TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Thác triển chỉnh hình của các hàm chỉnh hình yếu trong không gian có trọng của các hàm chỉnh hình

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Ngày nhận bài: 17/09/2020; Ngày đăng bài: 17/10/2020

TÓM TẮT

Cho v là một trọng trên một miền D trong không gian lồi địa phương khả mêtric E và F là không gian lồi địa phương đầy đủ. Ký hiệu $H_v(D,F)$ là không gian có trọng của các hàm chỉnh hình trên D nhận giá trị trong F và $A_v(D)$ là không gian con của $H_v(D,\mathbb{C})$ sao cho hình cầu đơn vị đóng của nó là compact theo tôpô compact mở. Trong bài báo này, áp dụng một định lý về tuyến tính hóa trong không gian có trọng các hàm chỉnh hình, chúng tôi đưa ra các điều kiện đối với các tập $M \subset D$ và $W \subset F'$ để mỗi hàm $f : M \to F$ được thác triển chỉnh hình đến D nếu $u \circ f$ có một thác triển chỉnh hình đến D với mỗi $u \in W$.

Từ khóa: *Tuyến tính hóa, không gian lồi địa phương, bất biến tôpô tuyến tính, chỉnh hình vô hạn chiều, hàm chỉnh hình.*

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JOURNAL OF SCIENCE
QUY NHON UNIVERSITY

Holomorphic extensions of weakly holomorphic functions in weighted spaces of holomorphic functions

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Received: 17/09/2020; Accepted: 17/10/2020

ABSTRACT

Let v be a weight on a domain D in a metrizable locally convex space E and F be a complete locally convex space. Denote $H_v(D, F)$ the weighted space of F-valued holomorphic functions on D, and $A_v(D)$ is a subspace of $H_v(D, \mathbb{C})$ with the closed unit ball which is compact for the open-compact topology. Using a linearization theorem of weighted spaces of holomorphic functions in this paper, we set up characterizations for $M \subset D$ and $W \subset F'$ such that every function $f : M \to F$ can be holomorphically extended to the whole domain D in the case $u \circ f$ admits a holomorphic extension to D for every $u \in W$.

Keywords: *Linearization, locally convex spaces, topological linear invariants, infinite- dimensional holomorphy, holomorphic functions*

1. INTRODUCTION

Let E, F be locally convex spaces and D a domain (open, connected set) in E. Holomorphic functions on D with values in F are useful in analytic functional calculus and in the theory of 1-parameter semigroups. As far as we know, in functional analysis, essentially two approaches to analyticity of vector-valued functions are through the notions of a (very) weakly holomorphic and a (strongly) holomorphic function, and the first way is easier to check in practical examples. Here, a function $f : D \to F$ is (very) weakly holomorphic if $u \circ f$ is holomorphic for each $u \in F'$. It is known that holomorphic functions are always (very) weakly holomorphic. It seems that the first answer of the question "what conditions decide the holomorphy of a (very) weakly holomorphic function" belongs to Dunford,[1] who proved that the class of Banach-valued (very) weakly holomorphic functions defining on a domain in $\mathbb{C}$ satisfies the requirement. Grothendieck [2] extended the result for the case the underlying spaces are quasicomplete. The assertion, in fact, is true in the case where E, F are Hausdorff locally convex spaces and E is metrizable (see [3] [Theorème 1.2.10]). Then, it is natural to ask (addressed by Grosse-Erdmann, [4,5] Arendt and Nikolski [6]) whether or not (proper/smallest) subsets W of the dual F' of the underlying spaces F such that a function f is holomorphic whenever f is (not very) weak holomorphic, i.e. $u \circ f$ is holomorphic for all $u \in W$. In other

