TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Một số bất đẳng thức cho đa thức ma trận

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TÓM TẮT

Đa thức ma trận (đôi khi được gọi là λ-ma trận) là đa thức một biến phức với hệ số là các ma trận. Đa thức ma trận đóng một vai trò quan trọng trong khoa học và kỹ thuật. Mục đích chính của bài báo này là nghiên cứu một số bất đẳng thức có liên quan đến giá trị riêng và chuẩn của đa thức ma trận.

Từ khóa: *Đa thức ma trận, bất đẳng thức, giá trị riêng, chuẩn.*

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Some inequalities for matrix polynomials

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ABSTRACT

A matrix polynomial (sometimes known as a λ-matrix) is a polynomial of a complex variable with matrix coefficients. Matrix polynomials play a crucial role in science and engineering. The main aim of this paper is to study some inequalities which are related to eigenvalues and norms of matrix polynomials.

Key words: *Matrix polynomial, inequalities, eigenvalue, norm.*

1. INTRODUCTION

Let $\mathbb{C}^{n\times n}$ denote the set of scalar matrices of size $n\times n$ whose entries are complex numbers. For a *matrix polynomial* we mean the matrix-valued function of a complex variable of the form

$$P(z) = A_d z^d + ... + A_1 z + A_0,$$

where $A_i \in \mathbb{C}^{n\times n}$ for all i = 0,...,d. If $A_d \neq 0$, $P(z)$ is called a matrix polynomial of *degree* d. When $A_d = I$, the identity matrix in $\mathbb{C}^{n\times n}$, the matrix polynomial P(z) is called a *monic*.

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Then the vector x is called, as usual, an *eigenvector* of $P(z)$ associated to the eigenvalue λ. Note that each eigenvalue of $P(z)$ is a root of the *characteristic polynomial* $\det(P(z))$.

The *polynomial eigenvalue problem (PEP)* is to find an eigenvalue λ and a non-zero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. For $m = 1$, (PEP) is actually the *generalized eigenvalue problem (GEP)*

$$Ax = \lambda Bx,$$

and, in addition, if $B = I$, we have the standard eigenvalue problem

$$Ax = \lambda x.$$

For $m = 2$ we have the *quadratic eigenvalue problem (QEP)*.

The theory of matrix polynomials was primarily devoted by two works, both of which are strongly motivated by the theory of *vibrating systems*: one by Frazer, Duncan, and Collar in 1938 [FDC], and the other by P. Lancaster in 1966.[1]

(QEPs), and more generally (PEPs), play an important role in applications to science and engineering. We refer to the book of I. Gohberg, P. Lancaster and L. Rodman[2] for a theory of matrix polynomials and their applications.

Matrix analysis is a research field of basic interest and has applications in scientific computing, control and systems theory, operations research, mathematical physics, statistics, economics and engineering disciplines. Sometimes it is also needed in other areas of pure mathematics.

A lot of theorems in matrix analysis appear in the form of inequalities. Given any complex-valued function defined on matrices, there are inequalities for it. We may say that matrix inequalities reflect the quantitative aspect of matrix analysis.

In this paper, we propose some inequalities which are related to eigenvalues and norms of matrix polynomials.

The paper is organized as follows. In the

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next section we recall some preliminaries. In Section 3 we establish some inequalities related to norms of matrix polynomials. Finally, in the last section we propose some inequalities for eigenvalues of matrix polynomials.

2. PRELIMINARIES

Throughout this paper, by a positive integer p we mean $p \geq 1$ or $p = \infty$.

For a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, a positive integer p and a vector p-norm $|.|_p$ on $\mathbb{C}^n$, the *matrix p-norm of A* is defined by

$$|A|_p := \begin{cases} \left(\sum_{i,j=1}^{n} |a_{ij}|^p\right)^{\frac{1}{p}}, & (1 \leq p < \infty), \\ \max_{i,j=1,\dots,n} |a_{ij}| \quad , & (p = \infty). \end{cases}$$

The *operator p-norm* of A is defined by $\|A\|_p := \max\{|Ax|_p \ : \ |x|_p = 1\}$.

For any matrix $A \in \mathbb{C}^{n\times n}$, the number

$$\rho(A) := \max\{|\lambda|, \lambda \in \sigma(A)\}$$

is called the *spectral radius* of A, where $\sigma(A)$ denotes the spectrum of A, i.e. the set of all eigenvalues of A.

The spectral radius can be compared to the operator p-norm, as follows.

Lemma 2.1.[3]. For any $A \in \mathbb{C}^{n\times n}$, $\rho(A) \leq \|A\|_p$.

