Phép chiếu Bergman tác động lên L^∞ trong hình cầu đơn vị $\mathbb{B}_n$

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TÓM TẮT

Với một hàm trọng cho trước, ta định nghĩa phép chiếu kiểu Bergman cảm sinh bởi hàm trọng đó. Chúng tôi đưa ra một đặc trưng của hàm trọng bán kính để phép chiếu kiểu Bergman đó là bị chặn từ L^∞ vào không gian Bloch $\mathcal{B}$ trên hình cầu đơn vị $\mathbb{B}_n$ của $\mathbb{C}^n, n > 1$.

Từ khóa: *Không gian Bergman, phép chiếu Bergman, không gian Bloch.*

On the Bergman projections acting on L^∞ in the unit ball $\mathbb{B}_n$

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ABSTRACT

Given a weight function, we define the Bergman type projection induced by this weight. We characterize the radial weights such that this projection is bounded from L^∞ to the Bloch space $\mathcal{B}$ on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n, n > 1$.

Keywords: *Bergman space, Bergman projection, Bloch space.*

1. INTRODUCTION AND MAIN RESULT

Let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean space. For any two points $z = (z_1, \dots, z_2)$, $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we use the well-known notation

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n},$$

and $|z| = \sqrt{\langle z, z\rangle}$.

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball, and let $\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$.

Denote by $H(\mathbb{B}_n)$ the space of all holomorphic functions on the unit ball $\mathbb{B}_n$. Let dv be the normalized volume measure on $\mathbb{B}_n$. The normalized surface measure on $\mathbb{S}_n$ will be denoted by $d\sigma$.

Let ρ be a positive and integrable function on $[0, 1)$. We extend it to $\mathbb{B}_n$ by $\rho(z) = \rho(|z|)$, and call such ρ a radial weight function. The weighted Bergman space A^2_ρ is the space of functions f in $H(\mathbb{B}_n)$ such that

$$\|f\|^2_\rho = \int_{\mathbb{B}_n} \left|f(z)\right|^2 \rho(z)dv(z) < \infty.$$

Let ρ be a radial weight and X be a space of measurable functions on $\mathbb{B}_n$. The Bergman type projection P_ρ acting on X is given by

$$P_\rho f(z) = \int_{\mathbb{B}_n} K_\rho(z, w)f(w)\rho(w)dv(w),$$

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for $z \in \mathbb{B}_n, f \in X$, where $K_\rho(z,w)$ is the reproducing kernel of the weighted Bergman space A^2_ρ.

When ρ is the standard radial weight $\rho(z) = (1-|z|^2)^\alpha, \alpha > -1$, the corresponding projection is denoted by P_α.

A radial weight ρ belongs to the class $\widehat{\mathcal{D}}$ if $\widehat{\rho}(r) \lesssim \widehat{\rho}(\frac{1+r}{2})$ for all $r \in [0,1)$, where $\widehat{\rho}(r) = \int_r^1 \rho(s)ds$.

The Bergman space and a lot of topics related to this famous space have attracted the attention of many mathematicians so far. When the weight function is the standard radial weight, the theory of Bergman space is well-known. We recommend the books of Zhu[9;10] for more details.

In 2018, we began the study of small Bergman spaces in higher dimensions[5]. The projections play a crucial role in studying operator theory on spaces of analytic functions. Bounded analytic projections can also be used to establish duality relations and to obtain useful equivalent norms in spaces of analytic functions. Hence the boundedness of projections is an interesting topic which has been studied by many authors in recent years[1;2;3;7;8]. In[7], Peláez and Rättyä considered the projection P_{ρ_1} acting on $L^p_{\rho_2}(\mathbb{D}), 1 \le p < \infty$, when two weights ρ_1, ρ_2 are in the class $\mathcal{R}$ of the so called regular weights. A radial weight ρ is regular if $\widehat{\rho}(r) \asymp (1-r)\rho(r), r \in (0,1)$. Recently, in 2019, they extended these results to the case where $\rho_1 \in \widehat{\mathcal{D}}$, ρ_2 is radial[8].

