=1}^{j} M_i^T\left(X\sharp B\right)M_i + \sum_{i=j+1}^{m} M_i^T\left(X^{-1}\sharp B\right)M_i,$$

where $p, m, j$ are positive integers such that $1 \le j \le m$, $A, B$ are positive definite matrices and $M_1, M_2, \ldots, M_m$ are nonsingular real matrices.

Recently, Dinh and co-authors [7] studied a more general case of these two equations. They considered similar matrix equations for the weighted matrix geometric mean

$$A\sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

Namely, they studied the following matrix equations

$$X^p = A + \sum_{i=1}^{m} M_i^T\left(X\sharp_t B\right)M_i,$$

and

$$X^p = A + \sum_{i=1}^{j} M_i^T\left(X\sharp_t B\right)M_i + \sum_{i=j+1}^{m} M_i^T\left(X^{-1}\sharp_t B\right)M_i,$$

where $p, m$ are positive integers, $A, B$ are $n\times n$ positive definite matrices and $M_1, M_2, \ldots, M_m$ are $n \times n$ nonsingular real matrices. At the end of the paper, they not only mentioned that the weighted geometric mean $A\sharp_t B$ is a matrix generalization of $a^{1-t}b^t$ for two non-negative numbers $a$ and $b$ but also noticed that there is another symmetric generalization such as $\left(A^{\frac{1-t}{2t}}BA^{\frac{1-t}{2t}}\right)^t$ which appears in the definition of the sandwiched quasi-relative entropy $Tr\left(A^{\frac{1-t}{2t}}BA^{\frac{1-t}{2t}}\right)^t$ (see [8]).

The following theorem was discussed in [7] without a proof.

**Theorem.** ([7] [Theorem 7]) *Let $A, B \in \mathcal{P}_n$, $m$ be positive integers greater than 2, and $p \ge 1$.*

*Then, for nonsingular matrices $M_1, M_2, \ldots, M_m$ in $\mathbb{M}_n$, the following matrix equation*

$$X^p = A + \sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i$$

*has a unique positive definite solution $X^*$ in $\mathcal{P}_n$.*

In this note, we give a detail proof of this theorem. We also study the multi-step stationary iterative method for this equation and prove the corresponding convergence.

## 2. Main results

**Definition 1.** ([9] [Definition 2.1.1]) Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be a operator, we say that $T$ is increasing if $0 < x \leq y$ implies $Tx \leq Ty$.

The following lemma is crucial for us to prove the main results in this paper.

**Lemma 2.** ([9] [Theorem 2.1.6]) *Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be an increasing operator, suppose that there exists $r \in (0,1)$ such that*

$$T(sx) \geq s^r T(x),\ x \in \mathcal{P}_n,\ s \in (0,1).$$

*Then $T$ has a unique fixed point $x^* \in \mathcal{P}_n$.*

**Theorem 3.** *Let $A, B \in \mathcal{P}_n$, $m$ be positive integers greater than 2, and $p \geq 1$. Then, for nonsingular matrices $M_1, M_2, \ldots, M_m$ in $\mathbb{M}_n$, the following matrix equation*

$$X^p = A + \sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i \tag{2}$$

*has a unique positive definite solution $X^*$ in $\mathcal{P}_n$.*

*Proof.* Let consider the function

$$T(X) = \left(A + \sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}}.$$

We will show that $T(X)$ satisfies the conditions of Lemma 2, so it has a unique fix point $X^*$ in $\mathcal{P}_n$. That leads to the fact that the equation (2) has a unique positive definite solution $X^*$ in $\mathcal{P}_n$.

Let $0 < X_1 \leq X_2$, we have $\left(B^{\frac{1-t}{2t}} X_1 B^{\frac{1-t}{2t}}\right)^t \leq \left(B^{\frac{1-t}{2t}} X_2 B^{\frac{1-t}{2t}}\right)^t$. Consequently,

$$M_i^T \left(B^{\frac{1-t}{2t}} X_1 B^{\frac{1-t}{2t}}\right)^t M_i \leq M_i^T \left(B^{\frac{1-t}{2t}} X_2 B^{\frac{1-t}{2t}}\right)^t M_i, \quad i = \overline{1,m}.$$

Therefore,

$$A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X_1 B^{\frac{1-t}{2t}}\right)^t M_i \leq A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X_2 B^{\frac{1-t}{2t}}\right)^t M_i.$$

Since $p \geq 1$, the function $x^{\frac{1}{p}}$ is a monotone operator on $(0,+\infty)$. We have

$$\begin{aligned} T(X_1) &= \left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X_1 B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}} \\ &\leq \left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X_2 B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}} \\ &= T(X_2), \end{aligned}$$

so the function $T(X)$ is increasing.

