Bất đẳng thức kiểu Morrey cho các hàm có giá trị trung bình bằng 0

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu một bất đẳng thức kiểu Morrey cho các hàm Sobolev có giá trị trung bình bằng 0. Chứng minh của bất đẳng thức này đã được nhắc đến trong[4,7,9] và trong bài báo này, chúng tôi sẽ cải thiện hằng số $C(d, n, p)$ đã được đề cập tường minh ở Bổ đề B.1.16 trong[7]. Sau đó, chúng tôi nghiên cứu sâu hơn về một ứng dụng của bất đẳng thức kiểu Morrey này đối với sự hội tụ yếu của dãy nghiệm của phương trình p-Laplace với điều kiện biên Neumann khi $p \to \infty$.

Từ khóa: *không gian Sobolev, phương trình p-Laplace, bất đẳng thức kiểu Morrey.*

A note on Morrey-type inequality for functions of mean value zero

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ABSTRACT

In the present article, we study a Morrey-type inequality for Sobolev functions of mean value zero. The proof of this inequality has been mentioned in [4,7,9] and in this paper we will improve the constant $C(d, n, p)$ given explicitly in [7] Lemma B.1.16. Then we study further an application of the Morrey-type inequality for the weak convergence of solutions to p-Laplace equations with a Neumann boundary condition as $p \to \infty$.
Key words: *Sobolev spaces, p-Laplace equations, Morrey-type inequality.*

1. Introduction

Let Ω be a bounded, smooth domain of $\mathbb{R}^n$. This paper is concerned with a Morrey-type inequality for Sobolev functions of mean value zero in Sobolev space $W^{1,p}(\Omega)$. Sobolev spaces consist of L^p functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of partial differential equations.

The well-known Morrey inequality in $\mathbb{R}^n$ (see, for example, [1,4,5]) states that if $p > n$ then for all $v \in W^{1,p}(\mathbb{R}^n)$ and all $x, y \in \mathbb{R}^n$

$$|v(x) - v(y)| \leq \overline{C}_{p,n}|x-y|^{1-n/p}\left(\int_{\mathbb{R}^n}|\nabla v|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{1}$$

where $\overline{C}_{p,n}$ is a positive constant depending only on p and n.

Now, let d_Ω denote the distance function to the boundary $\partial\Omega$, that is

$$d_\Omega(x) := \inf_{y\in\partial\Omega}|x-y|\ ,\ x \in \overline{\Omega}.$$

Taking an arbitrary $y \in \partial\Omega$ in (1) one arrives at the following pointwise inequality, for all $(x, v) \in \overline{\Omega} \times W_0^{1,p}(\Omega)$,

$$|v(x)| \leq \overline{C}_{p,n}\left(d_\Omega(x)\right)^{1-n/p}\|\nabla v\|_p, \tag{2}$$

where $\|.\|_p$ stands for the standard norm of $L^p(\Omega)$. Passing to the maximum value in the left-hand side of (2) we arrive at the well-known Morrey-Sobolev inequality

$$\|v\|_\infty \leq \overline{C}_{p,n,\Omega}\|\nabla v\|_p,\ \forall v \in W_0^{1,p}(\Omega), \tag{3}$$

where the constant $\overline{C}_{p,n,\Omega}$ depends only on p, n and Ω. The above inequality is devoted to Sobolev functions vanishing on the boundary and useful for studying partial differential equations with a Dirichlet boundary condition. The counterpart for the Morrey-Sobolev inequality (3) for Sobolev functions of mean value zero can be written as

$$\|u\|_\infty \leq C_{p,n,\Omega}\|\nabla u\|_p, \tag{4}$$

for all Sobolev function $u \in W^{1,p}(\Omega)$ of mean value zero, that is, it satisfies $\int_\Omega u \mathrm{d}x = 0$.

In [6], the authors make use of this inequality (with a smooth and convex domain Ω) in studying limits as $p \to \infty$ of solutions to p-Laplace equations coupled with a Neumann boundary condition.

Such a constant $C_{p,n,\Omega}$ has been explicitly mentioned in [7] Lemma B.1.16. Finding a smaller constant of the Morrey-type inequality (4) is an interesting issue.

In this short paper, we provide a better estimate for the Morrey-type inequality (4) with a Lipschitz and convex domain Ω. Furthermore, its application in studying the weak convergence of solutions to p-Laplace equations with a Neumann boundary condition as $p \to \infty$ is also considered in detail.

The article is organized as follows. In section 2, we give a new constant C of the Morrey-type inequality (4) for Sobolev functions of mean value zero. Then, in section 3, we derive the key bound of the gradients of solutions to p-Laplace equations coupled with a Neumann boundary condition. As a consequence, we obtain the weak convergence of solutions in Sobolev spaces as $p \to \infty$.