For a (d + 1) - tuple $\mathbf{A} = (A_0, ..., A_d)$ of matrices $A_i \in \mathbb{C}^{n\times n}$, the matrix polynomial

$$P_{\mathbf{A}}(z) = A_d z^d + ... + A_1 z + A_0$$

is called the *matrix polynomial associated* to $\mathbf{A}$. The spectrum of the matrix polynomial $P_{\mathbf{A}}(z)$ is defined by

$$\sigma(\mathbf{A}) := \sigma(P_{\mathbf{A}}(z)) = \{\lambda \in \mathbb{C} | det(P_{\mathbf{A}}(z)) = 0\},$$

which is the set of all its eigenvalues.

For a monic matrix polynomial $P_{\mathbf{A}}(z) = I.z^d + A_{d-1}z^{d-1} + \ldots + A_1 z + A_0$, with $A_i \in \mathbb{C}^{n\times n}$, the$(dn \times dn)-$matrix

$$C_{\mathbf{A}} := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{d-1} \end{bmatrix}$$

is called the *companion matrix* of the matrix polynomial $P_{\mathbf{A}}(z)$ or of the tuple $(A_0, \ldots, A_{d-1}, I)$.

Note that the spectrum $\sigma(A)$ of $P_{\mathbf{A}}(z)$ coincides to the spectrum $\sigma(C_{\mathbf{A}})$ of $C_{\mathbf{A}}$ [2]

For two (d+1)-tuples $\mathbf{A} = (A_0, \ldots, A_{d-1}, I)$ and $\bar{\mathbf{A}} = (\bar{A}_0, \ldots, \bar{A}_{d-1}, I)$, the relation between the operator norms of their difference and those of their companion matrices is given in the following lemma.

Lemma 2.2.[4]. Let $\mathbf{A} = (A_0, \ldots, A_{d-1}, I)$ and $\bar{\mathbf{A}} = (\bar{A}_0, \ldots, \bar{A}_{m-1}, I)$ be $(d+1)$-tuples. Then for any integer $p > 0$, we have

$$\|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p = \|\mathbf{A} - \bar{\mathbf{A}}\|_p.$$

Theorem 2.1.[5]. Let $Z \in \mathbb{C}^{n\times n}$ be a positive definite matrix with extremal eigenvalues a, b. Then for all vectors $h \in \mathbb{C}^n$,

$$|h|_p.|Z.h|_p \leq \frac{(\frac{a}{b})^{\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}}}{2} \langle h, Zh \rangle ,$$

where $|\cdot|_p$ denotes the matrix p-norm.

For a vector $x \in \mathbb{C}^n$. Note that x^* is the conjugate transpose of x.

Concerning inequalities of eigenvalues of scalar matrices, we have the following results obtained by Hassan (2014).

Theorem 2.2.[6]. Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and x and y $\in \mathbb{C}^n$ such that $|x| = |y| = 1$ and $x^*y = 0$. Assume that the eigenvalues of A, in increasing order, are $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Then we have the following Wielandt inequalities.

(1) $|x^*Ay| \leq \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\sqrt{(x^*Ax)(y^*Ay)}$;

(2) $|x^*Ax - y^*Ay| \leq \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}((x^*Ax + y^*Ay)^2 - 4(x^*Ay)^2)^{\frac{1}{2}}$;

(3) $|x^*Ay|^2 \leq \max\left\{\left(\frac{\lambda_i - \lambda_j}{\lambda_i^r + \lambda_j^r}\right)^2\right\}_{\substack{i,j=1 \\ i\neq j}}^{n} (x^*A^r x) \cdot (y^*A^r y)$.

Theorem 2.3.[6]. Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and x and y $\in \mathbb{C}^n$ such that $|x| = |y| = 1$ and $x^*y = 0$. Assume that the eigenvalues of A, in increasing order, are $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Then

$$|x^*Ay| \leq \frac{1}{2}\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}(x^*Ax + y^*Ay).$$

Theorem 2.4.[6]. Assume that A is a positive definite matrix of size n, and let the eigenvalues of A, in increasing order, be $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Then

$$|x^*Ax - y^*Ay| \leq \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}(x^*Ax + y^*Ay).$$

3. SOME INEQUALITIES FOR NORMS OF MATRIX POLYNOMIALS

In this section, we will propose some inequalities for norms of matrix polynomials.

For an (d+1)-tuple $\mathbf{A} = (A_0, \cdots, A_{d-1}, A_d)$ of matrices $A_i \in \mathbb{C}^{n\times n}$. $\|\mathbf{A}\|_p := \max\{|\mathbf{A}x|_p : |x|_p = 1\}$ with $\mathbf{A}$ is a matrix of dimension $(d+1)n \times n$.

Firstly, we establish some properties of norms of tuples of matrices.