In this paper, we are going to study the projections acting on the space L^∞. Let us recall that the Bloch space of $\mathbb{B}_n$, denoted by $\mathcal{B}(\mathbb{B}_n)$, or simply by $\mathcal{B}$, is the space of holomorphic functions f in $\mathbb{B}_n$ such that

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)\left|Rf(z)\right| < \infty,$$

where

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

is the radial derivative of f at $z \in \mathbb{B}_n$. In the one dimensional case, the Bloch space consists of analytic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right| < \infty,$$

and is denoted by $\mathcal{B}(\mathbb{D})$.

In the case of standard radial weight, we have the following result.

Theorem A. *For any $\alpha > -1$, the Bergman type projection P_α is a bounded linear operator from L^∞ onto the Bloch space $\mathcal{B}$.*

This theorem is also valid for the case of one dimension $\mathbb{C}$ and of higher dimension $\mathbb{C}^n$. See[10] Theorem 5.2 for the proof in the case of one variable and[9] Theorem 3.4 for the proof in the case of several variables.

When the weight functions is more general, in[8], Peláez and Rättyä obtained an interesting result in the one dimensional case.

Theorem B. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

The aim of this paper is to study the Bergman type projections acting on L^∞ in the case of higher dimension. In this case, what will be the target space and the characterizations of the weight functions such that the projection is bounded? We extend Theorem B to the case of several variables and obtain the following result.

Theorem 1.1. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

The paper is organized as follows. In Section 2, we give some important lemmas which are used in the proof of Theorem 1.1 and Section 3 gives us an explicit explanation for our main result.

Throughout this text, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a positive constant C such that $U(z) \leq C.V(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, then we write $U(z) \asymp V(z)$.

2. SOME AUXILIARY LEMMAS

To prove Theorem 1.1 we need several auxiliary lemmas.

Lemma 2.1. *Let ρ be a radial weight. Then the following conditions are equivalent:*

(i) $\rho \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\rho) > 0$ and $\beta_0 = \beta_0(\rho) > 0$ such that

$$\widehat{\rho}(r) \leq C \left(\frac{1-r}{1-t}\right)^\beta \widehat{\rho}(t),$$

where $0 \leq r \leq t < 1$, for all $\beta \geq \beta_0$;

(iii) The asymptotic equality

$$\int_0^1 s^x \rho(s) ds \asymp \widehat{\rho}\left(1 - \frac{1}{x}\right),$$

where $x \in [1, \infty)$, is valid;

(iv) There exist $C_0 = C_0(\rho) > 0$ and $C = C(\rho) > 0$ such that

$$\widehat{\rho}(0) \leq C_0 \widehat{\rho}(\frac{1}{2})$$

and $\rho_n \leq C\rho_{2n}$ for all $n \in \mathbb{N}$.

Lemma 2.1 gives us the characterizations of the weight function $\rho \in \widehat{\mathcal{D}}$. The proof of this lemma can be found in[6].

Lemma 2.2. *If*

$$f(z) = \sum_{n=0}^{\infty} a_j z^j \in H^p, \qquad 0 < p \leq 2,$$

then

$$\sum_{j=0}^{\infty} (j+1)^{p-2} \left|a_j\right|^p \lesssim \|f\|_p^p .$$

Lemma 2.3. *Let $\{a_j\}$ be a sequence of complex numbers such that $\sum j^{q-2} \left|a_j\right|^q < \infty$ for some $q, 2 \leq q < \infty$. Then the function $f(z) = \sum_{n=0}^\infty a_j z^j$ is in H^q, and*

$$\|f\|_q^q \lesssim \sum_{j=0}^{\infty} (j+1)^{q-2} \left|a_j\right|^q .$$

Two above lemmas are the classical Hardy-Littlewood inequalities, which can be found, for example, in Duren's book[4] Theorem 6.2 and 6.3.

In the following lemma, we give the explicit formula for the reproducing kernel of A^2_ρ.