2. Morrey-type inequality for Sobolev functions of mean value zero

Theorem 1. *Let Ω be a Lipschitz and convex domain of $\mathbb{R}^n$ and $p > n$. Then every Sobolev function $u \in W^{1,p}(\Omega)$ of mean value zero, i.e. $\int_\Omega u\mathrm{d}x = 0$, obeys the following inequality*

$$\|u\|_{L^\infty} \leq \frac{d^{n+1-n/p}}{|\Omega|\,(n+1-n/p)} \omega_n^{1-1/p} \left(\frac{p-1}{p-n}\right)^{1-1/p} \|\nabla u\|_{L^p}, \tag{5}$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$, Γ is the gamma function, $|\Omega|$ is Lebesgue measure of Ω and $d = \mathrm{diam}(\Omega)$ is the diameter of Ω.

Proof. Set

$$C(d,n,p) := \frac{d^{n+1-n/p}}{n+1-n/p} \omega_n^{1-1/p} \left(\frac{p-1}{p-n}\right)^{1-1/p}.$$

We divide the proof into three steps.
Step 1. Fix any $x \in \mathbb{R}^n$. Let us prove that for any $\mathbb{R}^n$-valued measurable function $W \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we have for all $p > n$

$$\int\limits_{B(0,d)} \int\limits_0^1 |W(x+tz)||z|\mathrm{d}t\mathrm{d}z \leq C(d,n,p)\|W\|_{L^p(B(x,d))}, \tag{6}$$

where $B(y,d)$ is the Euclidean ball of radius d and center y in $\mathbb{R}^n$. To this aim, we make use of the change of variables in polar coordinates by the bijection $\Phi : B(0,d)\setminus\{0\} \longrightarrow (0,d] \times \partial B(0,1)$ defined as $\Phi(z) := (r,z') = \left(|z|, \frac{z}{|z|}\right)$. More precisely, one has

$$\int\limits_{B(0,d)} g(z)\mathrm{d}z = \int\limits_0^d r^{n-1} \int\limits_{\partial B(0,1)} g(rz')\mathrm{d}S(z')\mathrm{d}r. \tag{7}$$

Now applying (7) with $g(z) := \int\limits_0^1 |W(x+tz)||z|\mathrm{d}t$ on $B(0,d)$ for the second line below, and $g(z) := |W(x+z)||z|^{1-n}$ on $B(0,r)$ for the fifth, we arrive the following estimate

$$\begin{aligned}
&\int\limits_{B(0,d)} \int\limits_0^1 |W(x+tz)||z|\mathrm{d}t\mathrm{d}z \\
&= \int\limits_0^d r^{n-1} \int\limits_{\partial B(0,1)} \int\limits_0^1 |W(x+trz')|r\mathrm{d}t\mathrm{d}S(z')\mathrm{d}r \\
&= \int\limits_0^d r^{n-1} \int\limits_{\partial B(0,1)} \int\limits_0^r |W(x+\tau z')|\mathrm{d}\tau\mathrm{d}S(z')\mathrm{d}r \text{ (set } \tau = rt) \\
&= \int\limits_0^d r^{n-1} \int\limits_0^r \tau^{n-1} \int\limits_{\partial B(0,1)} |W(x+\tau z')|\tau^{1-n}\mathrm{d}S(z')\mathrm{d}\tau\mathrm{d}r \\
&= \int\limits_0^d r^{n-1} \int\limits_{B(0,r)} |W(x+z)||z|^{1-n}\mathrm{d}z\mathrm{d}r \\
&\leq \int\limits_0^d r^{n-1} \left(\int\limits_{B(0,r)} |W(x+z)|^p\mathrm{d}z\right)^{1/p} \left(\int\limits_{B(0,r)} |z|^{q(1-n)}\mathrm{d}z\right)^{1/q} \mathrm{d}r.
\end{aligned} \tag{8}$$

Here, for the last line, we used Hölder's inequality with $q = \frac{p}{p-1}$.
By $r \leq d$, observe that

$$\left(\int\limits_{B(0,r)} |W(x+z)|^p\mathrm{d}z\right)^{1/p} \leq \|W\|_{L^p(B(x,d))}.$$

On the other hand, using again (7) with $g(z) = |z|^{q(1-n)}$ we can compute explicitly

$$\begin{aligned}
\int\limits_{B(0,r)} |z|^{q(1-n)}\mathrm{d}z &= \int\limits_0^r \tau^{n-1} \int\limits_{\partial B(0,1)} \tau^{q(1-n)}\mathrm{d}S(z')\mathrm{d}\tau \\
&= \omega_n \int\limits_0^r \tau^{\frac{1-n}{p-1}}\mathrm{d}\tau \\
&= \frac{\omega_n}{\frac{1-n}{p-1}+1} r^{\frac{1-n}{p-1}+1} = \omega_n \frac{p-1}{p-n} r^{\frac{p-n}{p-1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int\limits_{B(0,d)} \int\limits_0^1 |W(x+tz)||z|\mathrm{d}t\mathrm{d}z \\
&\leq \|W\|_{L^p(B(x,d))} \left(\omega_n \frac{p-1}{p-n}\right)^{1/q} \int\limits_0^d r^{n-1} \left(r^{\frac{p-n}{p-1}}\right)^{1/q} \mathrm{d}r \\
&= \|W\|_{L^p(B(x,d))} \left(\omega_n \frac{p-1}{p-n}\right)^{1/q} \int\limits_0^d r^{n-n/p}\mathrm{d}r \\
&= \|W\|_{L^p(B(x,d))} \left(\omega_n \frac{p-1}{p-n}\right)^{1-1/p} \frac{d^{n+1-n/p}}{n+1-n/p}.
\end{aligned}$$