Proposition 3.1. Let $\mathbf{A} = (A_0, \cdots, A_{d-1}, A_d)$ and $\mathbf{B} = (B_0, \cdots, B_{d-1}, B_d)$, where $A_i, B_j \in \mathbb{C}^{n\times n}$. Then for any positive integer p, the followings hold true.

(1) There is a constant $c \geq 0$ such that $|\mathbf{A}v|_p \leq c|v|_p$ for any vector $v \in \mathbb{C}^n$.

(2) $\|\mathbf{A}\|_p \geq 0$, with equality if and only if $\mathbf{A} = \mathbf{0}$.

(3) $\|\mathbf{A} + \mathbf{B}\|_p \leq \|\mathbf{A}\|_p + \|\mathbf{B}\|_p$.

(4) $\|c\mathbf{A}\|_p = |c|.\|\mathbf{A}\|_p$, for $c \in \mathbb{R}$.

(5) $\|\mathbf{A}^T\mathbf{B}\|_p \leq \|\mathbf{A}^T\|_p\|\mathbf{B}\|_p$.

(6) $\|\mathbf{A}\|_p = \|\mathbf{A}^T\|_p$.

(7) $\|\mathbf{A}^T\mathbf{A}\|_p = \|\mathbf{A}\mathbf{A}^T\|_p = \|\mathbf{A}\|_p^2$.

(8) $|(\mathbf{A}v, w)| \leq \|\mathbf{A}\|_p.|v|_p.|w|_p$, for any $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^{(d+1)n\times n}$.

(9) $\|\mathbf{A}\|_1 \leq n^2\|\mathbf{A}\|_p$.

Proof. Note that for a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}, p \in \mathbb{Z}^+$
$\|A\|_p = \inf\{c \geq 0 : |Ax|_p \leq c|x|_p,\ x \in \mathbb{C}^n\}$.
(1) For any vector $v \in \mathbb{C}^n$, we have

$$\mathbf{A}v = \begin{bmatrix} A_0v \\ A_1v \\ \vdots \\ A_dv \end{bmatrix}.$$

For $1 \leq p < \infty$, we have

$$|\mathbf{A}v|_p^p = \sum_{i=0}^{d} |A_iv|_p^p.$$

For each $i = 0, \ldots, d$, there exists $c_i > 0$ such that $|A_iv|_p \leq c_i|v|_p$, for all $i = 0, \ldots, d$.
Taking $c' = \max\limits_{i=0,\ldots,d}\{c_i\}$, we get $|\mathbf{A}v|_p \leq c|v|_p$, where $c = c'(d+1)^{1/p}$.
For $p = \infty$, we have
$|\mathbf{A}v|_\infty = \max\limits_{i=0,\ldots,d} |A_iv|_\infty$.
For each $i = 0, \ldots, d$, there exists $c_i > 0$ such that $|A_iv|_\infty \leq c_i|v|_\infty$, for all $i = 0, \ldots, d$.
Taking $c = \max\limits_{i=0,\ldots,d}\{c_i\}$, we have $|\mathbf{A}v|_\infty \leq c|v|_\infty$.
(2) It is obvious that $\|\mathbf{A}\|_p \geq 0$.
$\|\mathbf{A}\|_p = 0$ if and only if

$$|\mathbf{A}x|_p = 0 \Leftrightarrow \left|\begin{bmatrix} A_0x \\ \vdots \\ A_dx \end{bmatrix}\right|_p = 0$$

$\Leftrightarrow\ |A_ix|_p = 0, \forall i = 0, ..., d.$
$\Leftrightarrow\ \|A_i\|_p = 0,\ \forall i = 0, \ldots, d.$
$\Leftrightarrow\ A_i = 0,\ \forall i = 0, \ldots, d.$
Thus $\mathbf{A} = \mathbf{0}$.
(3) We have
$\|\mathbf{A} + \mathbf{B}\|_p = \max\{|(\mathbf{A} + \mathbf{B})x|_p : |x|_p = 1\}$.
For any $x \in \mathbb{C}^n$ such that $|x|_p = 1$, we have
$|(\mathbf{A} + \mathbf{B})x|_p = |\mathbf{A}x + \mathbf{B}x|_p \leq |\mathbf{A}x|_p + |\mathbf{B}x|_p$.
Taking maximum over all $x \in \mathbb{C}^n$ such that $|x|_p = 1$, we obtain $\|\mathbf{A}+\mathbf{B}\|_p \leq \|\mathbf{A}\|_p+\|\mathbf{B}\|_p$.
(4) We have $\|c\mathbf{A}\|_p = \max\{|c\mathbf{A}x|_p : |x|_p = 1\}$
$= \max\{|c|.|\mathbf{A}x|_p : |x|_p = 1\}$
$= |c|. \max\{|\mathbf{A}x|_p : |x|_p = 1\}$
$= |c|.\|\mathbf{A}\|_p$.
(5) For any $x \in \mathbb{C}^n$ such that $|x|_p = 1$, we have
$|(\mathbf{A}^T\mathbf{B})x|_p = |\mathbf{A}^T(\mathbf{B}x)|_p$