Lemma 2.4. *Let ρ be a radial weight. Then the reproducing kernel $K_\rho(z, w)$ is given by*

$$K_\rho(z, w) = \frac{1}{2} \sum_{d=0}^{\infty} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z, w\rangle^d,$$

for $z, w \in \mathbb{B}_n$, where

$$\rho_x = \int_0^1 t^x \rho(t) dt, \qquad x \geq 1.$$

Proof. By the multinomial formula (see[9] (1.1)), we have that

$$\langle z,w\rangle^d = \sum_{\beta\in\mathbb{N}^n,|\beta|=d} \frac{d!}{\beta!} z^\beta \bar{w}^\beta, \qquad z,w\in\mathbb{C}^n.$$

Hence, for $\alpha\in\mathbb{N}^n, |\alpha|=d$,

$$\int_{\mathbb{S}_n} \xi^\alpha \langle z,\xi\rangle^d d\sigma(\xi) = \sum_{\beta\in\mathbb{N}^n,|\beta|=d} \frac{d! z^\beta}{\beta!} \int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\beta d\sigma(\xi),$$

where $z\in\mathbb{B}_n$. By Lemma 1.11 in[9],

$$\int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\beta d\sigma(\xi) = \begin{cases} 0 & \text{if } \alpha\neq\beta, \\ \dfrac{\alpha!(n-1)!}{(d+n-1)!} & \text{if } \alpha=\beta, \end{cases}$$

and we obtain

$$\begin{aligned} \int_{\mathbb{S}_n} \xi^\alpha \langle z,\xi\rangle^d d\sigma(\xi) &= \frac{d!}{\alpha!} z^\alpha \int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\alpha d\sigma(\xi) \\ &= \frac{d!}{\alpha!}\frac{\alpha!(n-1)!}{(d+n-1)!} z^\alpha \\ &= \frac{d!(n-1)!}{(d+n-1)!} z^\alpha, \end{aligned}$$

for $z\in\mathbb{B}_n$. Therefore, for $\alpha\in\mathbb{N}^n, |\alpha|=d$ we have

$$\begin{aligned} &\int_{\mathbb{B}_n} w^\alpha \langle z,w\rangle^d \rho(w) dv(w) \\ &= 2n \int_0^1 t^{2n-1+2d}\rho(t)dt \int_{\mathbb{S}_n} \xi^\alpha \langle z,\xi\rangle^d d\sigma(\xi) \\ &= \frac{2d!n!\rho_{2n-1+2d}}{(d+n-1)!} z^\alpha, \qquad z\in\mathbb{B}_n. \end{aligned}$$

It follows that

$$z^\alpha = \frac{(d+n-1)!}{2d!n!\rho_{2n-1+2d}} \int_{\mathbb{B}_n} w^\alpha \langle z,w\rangle^d \rho(w) dv(w), \tag{1}$$

for any $z\in\mathbb{B}_n$.

Since $\rho(t)>0, 0<t<1$, we have $\rho_s \geq C_\varepsilon(1-\varepsilon)^s$ for every $\varepsilon>0$. Given $z\in\mathbb{B}_n$, we have