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words, we have to determine the minimal assumptions under which a weakly holomorphic function is (strongly) holomorphic as well. Arendt and Nikolski considered this problem when $D \subset \mathbb{C}$ and F is a complex Banach space. Let $W \subset F'$ be a subset and $\sigma(F, W)$ be the weak topology on F induced by W. A result in [6] claimed that a $\sigma(F, W)$-holomorphic function $f : D \to F$ is holomorphic if and only if W determines boundedness, in the sense that each $\sigma(F, W)$-bounded set in F is bounded. If $f : D \to F$ is additionally assumed to be locally bounded then f is holomorphic when W is just a separating subspace of F'. A generalization of this result to the case where F is a locally complete locally convex space was obtained by Grosse-Erdmann.[5] Under additional assumptions that E and F satisfy some linear topological invariants, in [7], Hai extended the results of Arendt and Nikolski in [6] for the case where f defines on an open set D either in a Schwartz-Fréchet space $E \in (\Omega)$ with values in a Schwartz-Fréchet space $F \in (LB_\infty)$ or in $\mathbb{C}$ with values in a Fréchet space $F \in (LB_\infty)$ [7] [Theorems 4.1, 4.2]. Note that, in the case of $E = \mathbb{C}$ the hypothesis "Schwartz" for F is superfluous. Recently, Quang, Lam and Dai [8] have considered the above problem in the cases where Fréchet spaces E, F have stronger conditions, $E \in (\Omega)$ and $F \in (LB_\infty)$ or $F \in (DN)$; but the locally boundedness of f is dropped to a weaker property, the boundedness on bounded sets in D.

This question is closely related to the problem of holomorphic extension. One of the extension results given by Bogdanowicz [9] through weak extension says that a function f defined on a domain D_1 in $\mathbb{C}$ with values in a sequentially complete, complex locally convex Hausdorff space F such that $u \circ f$ can be holomorphically extended to a domain $D_2 \supset D_1$ for each $u \in F'$, must admit a holomorphic extension to D_2.

More recently, Grosse - Erdmann,[5] Arendt and Nikolski,[6] Bonet, Frerick and Jordá,[10] Frerick and Jordá,[11] Frerick, Jordá and Wengenroth [12] have given results in this way but with requiring extensions of $u \circ f$ only for a proper subset $W \subset F'$ and the conditions on D_1 are smoother. Also, Laitila and Tylli [13] have recently discussed the difference between strong and weak definitions for important spaces of vector-valued functions. Most recently, this problem is also investigated for more general cases, for instance, on Fréchet spaces with the linear topological invariants by Quang, Lam and Dai,[8] Quang and Dai;[14,15] and for weakly meromorphic functions by Quang and Lam [16,17].

The aim of this paper is to investigate the holomorphic extension of holomorphic functions in a subspaces of the weighted space $H_v(D, F)$ of *weakly-type holomorphic functions* of a between locally convex spaces given by

$$A_v(D, F) := \{f : D \to F : \ u \circ f \in A_v(D) \ \forall u \in F'\}.$$

The rest of the paper is organized as follows. Section 2 presents some preliminaries. The main results will be introduced in Section 3. We give the conditions for subsets $W \subset F'$ under which every F-valued function f can be extended (unique) to a function $\widetilde{f} \in A_v(D, F)$ from a (thin) subset of uniqueness (Theorems 3.2, 3.3) and from a (fat) sampling subset (Theorem 3.4) M of D whenever $u \circ f$ admits a unique extension $f_u \in A_v(D)$ for all $u \in W$.

The main tool for our proofs will be the linearization theorems in $A_v(D, F)$ that has been proven by T. T. Quang et al. in [18].

Our notation for locally convex spaces and functional analysis is standard and we refer to the book of Jarchow.[19] A locally convex space is always a complex vector space with a locally convex Hausdorff topology.

For more background and details about the theory of infinite dimensional holomorphy we refer to [20] and the reference given therein.

2. PRELIMINARIES

2.1. General notations

We always assume that the locally convex structure of a Fréchet space E is generated by an increasing system $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\ker\|\cdot\|_k$, by $\omega_k : E \to E_k$ the canonical map and by U_k the set $\{x \in E : \|x\|_k < 1\}$. Sometimes it is convenient to assume that $\{U_k\}_{k\in\mathbb{N}}$ is a neighbourhood basis of zero (shortly $\mathcal{U}(E)$).