$\leq \|\mathbf{A}^T\||\mathbf{B}x|_p \leq \|\mathbf{A}^T\|\|\mathbf{B}\||x|_p.$

Thus $\|\mathbf{A}^T\mathbf{B}\| \leq \|\mathbf{A}^T\|\|\mathbf{B}\|.$

(6) For any vector $x \in \mathbb{C}^n$, we have
$|\mathbf{A}x|_p^2 = |(\mathbf{A}x, \mathbf{A}x)| = |(x, \mathbf{A}^T\mathbf{A}x)|$
$\leq |x|_p.|\mathbf{A}^T\mathbf{A}x|_p$ (Cauchy-Schwarz inequality)
$\leq \|\mathbf{A}^T\mathbf{A}\|_p.|x|_p^2.$
$\Rightarrow |\mathbf{A}x|_p^2 \leq \|\mathbf{A}^T\mathbf{A}\|_p.|x|_p^2.$
$\Rightarrow |\mathbf{A}x|_p \leq \sqrt{\|\mathbf{A}^T\mathbf{A}\|_p.|x|_p^2}.$
$\Rightarrow \|\mathbf{A}\|_p \leq \sqrt{\|\mathbf{A}^T\mathbf{A}\|_p}.$
$\Rightarrow \|\mathbf{A}\|_p^2 \leq \|\mathbf{A}^T\mathbf{A}\|_p \leq \|\mathbf{A}^T\|_p\|\mathbf{A}\|_p. \quad (3)$
When $\mathbf{A} \neq 0$, we receive
$\|\mathbf{A}\|_p \leq \|\mathbf{A}^T\|_p$ and $\|\mathbf{A}^T\|_p \leq \|(\mathbf{A}^T)^T\|_p.$
So $\|\mathbf{A}\|_p = \|\mathbf{A}^T\|_p.$
(7) From (5) and (6), we have
$\|\mathbf{A}\|_p^2 \leq \|\mathbf{A}^T\mathbf{A}\|_p \leq \|\mathbf{A}^T\|_p\|\mathbf{A}\|_p = \|\mathbf{A}\|_p^2$
Thus $\|\mathbf{A}\|_p^2 = \|\mathbf{A}^T\mathbf{A}\|_p.$
(8) For any $v \in \mathbb{C}^n, w \in \mathbb{C}^{(d+1)n\times n}$, apply the Cauchy-Schwarz inequality, we obtain
$|(\mathbf{A}v, w)| \leq |\mathbf{A}v|_p.|w|_p \leq \|\mathbf{A}\|_p.|v|_p.|w|_p.$
(9) From (8), let $v = e_i \in \mathbb{C}^n, w = e_j \in \mathbb{C}^{(d+1)n}$,we have
$|a_{ij}| = |(\mathbf{A}e_i, e_j)| \leq \|\mathbf{A}\|_p$
with $i = 1, \dots, n$ and $j = 1, \dots, (d+1)n.$
Hence, $\sum_{i,j=1}^n |a_{ij}| \leq n^2\|\mathbf{A}\|_p$ and $\|A_0\| \leq n^2\|\mathbf{A}\|.$
Similarly $\|A_i\| \leq n^2\|\mathbf{A}\|_p, \forall i = 0, \dots, d.$
Therefore, $\max_{i=0,\dots,d} \|A_i\| \leq n^2\|\mathbf{A}\|_p.$
Thus $\|\mathbf{A}\|_1 \leq n^2\|\mathbf{A}\|_p.$

In order to prepare for main results of this section, we need the following lemmas.