$$\begin{aligned} &\int_{\mathbb{B}_n} \left| \frac{1}{2}\sum_{d=0}^\infty \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z,w\rangle^d \right|^2 \rho(w) dv(w) \\ &= \frac{1}{4}\sum_{d_1,d_2\geq 0} \frac{(d_1+n-1)!(d_2+n-1)!}{d_1!d_2!(n!)^2\rho_{2n-1+2d_1}\rho_{2n-1+2d_2}} \times \\ &\qquad \times \int_{\mathbb{B}_n} \langle z,w\rangle^{d_1}\langle w,z\rangle^{d_2}\rho(w)dv(w) \\ &= \frac{1}{4}\sum_{d_1,d_2\geq 0} \frac{(d_1+n-1)!(d_2+n-1)!}{d_1!d_2!(n!)^2\rho_{2n-1+2d_1}\rho_{2n-1+2d_2}} \times \\ &\qquad \times \int_{\mathbb{B}_n} \sum_{|\beta|=d_2} w^\beta \bar{z}^\beta \frac{d_2!}{\beta!} \langle z,w\rangle^{d_1}\rho(w)dv(w) \\ &= \frac{1}{2}\sum_{d\geq 0} \left(\frac{(d+n-1)!}{d!n!}\right) \frac{1}{\rho^2_{2n-1+2d}} \times \\ &\qquad \times \sum_{|\beta|=d} \frac{(d!)^2}{\beta!} \frac{n!\rho_{2n-1+2d}}{(d+n-1)!} z^\beta \bar{z}^\beta \\ &= \frac{1}{2}\sum_{d\geq 0} \frac{(d+n-1)!}{n!\rho_{2n-1+2d}} \sum_{|\beta|=d} \frac{z^\beta\bar{z}^\beta}{\beta!} \\ &= \frac{1}{2}\sum_{d\geq 0} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} |z|^{2d} < \infty. \end{aligned}$$

Thus, the function

$$w \mapsto \frac{1}{2}\sum_{d=0}^\infty \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle w,z\rangle^d$$

belongs to A^2_ρ.

By (1) and by continuity, for every $f\in A^2_\rho(\mathbb{B}_n)$, we have

$$\begin{aligned} f(z) &= \int_{\mathbb{B}_n} f(w) \times \\ &\times \left(\frac{1}{2}\sum_{d=0}^\infty \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z,w\rangle^d\right) \rho(w)dv(w), \end{aligned}$$

for $z\in\mathbb{B}_n$.

Therefore,

$$K_\rho(z,w) = \frac{1}{2}\sum_{d=0}^{\infty} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z,w\rangle^d.$$

□

3. PROOF OF MAIN RESULT

It suffices to consider only the case $n > 1$.

Proposition 3.1. *If $\rho \in \widehat{\mathcal{D}}$, then the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded, where P_ρ is defined by*

$$P_\rho\varphi(z) = \int_{\mathbb{B}_n} K_\rho(z,w)\varphi(w)\rho(w)dv(w),$$

for $\varphi \in L^\infty, z \in \mathbb{B}_n$.

Proof. We have

$$K_\rho(z,w) = \frac{1}{2}\sum_{d=0}^{\infty} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z,w\rangle^d.$$

Hence, for a fixed $w \in \mathbb{B}_n$,

$$\begin{aligned}
&RK_\rho(z,w)\\
&= \sum_{j=1}^{n} z_j \frac{\partial K_\rho(z,w)}{\partial z_j}\\
&= \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\left(\frac{1}{2}\sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}}\langle z,w\rangle^d\right)\\
&= \frac{1}{2}\sum_{j=1}^{n} z_j \sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} d\bar{w}_j \langle z,w\rangle^{d-1}\\
&= \frac{1}{2}\sum_{d=1}^{\infty}\frac{(d+n-1)!}{(d-1)!n!\rho_{2n-1+2d}}\langle z,w\rangle^d\\
&= \frac{1}{2}\sum_{d=1}^{\infty}\frac{\Gamma(d+n)}{\Gamma(d)\Gamma(n+1)\rho_{2n-1+2d}}\langle z,w\rangle^d.
\end{aligned}$$

Now, given $\varphi \in L^\infty$, let

$$\begin{aligned}
f(z) &:= P_\rho\varphi(z)\\
&= \int_{\mathbb{B}_n} K_\rho(z,w)\varphi(w)\rho(w)dv(w),
\end{aligned}$$

where $z \in \mathbb{B}_n$. For all $z \in \mathbb{B}_n$ we have

$$\begin{aligned}
\left|Rf(z)\right| &= \left|\int_{\mathbb{B}_n} RK_\rho(z,w)\varphi(w)\rho(w)dv(w)\right|\\
&\le \int_{\mathbb{B}_n} \left|RK_\rho(z,w)\right|\left|\varphi(w)\right| \rho(w)dv(w)\\
&\le \|\varphi\|_\infty \int_{\mathbb{B}_n} \left|RK_\rho(z,w)\right| \rho(w)dv(w).
\end{aligned} \tag{2}$$