If B is an absolutely convex subset of E we define a norm $\|\cdot\|_B^*$ on E', the strongly dual space of E with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)|, x \in B\}.$$

Obviously $\|\cdot\|_B^*$ is the gauge functional of B°. Instead of $\|\cdot\|_{U_k}^*$ we write $\|\cdot\|_k^*$. By E_B we denote the linear hull of B which becomes a normed space in a canonical way if B is bounded. The space E is called *locally complete* if for every absolutely convex, closed and bounded B of E the space E_B is Banach.

Definition 2.1. Let F be a locally convex space. A subset W of F' is called

- *separating* if $u(x) = 0$ for each $u \in W$ implies $x = 0$. Clearly this is equivalent to the span of W being weak*-dense (or dense in the co-topology);
- *determining boundedness* if every subset $B \subset F$ is bounded whenever $u(B)$ is bounded in $\mathbb{C}$ for all $u \in W$. This holds if and only if every $\sigma(F, \mathrm{span}W)$-bounded set is F-bounded. Obviously, the linear span of such sets is $\sigma(F', F)$-dense;

Clearly, if $W \subset F'$ determines boundedness in F' then W is separating.

2.2. Weighted spaces of holomorphic functions

Let E and F be locally convex spaces and D be a domain in E. By $H(D, F)$ we denote the vector space of all holomorphic. Instead of $H(D, \mathbb{C})$, we write $H(D)$.

For a domain D in E, a *weight* $v : D \to (0, \infty)$ is a continuous function which is strictly positive. The space

$$\begin{aligned} &H_v(D, F) := \\ &\{f \in H(D, F) : (v.f)(D) \text{ is bounded on } D\} \\ &= \{f \in H(D, F) : \\ &\qquad \|f\|_{v,p} := \sup_{x\in D} v(x)p(f(x)) < \infty \\ &\qquad\qquad \forall p \in cs(F)\} \end{aligned}$$

equipped with the topology generated by the family $\{\|\cdot\|_{v,p}\}_{p\in cs(F)}$ of semi-norms. Then $H_v(D, F)$ is complete whenever F is complete, in particular, it is Banach if F is Banach. It is easy to check that

$$H_v(D, F) = \varprojlim_{p\in cs(F)} H_v(D, F_p)$$

where F_p is the completion of the canonically normed space $F/\ker p$.

We also write

$$H_v(D) := H_v(D, \mathbb{C}) = \\ = \{f \in H(D) : \\ \|f\|_v := \sup_{x \in D} v(x)|f(x)| < \infty\}.$$

Remark 2.1. In the case E is metrizable we have

$$H(D, F) = H(D, F)_w \\ := \{f : D \to F :\ u \circ f \in H(D)\ \forall u \in F'\}$$

(see [3] [Theorem 1.2.10], also ([20] [Example 3.8(g)]).

Let $A_v(D)$ be a subspace of $H_v(D)$ such that the closed unit ball $B_{A_v(D)}$ is compact for the topology τ_0. Note that this condition implies that $A_v(D)$ is norm-closed and hence Banach because $H_v(D)$ is Banach.

In [21] [Theorem 7] Jordá showed that if, for $m \in \mathbb{N}$, the space $\mathscr{P}(^m E)$ of continuous m-homogeneous polynomials on a Fréchet E, endowed with its norm topology, is contained $H_v(D)$ then the closed unit ball of $\mathscr{P}(^m E)$ is compact for the topology τ_0.

Another illustrative example of $A_v(D, F)$ has been built by Quang in [22]. We briefly sketch this example as follows.

Let D be a balanced, bounded open set in a Fréchet space E. Let v be a *weight* on D satisfying the following conditions:

(i) v is *radial* in the sense that

$$v(\lambda z) = v(z)$$
$$\forall z \in D, \forall \lambda \in \partial\Delta := \{\lambda \in \mathbb{C} : |\lambda| = 1\},$$

(ii) v *vanishes at infinity* outside compact sets of D in the sense, if for every $\varepsilon > 0$ there exists a compact subset K of D such that $|v(z)| < \varepsilon$ for all $z \in D \setminus K$.