Lemma 3.1. Let $P_{\bar{\mathbf{A}}}(\lambda) = I.\lambda^d$ be a monic matrix polynomial whose corresponding companion matrix is $C_{\bar{\mathbf{A}}}$. Then for any integer p $>$ 0, we have

$$\|C_{\bar{\mathbf{A}}}\|_p \leq 1.$$

Proof. We have

$$C_{\bar{\mathbf{A}}} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For any $x = (x_1, x_2, ..., x_{dn})^T \in C^{dn}$ such that $|x|_p = 1$, we have

$$C_{\bar{\mathbf{A}}}x = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_{dn} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ . \\ x_{dn} \\ 0 \\ . \\ 0 \end{bmatrix}$$

and

$$\left| \begin{bmatrix} x_{n+1} \\ . \\ x_{dn} \\ 0 \\ . \\ 0 \end{bmatrix} \right|_p = \begin{cases} (\sum_{i=n+1}^{dn} |x_i|^p)^{\frac{1}{p}}, & p \geq 1 \\ \max\limits_{i=n+1,\dots,dn} |x_i|, & p = \infty. \end{cases}$$

Moreover,

$$\left(\sum_{i=n+1}^{dn} |x_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{dn} |x_i|^p \right)^{\frac{1}{p}} = |x|_p = 1$$

and

$$\max_{i=n+1,\dots,dn} |x_i| \leq 1.$$

Thus

$$\left| \begin{bmatrix} x_{n+1} \\ . \\ x_{dn} \\ 0 \\ . \\ 0 \end{bmatrix} \right|_p \leq 1, \quad \text{for any} \quad p \geq 1.$$

It follows that $|C_{\bar{\mathbf{A}}}x|_p \leq 1$ for any $p \geq 1$.
Since $||C_{\bar{\mathbf{A}}}||_p = \max\{|C_{\bar{\mathbf{A}}}x|_p : |x|_p = 1\}$, we have $||C_{\bar{\mathbf{A}}}||_p \leq 1$.

The proof is complete.

Lemma 3.2. Let $\mathbf{A} = (A_0, \dots, A_{d-1}, I)$ and $\bar{\mathbf{A}} = (0, \dots, 0, I)$ be (d + 1)-tuples. Then for any integer p $>$ 0, we have

$$\|\mathbf{A} - \bar{\mathbf{A}}\|_p \leq \|\mathbf{A}\|_p.$$

Proof. We have

$$\|\mathbf{A} - \bar{\mathbf{A}}\|_p = \max\{|(\mathbf{A} - \bar{\mathbf{A}})x|_p : |x|_p = 1\}.$$

Note that the (d+1)-tuple $(\mathbf{A} - \bar{\mathbf{A}}) = (A_0, A_1, \cdots, A_{d-1}, 0)$ is a matrix of dimension

$(d+1)n \times n$:

$$\mathbf{A} - \bar{\mathbf{A}} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \\ 0 \end{bmatrix}.$$

Then, with $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, we have

$$(\mathbf{A} - \bar{\mathbf{A}})x = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \\ 0 \end{bmatrix} x = \begin{bmatrix} A_0 x \\ A_1 x \\ \vdots \\ A_{d-1} x \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \\ \vdots \\ t_{dn} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

with $\begin{bmatrix} t_{in+1} \\ \vdots \\ t_{(i+1)n} \end{bmatrix} = A_i x \quad , \forall i = 0, ..., d-1.$

Thus

$$|(\mathbf{A} - \bar{\mathbf{A}})x|_p = \left(\sum_{i=1}^{dn} |t_i|^p \right)^{1/p}$$
$$\leq \left(\sum_{i=1}^{dn} |t_i|^p + \sum_{j=1}^{n} |x_j|^p \right)^{1/p} (1).$$

We also have

$$\mathbf{A}x = \begin{bmatrix} A_0 x \\ A_1 x \\ \vdots \\ A_{d-1} x \\ Ix \end{bmatrix} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \\ \vdots \\ t_{dn} \\ x_1 \\ \vdots \\ x_n \end{bmatrix},$$

and $|\mathbf{A}x|_p = \left(\sum_{i=1}^{dn} |t_i|^p + \sum_{j=1}^{n} |x_j|^p \right)^{1/p}$ (2).

It follows from (1) and (2) that

$$|(\mathbf{A} - \bar{\mathbf{A}})x|_p \leq |\mathbf{A}x|_p.$$

Then $\max\{|(\mathbf{A} - \bar{\mathbf{A}})x|_p : |x|_p = 1\} \leq \max\{|\mathbf{A}x|_p : |x|_p = 1\}$.

Thus $\quad \|\mathbf{A} - \bar{\mathbf{A}}\|_p \leq \|\mathbf{A}\|_p$.

The proof is complete.

The first main result in this section is presented as follows, which is a version for matrix polynomials of Lemma 2.1.