Set

$$g(\lambda) = \sum_{d=1}^{\infty} \frac{\Gamma(d+n)}{\Gamma(d)} \frac{\lambda^{d-1}}{\rho_{2n-1+2d}}, \qquad \lambda \in \mathbb{D}.$$

Since $\rho(t) > 0, 0 < t < 1$, g is analytic in the unit disk. Then

$$RK_\rho(z,w) = \frac{\langle z,w\rangle}{2\Gamma(n+1)} g(\langle z,w\rangle). \tag{3}$$

Next we consider the reproducing kernel $K_\rho^1(z,w)$ of the Bergman space in the unit disk with the weight ρ. We have

$$K_\rho^1(z,w) = \frac{1}{2}\sum_{d=0}^{\infty} \frac{(z\overline{w})^d}{\rho_{2d+1}}.$$

Furthermore,

$$\begin{aligned}
\frac{\partial^n}{\partial z^n} K_\rho^1(z,w) &= \frac{1}{2}\sum_{d=n}^{\infty} \frac{\Gamma(d+1)(z\overline{w})^{d-n}\overline{w}^n}{\Gamma(d-n+1)\rho_{2d+1}}\\
&= \frac{1}{2}\sum_{s=1}^{\infty} \frac{\Gamma(s+1)}{\Gamma(s)} \frac{(z\overline{w})^{s-1}\overline{w}^n}{\rho_{2s+2n-1}}\\
&= \frac{1}{2} g(z\overline{w})\overline{w}^n.
\end{aligned}$$

By a result of Peláez and Rättyä ([7] Theorem 1 (ii)), we have

$$\int_{\mathbb{D}} \left| \frac{\partial^n}{\partial z^n} K^1_\rho(z,w) \right| (1-|z|^2)^{n-2} dA(z) \asymp \int_0^{|w|} \frac{dt}{\widehat{\rho}(t)(1-t)^2}, \qquad \frac{1}{2} \leq |w| < 1,$$

where $\widehat{\rho}(t) = \int_t^1 \rho(s)ds$.

Thus,

$$\int_{\mathbb{D}} \left| g(z\overline{w}) \right| (1-|z|^2)^{n-2} dA(z) \asymp \int_0^{|w|} \frac{dt}{\widehat{\rho}(t)(1-t)^2}, \qquad \frac{1}{2} \leq |w| < 1.$$

Since g is analytic in the unit disk, we have

$$\int_{\mathbb{D}} \left| g(z\overline{w}) \right| (1-|z|^2)^{n-2} dA(z) \lesssim 1 + \int_0^{|w|} \frac{dt}{\widehat{\rho}(t)(1-t)^2}, \qquad w \in \mathbb{D}. \quad (4)$$

Now, by (3), we have

$$\begin{aligned} &\int_{\mathbb{B}_n} \left| RK_\rho(z,w) \right| \rho(w) dv(w) \\ &\lesssim \int_{\mathbb{B}_n} \left| g(\langle z,w \rangle) \right| \rho(w) dv(w) \\ &\asymp \int_0^1 r^{2n-1} \rho(r) \left(\int_{\mathbb{S}_n} \left| g(\langle rz, \xi \rangle) \right| d\sigma(\xi) \right) dr. \end{aligned}$$

By[9] Lemma 1.9 and the unitary invariance of $d\sigma$, we have

$$\int_{\mathbb{S}_n} \left| g(\langle rz, \xi \rangle) \right| d\sigma(\xi) \asymp \int_{\mathbb{D}} \left| g(r|z|\,\lambda) \right| (1-|\lambda|^2)^{n-2} dA(\lambda).$$