For each $f \in H(D)$ we consider the Taylor series representation at zero

$$f(z) = \sum_{k=0}^{\infty} (P_k f)(z), \quad z \in D,$$

where $P_n f$ is a k-hemogeneous polynomial, $k = 0, 1, \dots$. The series converges to f uniformly on each compact subset of D.

Now for each $n \geq 0$ we consider the linear operator

$$C_n : H_v(D) \to H_v(D)$$

given by Cesàro means $C_n f$ of the partial sums of the Taylor series of f :

$$(C_n f)(z) = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=0}^{j} (P_k f)(z),\ z \in D.$$

Put

$$\mathcal{C}_{n,v}(D) = C_n(H_v(D)),$$
$$\mathcal{C}_v(D) = \bigcup_{n \geq 0} \mathcal{C}_{n,v}(D),$$
$$A_v(D) = \overline{\mathcal{C}_v(D)}^{\tau_v}.$$

Then we have the following

Proposition 2.1 ([22] Proposition 4). *Let D be a balanced, bounded open set in a Fréchet-Montel space E having the $(BB)_\infty$-property and v be a weight on D which vanishes at infinity outside compact sets of D. Then, the closed unit ball of $(A_v(D), \|\cdot\|_v)$ is τ_0-compact.*

Here, we recall that a locally convex space E has the $(BB)_n$-property if each bounded subset of $\widehat{\bigotimes_{n,s,\pi}} E$ is contained in $\overline{\mathrm{acx}}\big(\bigotimes_{n,s} B\big)$ for some bounded subset B of E, where

$$\bigotimes_{n,s} B := \{x \otimes \cdots \otimes x :\ x \in B\}$$

and $\underset{n,s,\pi}{\otimes} E$ denotes the space $\underset{n,s}{\otimes} E$ endowed with the projective topology τ_π and $\widehat{\underset{n,s,\pi}{\otimes}} E$ is its completion.

If E has $(BB)_n$ for all n we say that E has $(BB)_\infty$-property.

3. THE PROBLEM OF WEIGHTED HOLOMORPHIC EXTENSIONS

In this section we present the main resuts of the paper. First, we recall the linearization theorem which has been proven in [18].

Let us denote by $P_{A_v(D)}$ the closed subspace of all linear functionals $u \in (A_v(D), \|\cdot\|_v)'$ such that $u\big|_{B_{A_v(D)}}$ is τ_0-continuous.

By the Ng Theorem [23] [Theorem 1] the evaluation mapping

$$J : (A_v(D), \|\cdot\|_v) \to (P_{A_v(D)})'$$

given by

$$(Jf)(u) = u(f) \quad \forall u \in P_{A_v(D)},$$

is a topological isomorphism. This is why the space $P_{A_v(D)}$ is called the predual of $A_v(D)$.

Theorem 3.1. *Let v be a weight on a domain D in a metrizable locally convex space E and $A_v(D)$ be a subspace of $H_v(D)$ such that the closed unit ball is τ_0-compact. Then there exists a mapping $\delta_D \in H(D, P_{A_v(D)})$ with the following universal property: For each complete locally convex space F, a function $f \in A_v(D,F)$ if and only if there is a unique mapping $T_f \in L(P_{A_v(D)}, F)$ such that $T_f \circ \delta_D = f$. This property characterize $P_{A_v(D)}$ uniquely up to an isometric isomorphism.*

Moreover, the mapping

$$\Phi : f \in A_v(D,F) \mapsto T_f \in (L(P_{A_v(D)}, F), \tau_c)$$

$$:= \underset{\overleftarrow{p \in cs(F)}}{\lim} (L(P_{A_v(D)}, F_p)$$

is a topological isomorphism.

Note that, $\delta_D : D \to P_{A_v(D)}$ is the evaluation mapping given by

$$\delta_D(x) = \delta_x$$

with $\delta_x(g) := g(x)$ for all $g \in A_v(D)$.