Theorem 3.1. Let $P_{\mathbf{A}}(\lambda) = I.\lambda^d + A_{d-1}\lambda^{d-1} + \ldots + A_1\lambda + A_0$ be a monic matrix polynomial whose corresponding (d + 1)- tuple is $\mathbf{A} = (A_0, ..., A_{d-1}, I)$. Then for any integer p > 0, we have

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|_p + 1.$$

Proof. It follows from Lemma 2.1 that

$$\rho(C_{\mathbf{A}}) \leq \|C_{\mathbf{A}}\|_p.$$

From the sub-additivity of the operator p-norm, we have

$$\|C_{\mathbf{A}}\|_p \leq \|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p + \|C_{\bar{\mathbf{A}}}\|_p,$$

where $C_{\bar{\mathbf{A}}}$ is the companion matrix of the matrix polynomial $P_{\bar{\mathbf{A}}}(\lambda) = I.\lambda^d$.

It follows from Lemma 3.1 and Lemma 3.2 that

$$\|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p \leq \|\mathbf{A}\|_p \quad and \quad \|C_{\bar{\mathbf{A}}}\|_p \leq 1.$$

Since $\rho(\mathbf{A}) = \rho(C_{\mathbf{A}})$, we have

$$\rho(\mathbf{A}) \leq ||\mathbf{A}||_p + 1.$$

The proof is complete.

For a matrix $A \in \mathbb{C}^{nxn}$. If A is positive definite matrix then we write A>0.

Now we establish a version for matrix polynomials of Theorem 2.1.

Theorem 3.2. Let $P_{\mathbf{A}}(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial with $A_i > 0$. Let a_i, b_i be extremal eigenvalues of A_i, for all i=0,...,d. Then for any vector $h \in \mathbb{C}^n$, we have

$$|h|_p.|P_{\mathbf{A}}(\lambda).h|_p \leq H \langle h, P_{\mathbf{A}}(|\lambda|)h \rangle,$$

where $H = \max\left\{ \frac{a_i + b_i}{2\sqrt{a_i b_i}} \right\}_{i=0}^{d}$.

Proof. We have

$|h|_p|.P_{\mathbf{A}}(\lambda).h|_p = |h|_p.|\sum_{i=0}^{d} A_i\lambda^i.h|_p,$

$\leq |h|_p. \sum_{i=0}^{d} |A_i\lambda^i.h|_p,$

$\leq |h|_p. \sum_{i=0}^{d} (|A_i.h|_p)(|\lambda^i|),$

$= \sum_{i=0}^{d} (|h|_p.|A_i.h|_p).(|\lambda^i|).$

It follows from Theorem 2.1 that

$|h|_p.|A_i.h|_p \leq \frac{(\frac{a_i}{b_i})^{\frac{1}{2}} + (\frac{b_i}{a_i})^{\frac{1}{2}}}{2}. \langle h, A_i h \rangle.$

Thus

$|h|_p|P_{\mathbf{A}}(\lambda).h|_p$
$\leq \sum_{i=0}^{d}\left(\frac{(\frac{a_i}{b_i})^{\frac{1}{2}}+(\frac{b_i}{a_i})^{\frac{1}{2}}}{2}\right)\langle h, A_ih\rangle .|\lambda^i|$
$\leq H.\sum_{i=0}^{d}\langle h, A_ih\rangle .|\lambda^i|$
$= H.\sum_{i=0}^{d}\langle h, A_ih\rangle$
$= H.\left\langle h, \sum_{i=0}^{d} A_i|\lambda|^i.h\right\rangle$
$= H.\langle h, P_{\mathbf{A}}(|\lambda|)h\rangle .$
The proof is complete.

4. SOME INEQUALITIES FOR EIGENVALUES OF MATRIX POLYNOMIALS

In this section we establish some inequalities for eigenvalues of matrix polynomials.

Firstly, we propose a version of Theorem 2.2 for matrix polynomials.

Theorem 4.1. Let $P_A(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial whose coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ are positive definite. Let $x, y \in \mathbb{C}^n$ be such that $|x| = |y| = 1$ and $x^*y = 0$. Assume $\sigma(A_i) = \{\lambda_1^i, \dots, \lambda_{n_i^k}^i\}$ is the spectrum of the matrix A_i and $\lambda_1^i \leq \lambda_2^i \leq \dots \leq \lambda_{n_i}^i$, for all $i = 0, \dots, d$. Then the followings hold true.

(1) $|x^*P_A(\lambda)y| \leq K((x^*P_A(|\lambda|)x)(y^*P_A(|\lambda|y))^{\frac{1}{2}}$.

(2) $|x^*P_A(\lambda)x - y^*P_A(\lambda)y|$
$\leq K((d+1)(x^*P_A(|\lambda|)x + y^*P_A(|\lambda|)y)^2$
$-4(x^*P_A(|\lambda|)y)^2)^{\frac{1}{2}}$.

Here $K = \max\left\{\frac{\lambda_{n_i}^i-\lambda_1^i}{\lambda_{n_i}^i+\lambda_1^i}\right\}_{i=0}^{d}$.