Thus, by (4) we obtain

$$\begin{aligned} &\int_{\mathbb{B}_n} \left| RK_\rho(z,w) \right| \rho(w) dv(w) \\ &\lesssim \int_0^1 r^{2n-1} \rho(r) \left(1 + \int_0^{r|z|} \frac{dt}{\widehat{\rho}(t)(1-t)^2} \right) dr \\ &\lesssim 1 + \int_0^{|z|} \frac{1}{\widehat{\rho}(t)(1-t)^2} \left(\int_{t/|z|}^1 r^{2n-1} \rho(r) dr \right) dt \\ &\lesssim 1 + \int_0^{|z|} \frac{\widehat{\rho}(t/|z|)}{\widehat{\rho}(t)} \frac{dt}{(1-t)^2} \\ &\lesssim \frac{1}{1-|z|}, \qquad z \in \mathbb{B}_n. \end{aligned}$$

By (2) we obtain now that

$$\left| Rf(z) \right| \lesssim \|\varphi\|_\infty \frac{1}{1-|z|^2}, \quad z \in \mathbb{B}_n.$$

Hence,

$$\sup_{z \in \mathbb{B}_n} (1-|z|^2) \left| Rf(z) \right| \lesssim \|\varphi\|_\infty .$$

It is easy to see that

$$\left| f(0) \right| \lesssim \|\varphi\|_\infty .$$

Therefore, P_ρ is bounded. The Proposition 3.1 is proved. □

Proposition 3.2. *Suppose that the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded. Then $\rho \in \widehat{\mathcal{D}}$.*

Proof. Given $\xi \in \mathbb{S}_n$ and $w \in \mathbb{B}_n$, let us consider a function g given by

$$g(\lambda) = RK_\rho(\lambda\xi, w), \qquad \lambda \in \mathbb{D}.$$

Then

$$g(\lambda) = \sum_{d=1}^\infty c_d \langle \xi, w \rangle^d \lambda^d,$$

where $c_d = \dfrac{1}{2n} \dfrac{\Gamma(d+n)}{\Gamma(d)\Gamma(n)\rho_{2n-1+2d}}$.

By the Hardy–Littlewood inequality (see Lemma 2.2) we have

$$\sum_{d=1}^{\infty} \frac{c_d\left|\langle \xi, w\rangle\right|^d}{d+1} \lesssim \int_0^{2\pi} \left|g(e^{i\theta})\right| \frac{d\theta}{2\pi}$$
$$= \int_0^{2\pi} \left|RK_\rho(e^{i\theta}\xi, w)\right| \frac{d\theta}{2\pi}.$$

Integrating both sides of the above inequality over $\xi \in \mathbb{S}_n$ we obtain

$$\sum_{d=1}^{\infty} \frac{c_d}{d+1} \int_{\mathbb{S}_n} \left|\langle \xi, w\rangle\right|^d \, d\sigma(\xi)$$
$$\lesssim \int_{\mathbb{S}_n} \int_0^{2\pi} \left|RK_\rho(e^{i\theta}\xi, w)\right| \frac{d\theta}{2\pi} d\sigma(\xi)$$
$$= \int_{\mathbb{S}_n} \left|RK_\rho(\xi, w)\right| \, d\sigma(\xi).$$

By the unitary invariance of $d\sigma$ and[9] Lemma 1.9, we have

$$\int_{\mathbb{S}_n} \left|\langle \xi, w\rangle\right|^d \, d\sigma(\xi)$$
$$= |w|^d \int_{\mathbb{S}_n} |\xi_1|^d \, d\sigma(\xi)$$
$$= (n-1)|w|^d \int_{\mathbb{D}} (1-|z|^2)^{n-2} |z|^d \, dA(z)$$
$$= (n-1)\pi |w|^d \int_0^1 (1-t)^{n-2} t^{d/2} dt$$
$$\asymp \frac{\Gamma(\frac{d}{2}+1)\Gamma(n)}{\Gamma(\frac{d}{2}+n)} |w|^d.$$