First we present some notions which will be needed in subsequent sections.

A subset $M \subseteq D$ is said to be a *set of uniqueness for* $A_v(D)$ if each $f \in A_v(D)$ such that $f\big|_M = 0$ then $f \equiv 0$.

A subset $M \subset D$ is said to be *sampling for* $A_v(D)$ if there exists a constant $C \geq 1$ such that for every $f \in A_v(D)$ we have

$$\sup_{z\in D} v(z)|f(z)| \leq C \sup_{z\in M} v(z)|f(z)|. \quad (3.1)$$

Remark 3.1. For $M \subset D$, denote

$$M_v^* := \{v(x)\delta_x : \ x \in M\} \subset B_{P_{A_v(D)}}$$

where $B_{P_{A_v(D)}}$ denotes the unit ball of $P_{A_v(D)}$.

1. Since the closed unit ball $B_{A_v(D)}$ of the space $A_v(D)$ is τ_0-compact, by the Hahn-Banach theorem, it is easy to check that the following are equivalent:
   - (i) M is a set of uniqueness for $A_v(D)$;
   - (ii) M_v^* is separating in $A_v(D)$;
   - (iii) $\langle M_v^* \rangle := \mathrm{span} M_v^*$ is $\sigma(P_{A_v(D)}, A_v(D))$-dense.
2. For the norm given by

   $$\|f\|_{M,v} := \sup_{z\in M} v(z)|f(z)|$$

   on $A_v(D)$, it is obvious that the following are equivalent:
   - (i) M is a sampling for $A_v(D)$;

(ii) $\|\cdot\|_v \simeq \|\cdot\|_{M,v}$ on $A_v(D)$.

3. Obviously, if M is sampling for $A_v(D)$ then M_v^* is separating in $A_v(D)$, hence, M is a set of uniqueness for $A_v(D)$.

Given a Fréchet space E, an increasing sequence $\{B_n\}_n$ of bounded subsets of E' is said to *fix the topology* if the polars $\{B_n^\circ\}_n$ taken in E form a fundamental system of 0-neighbourhoods of E. Let us recall the following result.

Lemma 3.1 ([11], Proposition 7)**.** *Let $\{B_n\}_n$ be an increasing sequence of bounded subsets of F'. The following assertions are equivalent:*

(i) *$\{B_n\}_n$ fixes the topology of F;*

(ii) *$F'((B_n)_{n\in\mathbb{N}})$ determines boundedness in F;*

(iii) $\operatorname{span}\bigcup_n \overline{B_n}^{\sigma(F',F)}$ *determines boundedness in F.*

Theorem 3.2. *Let v be a weight on a domain D in a metrizable locally convex space and $A_v(D)$ a subspace of $H_v(D)$ such that the closed unit ball $B_{A_v(D)}$ is τ_0-compact. Let $M \subset D$ be a set of uniqueness for $A_v(D)$. Let F be a complete locally convex space and $W \subset F'$ be a subspace which determines boundedness in F. If $f : M \to F$ is a function such that $u\circ f$ has an extension $f_u \in A_v(D)$ for each $u \in W$ then f admits a unique extension $\widetilde{f} \in A_v(D, F)$.*

Chứng minh. By Remark 3.1, $\langle M_v^*\rangle$ is $\sigma(P_{A_v(D)}, A_v(D))$-dense, and hence it is norm-dense.

Now we take a function $f : M \to F$ satisfying the assumptions of Theorem. Put $T : \langle M_v^*\rangle \to F$, $T(\delta_x) := f(x)$. Since W is separating, T is well defined.