Proof. (1) We have

$$|x^*P_A(\lambda)y| = |\sum_{i=0}^{d} x^*A_i\lambda^iy| \leq \sum_{i=0}^{d}|x^*A_iy|\cdot|\lambda|^i.$$

It follows from Theorem 2.2 (1) that

$$\sum_{i=0}^{d}|x^*A_iy||\lambda|^i \leq \sum_{i=0}^{d}(\frac{\lambda_{n_i}^i-\lambda_1^i}{\lambda_{n_i}^i+\lambda_1^i})\cdot(x^*A_ix)^{\frac{1}{2}}\cdot (y^*A_iy)^{\frac{1}{2}}\cdot|\lambda|^i$$
$$\leq K\cdot\sum_{i=0}^{d}(x^*A_i|\lambda|^ix)^{\frac{1}{2}}\cdot(y^TA_i|\lambda|^iy)^{\frac{1}{2}}$$
$$\leq K\cdot(\sum_{i=0}^{d}x^*A_i|\lambda|^ix)^{\frac{1}{2}}\cdot(\sum_{i=0}^{d}y^*A_i|\lambda|^iy)^{\frac{1}{2}}$$
$$= K\cdot(x^*(\sum_{i=0}^{d}A_i|\lambda|^i)x)^{\frac{1}{2}}\cdot(y^*(\sum_{i=0}^{d}A_i|\lambda|^i)y)^{\frac{1}{2}}$$
$$= K\cdot\sqrt{(x^*P_A(|\lambda|)x)(y^*P_A(|\lambda|)y)}.$$

The proof is complete.

(2) Similarly to the arguments given in (1).

Theorem 4.2. Let $P_A(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial whose coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ are positive definite for all $i = 0, \dots, d$ and $A_iA_j > 0$ for all $i, j = 0, \dots, d$. Let $x, y \in \mathbb{C}^n$ such that $|x| = |y| = 1$ and $x^*y = 0$. Assume $\sigma(A_i) = \{\lambda_1^i, \dots, \lambda_{n_i^k}^i\}$ is the spectrum of the matrix A_i and $\lambda_1^i \leq \lambda_2^i \leq \dots \leq \lambda_{n_i}^i$, for all $i = 0, \dots, d$. Then, we have

$$|x^*P_A(\lambda)y|^2 \leq \begin{cases} M.(x^*P_A^2(|\lambda|)x)(y^*P_A^2(|\lambda|)y) & , |\lambda| \geq 1\\ M.(x^*P_A^2(|\lambda+2|)x)(y^*P_A^2(|\lambda+2|)y), & |\lambda| < 1,\end{cases}$$

where

$$M = (d+1)\cdot\max\left\{\max\left\{\left(\frac{\lambda_m^i-\lambda_k^i}{(\lambda_m^i)^2+(\lambda_k^i)^2}\right)^2\right\}_{\substack{m,k=1\\ m\neq k}}^{n_i}\right\}_{i=0}^{d}.$$

Proof. We have

$$|x^*P_A(\lambda)y|^2 = \left(\sum_{i=0}^{d}x^*A_i\lambda^iy\right)^2 \leq (d+1)\sum_{i=0}^{d}(x^*A_iy)^2(\lambda^i)^2.$$

It follows from Theorem 2(3) that

$$(d+1)\sum_{i=0}^{d}(x^*A_iy)^2(\lambda^i)^2 \leq (d+1).\sum_{i=0}^{d}\max\left\{\left(\frac{\lambda_m^i-\lambda_k^i}{(\lambda_m^i)^2+(\lambda_k^i)^2}\right)^2\right\}_{\substack{m,k=1\\ m\neq k}}(x^*A_i^2x)(y^*A_i^2y)(\lambda^i)^2$$
$$\leq M\cdot\sum_{i=0}^{d}(x^*A_i^2x)(y^*A_i^2y)(|\lambda|^i)^2.$$

When $|\lambda| \geq 1$:

$$|x^*P_A(\lambda)y|^2 \leq M\cdot\sum_{i=0}^{d}(x^*A_i^2x)(y^TA_i^2y)(|\lambda|^i)^4$$
$$= M\cdot\sum_{i=0}^{d}(x^*A_i^2|\lambda^i|^2x)(y^*A_i^2|\lambda^i|^2y)$$
$$\leq M\cdot\sqrt{\left(\sum_{i=0}^{d}(x^*A_i^2|\lambda^i|^2x)^2\right)\cdot\left(\sum_{i=0}^{d}(y^*A_i^2|\lambda^i|^2y)^2\right)}.$$