Hence,

$$\int_{\mathbb{S}_n} \left|RK_\rho(\xi, w)\right| \, d\sigma(\xi)$$
$$\gtrsim \sum_{d=1}^{\infty} \frac{c_d}{d+1} \frac{\Gamma(\frac{d}{2}+1)\Gamma(n)}{\Gamma(\frac{d}{2}+n)} |w|^d$$
$$= \frac{1}{2n} \sum_{d=1}^{\infty} \frac{\Gamma(d+n)\Gamma(\frac{d}{2}+1)}{(d+1)\Gamma(d)\Gamma(\frac{d}{2}+n)\rho_{2n-1+2d}} |w|^d.$$

Since

$$\frac{\Gamma(d+n)\Gamma(\frac{d}{2}+1)}{(d+1)\Gamma(d)\Gamma(\frac{d}{2}+n)} \asymp 1,$$

we get

$$\int_{\mathbb{S}_n} \left|RK_\rho(\xi, w)\right| \, d\sigma(\xi)$$
$$\gtrsim \frac{1}{2n} \sum_{d=1}^{\infty} \frac{|w|^d}{\rho_{2n-1+2d}}, \qquad w \in \mathbb{B}_n.$$

Therefore, for $z \in \mathbb{B}_n$, we have

$$\int_{\mathbb{B}_n} \left|RK_\rho(z, w)\right| \rho(w) dv(w)$$
$$= 2n \int_0^1 r^{2n-1} \rho(r) \int_{\mathbb{S}_n} \left|RK_\rho(z, r\xi)\right| \, d\sigma(\xi) \, dr$$
$$= 2n \int_0^1 r^{2n-1} \rho(r) \int_{\mathbb{S}_n} \left|RK_\rho(\xi, rz)\right| \, d\sigma(\xi) \, dr$$
$$\gtrsim \sum_{d=1}^{\infty} \frac{|z|^d}{\rho_{2n-1+2d}} \int_0^1 r^{2n-1+d} \rho(r) dr$$
$$= \sum_{d=1}^{\infty} \frac{\rho_{2n-1+d}}{\rho_{2n-1+2d}} |z|^d.$$

Thus,

$$\sup_{z \in Bn} (1-|z|^2) \int_{\mathbb{B}_n} \left|RK_\rho(z, w)\right| \rho(w) dv(w)$$
$$\gtrsim \sup_{z \in \mathbb{B}_n} (1-|z|) \sum_{d=1}^{\infty} \frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}} |z|^d$$
$$\geq \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{d=1}^{N} \frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}} \left(1 - \frac{1}{N}\right)^d$$
$$\gtrsim \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{d=1}^{N} \frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}.$$

Since P_ρ is bounded,

$$\sup_{z \in Bn} (1-|z|^2) \int_{\mathbb{B}_n} \left|RK_\rho(z, w)\right| \rho(w) dv(w) < \infty.$$

Given $N \geq 2n$, we obtain that

$$1 \gtrsim \frac{1}{4N-2n} \sum_{d=3N-n+1}^{4N-2n} \frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}$$
$$\geq \frac{1}{4N}(N-n) \frac{\rho_{4N}}{\rho_{6N}}.$$

Hence,

$$\rho_{6N} \gtrsim \rho_{4N}.$$

If $8N \leq k < 8N + 8, N \geq 2n + 8$, then

$$\rho_k \leq \rho_{8N} \lesssim \rho_{12N} \lesssim \rho_{18N} \leq \rho_{2k},$$

and by Lemma 2.1 we conclude that $\rho \in \widehat{\mathcal{D}}$. □

From Proposition 3.1 and Proposition 3.2, we obtain the conclusion of Theorem 1.1.

Remark 3.3. The method given herein combined with our results in[5] can be used to generalize to the unit ball case the L^p estimates proved in[7] in the unit disk case. This will be the object of a forthcoming paper.

ACKNOWLEDGMENTS

This study is conducted within the framework of science and technology projects at institutional level of Quy Nhon University under the project code T2021.700.03.

I am grateful to Alexander Borichev and El Hassan Youssfi for helpful remarks.

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