Let $x = \sum_{k=1}^n \alpha_k v(x_k)\delta_{x_k} \in B_{\langle M_v^*\rangle}$, the unit ball of $\langle M_v^*\rangle$. For each $u \in W$ we have the estimate

$$\begin{aligned}|\langle Tx, u\rangle| &= \Big|\Big\langle \sum_{k=1}^n \alpha_k v(x_k) f(x_k), u\circ f\Big\rangle\Big| \\ &= \Big|\Big\langle \sum_{k=1}^n \alpha_k v(x_k)\delta_{x_k}, u\circ f\Big\rangle\Big| \\ &\le \|u\circ f\|_v.\end{aligned}$$

This means that $T(B_{\langle M_v^*\rangle})$ is $\sigma(F, W)$-bounded and then it is bounded. Thus, T is a bounded linear mapping. Since $\langle M_v^*\rangle$ is norm-dense in $P_{A_v(D)}$ we can extend T to $\widetilde{T} : P_{A_v(D)} \to F$. Now, the proof is complete by applying Theorem 3.1. □

In the case where D is a domain in a Banach space, by Montel's theorem, the closed unit ball $B_{H_v(D)}$ of the space $H_v(D)$ is τ_0-compact. Therefore, from Theorem 3.1 and Remark 2.1 we get the following:

Corollary 3.1. *Let D be a domain in a Banach space and v be a weight on D and $M \subset D$ be a set of uniqueness for $H_v(D)$. Let F be a complete locally convex space and $W \subset F'$ be a subspace which determines boundedness in F. If $f : M \to F$ is a function such that $u\circ f$ has an extension $f_u \in H_v(D)$ for each $u \in W$ then f admits a unique extension $\widetilde{f} \in H_v(D, F)$.*

Theorem 3.3. *Let v be a weight on a domain D in a metrizable locally convex space and $A_v(D)$ a subspace of $H_v(D)$ such that the closed unit ball $B_{A_v(D)}$ is τ_0-compact. Let $M \subset D$ be a set of uniqueness for $A_v(D)$. Let F be a Fréchet space and $W = \cup_n B_n \subset F'$ where $\{B_n\}_n$ fixes the topology of F. If $f : M \to F$ is a function such that $u\circ f$ has an extension*

$f_u \in A_v(D)$ for each $u \in W$ and $\{f_u\}_{u\in B_n}$ is bounded in $A_v(D)$ for all n then f admits a unique extension $\widetilde{f} \in A_v(D,F)$.

Chứng minh. By Lemma 3.1,

$$\widetilde{W} := \operatorname{span}\bigcup_n \overline{B_n}^{\sigma(F',F)}$$

determines boundedness in F. For each $u \in \widetilde{W}$ we write u in the form

$$u = \sum_{i=1}^{n_0} \alpha_i u_i$$

with $u_i \in \overline{B_i}^{\sigma(F',F)}$, $i = 1,\dots,n_0$. Because $\{B_n\}_n$ is increasing we have $u \in (\sum_{i=1}^{n_0}\alpha_i)\overline{B_{n_0}}^{\sigma(F',F)}$. Without loss of generality we can assume that $\sum_{i=1}^{n_0}\alpha_i = 1$. Then there exists $\{u_k^0\}_k \subset B_{n_0}$ such that $u_k^0 \to u$ in the topology $\sigma(F',F)$. It follows from the hypothesis that $\{f_{u_k^0}\}_k$ is a bounded sequence in $A_v(D)$ and $\{f_{u_k^0}(x)\}$ converges to $u(f(x))$ for each $x \in M$. Therefore, by [18][Theorem 4.3], there exists $f_u \in A_v(D)$ such that $\{f_{u_k^0}(x)\}$ converges $f_u(x)$ for each $x \in D$. Then, since $u \in \widetilde{W}$ is arbitrary, the conclusion is a consequence of Theorem 3.2. □

Theorem 3.4. *Let v be a weight on a domain D in a metrizable locally convex space and $A_v(D)$ a subspace of $H_v(D)$ such that the closed unit ball $B_{A_v(D)}$ is τ_0-compact. Let M be a sampling set for $A_v(D)$ and W a $\sigma(F',F)$-dense subspace of the dual F' of a locally complete locally convex space F. If $f : M \to F$ is a function such that*

$$\sup_{x\in M} v(x)p(f(x)) < \infty \quad \textit{for all } p \in cs(F) \tag{3.2}$$

and $u \circ f$ has an extension $f_u \in A_v(D)$ for each $u \in W$ then f admits a unique extension $\widetilde{f} \in A_v(D,F)$.