Let $X \in \mathcal{P}_n$. For $t \in (0,1)$ and $p \geq 1$, there exists a constant $r \in (0,1)$ such that $r \geq \dfrac{t}{p}$. It is obvious that

$$\left(B^{\frac{1-t}{2t}} (sX) B^{\frac{1-t}{2t}}\right)^t = s^t \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t,$$

for any $s \in (0,1)$.

Since $rp \geq t$, we have $s^{rp} \leq s^t < 1$ for all $s \in (0,1)$. Therefore,

$$A+s^t\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i \geq s^{rp}\left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i\right).$$

By the monotonicity of the function $x^{\frac{1}{p}}$, we have

$$\begin{aligned} T(sX) &= \left(A+s^t\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}} \\ &\geq \left(s^{rp}\left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i\right)\right)^{\frac{1}{p}} \\ &= s^r\left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}} \\ &= s^r T(X). \end{aligned}$$

Thus, $T(X)$ satisfies all conditions of Lemma 2. In other words, equation (2) has a unique positive solution $X^* \in \mathcal{P}_n$. □

Now, let $X_1, X_2, \ldots, X_m$ be initial matrices in $\mathcal{P}_n$ and consider the multi-step stationary iterative method for the equation (2) as following

$$X_{lm+j} = \left(A+\sum_{i=1}^{m} M_i^T \left(B^{\frac{1-t}{2t}} X_{(l-1)m+j} B^{\frac{1-t}{2t}}\right)^t M_i\right)^{\frac{1}{p}} \tag{3}$$

for $l = 1, 2, 3, \ldots$ and $j = 1, 2, \ldots, m$.

In the following theorem, we will show that the matrix sequence $\{X_k\}$ generated by (3) converges to $X^*$.

**Theorem 4.** *For any $X_1, X_2, \ldots, X_m \in \mathcal{P}_n$, the matrix sequence $\{X_k\}$ generated by (3) converges to the unique positive definite solution $X^*$ of the equation (2).*

*Proof.* For matrices $X_1, X_2, \ldots, X_m$ and $X^*$, there exists $a \in (0, 1)$ such that

$$aX^* \leq X_j \leq a^{-1}X^*, \; j = 1, 2, \ldots, m. \tag{4}$$

We will show that for any $b \in \mathbb{N}$ we have

$$a^{r^b}X^* \leq X_k \leq a^{-r^b}, \; k = bm + j \; (j = 1, 2, \ldots, m) \tag{5}$$

for some $r \in (0, 1)$ and $r \geq \dfrac{t}{p}$. Then, according to the fact that $\lim\limits_{b\to\infty} a^{r^b} = \lim\limits_{b\to\infty} a^{-r^b} = 1$ and the Squeeze theorem in the normal cone $\mathcal{P}_n$, it implies that $\{X_k\}$ converges to $X^*$.

Now, we prove (5) by using the method of mathematical induction. For $b = 0$, the inequality (5) reduces to the case of (4). Assume that (5) is true for $b = q - 1$ for some positive interger $q$, it means

$$a^{r^{q-1}}X^* \leq X_{(q-1)m+j} \leq a^{-r^{q-1}}X^* \tag{6}$$

for $k = (q - 1)m + j$ and $j = 1, 2, \ldots, m$.

Since $X_{qm+j} = T(X_{(q-1)m+j})$ and $T(X)$ is increasing, it implies from (6) that

$$T(a^{r^{q-1}}X^*) \leq T(X_{(q-1)m+j}) = X_{qm+j} \leq T(a^{-r^{q-1}}X^*).$$

Moreover, $T(sX) \geq s^rT(X)$ in the case $s \in (0, 1)$ and $T(sX) \leq s^rT(X)$ in the case $s > 1$. Therefore,

$$T(a^{r^{q-1}}X^*) \geq \left(a^{r^{q-1}}\right)^r T(X^*) = a^{r^q}T(x^*) = a^{r^q}X^*$$

and

$$T\left(a^{-r^{q-1}}X^*\right) \leq a^{-r^q}T(X^*) = a^{-r^q}X^*.$$

So, we have

$$a^{r^q}X^* \leq X_{qm+j} \leq a^{-r^q}X^*.$$

Thus, (5) is true, and $\{X_k\}$ converges to $X^*$. □

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