Since $A_i^2 > 0, \forall i = 0, ..., d$, thus,

$$|x^*P_A(\lambda)y|^2 \leq M\cdot\sqrt{\left(\sum_{i=0}^{d}x^*A_i^2|\lambda^i|^2x\right)^2\cdot\left(\sum_{i=0}^{d}y^*A_i^2|\lambda^i|^2y\right)^2}$$
$$= M\cdot\left[x^*\left(\sum_{i=0}^{d}A_i^2|\lambda^i|^2\right)x\right]\cdot\left[y^*\left(\sum_{i=0}^{d}A_i^2|\lambda^i|^2\right)y\right].$$

Since $A_iA_j > 0, \forall i = 0, \ldots, d, i \neq j$ hence

$$|x^*P_A(\lambda)y|^2 \leq M \cdot \left[x^*\left(\sum_{i=0}^{d} A_i|\lambda^i|\right)^2 x\right] \cdot \left[y^*\left(\sum_{i=0}^{d} A_i|\lambda^i|\right)^2 y\right]$$

$$= M \cdot (x^*P_A^2(|\lambda|)x)(y^*P_A^2(|\lambda|)y).$$

When $|\lambda| < 1$:

It is easy to see that $|\lambda^i|^2 \leq (|\lambda + 2|^i)^4$. Hence

$$|x^*P_A(\lambda)y|^2 \leq M \cdot \sum_{i=0}^{d}(x^*A_i^2x)(y^*A_i^2y) \cdot (|\lambda + 2|^i)^4$$

$$= M \cdot \sum_{i=0}^{d}(x^*A_i^2(|\lambda + 2|^i)^2x) \cdot (y^*A_i^2(|\lambda + 2|^i)^2y).$$

Similarly, we receive

$$|x^*P_A(\lambda)y|^2 \leq M \cdot (x^*P_A^2(|\lambda+2|)x)(y^*P_A^2(|\lambda+2|)y).$$

The proof is complete.

Theorem 4.3. Let $P_A(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial whose coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ are positive definite for all $i = 0, \ldots, d$. Let $x, y \in \mathbb{C}^n$ such that $|x| = |y| = 1$ and $x^*y = 0$. Assume $\sigma(A_i) = \{\lambda_1^i, \ldots, \lambda_{n_i^k}^i\}$ is spectrum of matrix A_i and $\lambda_1^i \leq \lambda_2^i \leq \ldots \leq \lambda_{n_i}^i$, for all $i = 0, \ldots, d$. Then we have

$$|x^*P_A(\lambda)y| \leq \frac{1}{2}K \cdot \left(x^*P_A(|\lambda|)x + y^*P_A(|\lambda|)y\right),$$

where $K = \max\left\{\frac{\lambda_{n_i}^i - \lambda_1^i}{\lambda_{n_i}^i + \lambda_1^i}\right\}_{i=0}^{d}$.

Proof. We have

$$|x^*P_A(\lambda)y| = \left|\sum_{i=0}^{d} x^*A_i\lambda^i y\right| \leq \sum_{i=0}^{d} |x^*A_iy| \cdot |\lambda|^i$$

$$\leq \sum_{i=0}^{d} \frac{1}{2}\left(\frac{\lambda_{n_i}^i - \lambda_1^i}{\lambda_{n_i}^i + \lambda_1^i}\right)(x^*A_ix + y^*A_iy) \cdot |\lambda|^i$$

$$\leq \tfrac{1}{2}K \cdot \sum_{i=0}^{d}(x^*A_i|\lambda|^ix + y^*A_i|\lambda|^iy)$$

$$\leq \tfrac{1}{2}K \cdot \left(\sum_{i=0}^{d} x^*A_i|\lambda|^ix + \sum_{i=0}^{d} y^*A_i|\lambda|^iy\right)$$

$$= \tfrac{1}{2}K \cdot \left(x^*P_A(|\lambda|)x + y^*P_A(|\lambda|)y\right).$$

The proof is complete.

Theorem 4.4. Let $P_A(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial whose coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ are positive definite for all $i = 0, \ldots, d$. Assume $\sigma(A_i) = \{\lambda_1^i, \ldots, \lambda_{n_i^k}^i\}$ is spectrum of matrix A_i and $\lambda_1^i \leq \lambda_2^i \leq \ldots \leq \lambda_{n_i}^i$, for all $i = 0, \ldots, d$. Then we have

$$|x^*P_A(\lambda)x - y^*P_A(\lambda)y|$$

$$\leq K \cdot \left(x^*P_A(|\lambda|)x + y^*P_A(|\lambda|)y\right),$$

with $K = \max\left\{\frac{\lambda_{n_i}^i - \lambda_1^i}{\lambda_{n_i}^i + \lambda_1^i}\right\}_{i=0}^{d}$.

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