Chứng minh. Consider the Banach space

$$\ell_1(M) := \Big\{\alpha := (\alpha_x)_{x\in M} : M \to \mathbb{C} : \|\alpha\| := \sum_{x\in M} |\alpha_x| < \infty\Big\}$$

where $\alpha_x = \alpha(x)$ and the sum is defined by

$$\sum_{x\in M} |\alpha_x| := \sup\Big\{\sum_{x\in\gamma} |\alpha_x| : \gamma \text{ a finite subset of } M\Big\}.$$

Put

$$\ell_1(M_v^*) := \Big\{g \in P_{A_v(D)} :\ \exists(\alpha_x)_{x\in M} \in \ell_1(M) : g = \sum_{x\in M} \alpha_x v(x)\delta_x\Big\}$$

where

$$\sum_{x\in M} \alpha_x v(x)\delta_x := \lim_{\gamma\in\mathscr{F}_M} \sum_{x\in\gamma} \alpha_x v(x)\delta_x$$

with $\mathscr{F}_M$ be the directed set of all finite sets of M ordered inder inclusion. It is easy to check that the space $\ell_1(M_v^*)$ is Banach with the following norm

$$\|g\|_M := \inf\Big\{\sum_{x\in M} |\alpha_x|v(x)\|\delta_x\| : (\alpha_x)_{x\in M} \in \widehat{g}_{\ell_1}\Big\}$$

where

$$\widehat{g}_{\ell_1} = \{(\beta_x)_{x\in M} \in \ell_1(M) : g = \sum_{x\in M} \beta_x v(x)\delta_x = \sum_{x\in M} \alpha_x v(x)\delta_x\}.$$

We denote $P_{A_{M,v}(D)}$ the predual of $(A_v(D), \|\cdot\|_{M,v})$ and $B_{P_{M,v}(D)}$ is its unit ball. Define

$$S : \ell_1(M) \to P_{A_{M,v}(D)}$$

by

$$S((\alpha_x)_{x\in M}) := \sum_{x\in M} \alpha_x v(x)\delta_x.$$

It is easy to see that S is bounded linear, moreover

$$B_{P_{M,v}(D)} \subset \overline{\mathrm{acx}}(M_v^*) \subseteq \overline{S(B_{\ell_1(M)})}.$$

Since $\|\cdot\|_v \simeq \|\cdot\|_{M,v}$ (see Remark 3.1) $S : \ell_1(M) \to P_{A_v(D)}$ is bounded. Then, by the Schauder lemma [24] [Lemma 3.9], S is open and then surjective. This implies that the injection of $\ell_1(M_v^*)$ is an onto isomorphism. Therefore, for each $g \in P_{A_v(D)}$ there exists $\alpha := (\alpha_x)_{x\in M} \in \ell_1(M)$ such that

$$g = \sum_{x\in M} \alpha_x v(x)\delta_x.$$

Take f as in the hypothesis. We define $T : P_{A_v(D)} \to F$ with $T(g) := \sum_{x\in M} \alpha_x v(x) f(x)$. Since (3.2) the series is convergent. Moreover, if $\sum_{x\in M} \alpha_x v(x)\delta_x = 0$ then for all $u \in W$ we have

$$u\Big(\sum_{x\in M} \alpha_x v(x) f(x)\Big) = \Big\langle \sum_{x\in M} \alpha_x v(x)\delta_x, f_u\Big\rangle = 0,$$

and then, because W is separating, T is well defined. From (3.2) and $\|\cdot\|_v \simeq \|\cdot\|_{M,v}$, it implies that $T(B_{P_{A_v(D)}})$ is bounded. Finally, the proof is complete by applying Theorem 3.1. □

ACKNOWLEDGMENTS

This study is conducted within the framework of science and technology projects at institutional level of Quy Nhon University under the project code T2020.654.02.

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