Step 2. We are now in a position to show the assertion of Theorem 1 for the case of smooth function u. In this case, set

$$W(x) := \begin{cases} \nabla u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Since $\int_\Omega u(y)\mathrm{d}y = 0$, we get for any $x \in \Omega$

$$\begin{aligned} u(x) &= \frac{1}{|\Omega|}\int_\Omega (u(x)-u(y))\,\mathrm{d}y \\ &= \frac{1}{|\Omega|}\int_\Omega\int_0^1 \langle \nabla u(x+t(y-x)), x-y\rangle \mathrm{d}t\mathrm{d}y \\ &\le \frac{1}{|\Omega|}\int_\Omega\int_0^1 |\nabla u(x+t(y-x))||x-y|\mathrm{d}t\mathrm{d}y \\ &\le \frac{1}{|\Omega|}\int\limits_{B(x,d)}\int\limits_0^1 |W(x+t(y-x))||x-y|\mathrm{d}t\mathrm{d}y \\ &= \frac{1}{|\Omega|}\int_{B(0,d)}\int_0^1 |W(x+tz)||z|\mathrm{d}t\mathrm{d}z. \end{aligned}$$

Following Step 1, we obtain

$$\begin{aligned} |u(x)| &\le \frac{C(d,n,p)}{|\Omega|}\|W\|_{L^p(B(x,d))} \\ &= \frac{C(d,n,p)}{|\Omega|}\|\nabla u\|_{L^p(\Omega)}, \end{aligned}$$

which completes the proof for smooth functions.

Step 3. For the general case, we make use a smooth approximation. Fix any $u \in W^{1,p}(\Omega)$ satisfying $\int_\Omega u\mathrm{d}x = 0$. Then there exists a sequence of smooth functions u_ε of mean value zero such that u_ε converges to u strongly in $L^m(\Omega)$ $(1 \le m < \infty)$ and u_ε converges to u strongly in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$, from the inequality

$$\begin{aligned} \|u_\varepsilon\|_{L^m(\Omega)} &\le |\Omega|^{\frac{1}{m}}\|u_\varepsilon\|_{L^\infty(\Omega)} \\ &\le |\Omega|^{\frac{1}{m}}\frac{C(d,n,p)}{|\Omega|}\|\nabla u_\varepsilon\|_{L^p(\Omega)} \end{aligned}$$

we obtain

$$\|u\|_{L^m(\Omega)} \le |\Omega|^{\frac{1}{m}}\frac{C(d,n,p)}{|\Omega|}\|\nabla u\|_{L^p(\Omega)}.$$

Letting $m \to \infty$ we arrive to

$$\|u\|_{L^\infty(\Omega)} \le \frac{C(d,n,p)}{|\Omega|}\|\nabla u\|_{L^p(\Omega)},$$

which completes the proof. □

Remark 2. The above Morrey-type inequality is inspired by the estimate (B.1.3) of[7] p. 556 for smooth functions of mean value zero. However, it is worth noting that the inequality given in Theorem 1 is a little bit sharper than the one in[7] p. 556, where the constant is explicitly stated by

$$C = \frac{d^{n+1-n/p}}{|\Omega|n}\omega_n^{1-1/p}\left(\frac{p-1}{p-n}\right)^{1-1/p}.$$

3. Applications

Give a Lipschitz and convex domain Ω of $\mathbb{R}^n$, we consider limits as $p \to \infty$ of solutions u_p to the p-Laplace problems coupled with a Neumann boundary condition

$$\begin{cases} -\operatorname{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right) = f(x) & \text{in } \Omega \\ |\nabla u(x)|^{p-2}\dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

For a fixed $p > n$, the equation (9) has unique solution u_p of mean value zero, that is $\int_\Omega u_p \mathrm{d}x = 0$. This is a standard result in the field of calculus of variations and partial differential equations. In fact, consider the variational problem

$$\min_{u\in S_p}\left\{\int_\Omega \frac{|\nabla u|^p}{p}\,dx - \int_\Omega uf\,dx\right\}, \tag{10}$$

where $S_p := \left\{u \in W^{1,p}(\Omega) : \int_\Omega u\mathrm{d}x = 0\right\}$. The cost functional in (10) is lower semi-continuous, coercive and strictly convex on the non-empty convex set S_p. Therefore, there exists a unique minimizer u_p to (10), which is also a weak solution of problem (9), that is, it verifies

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p.\nabla\phi\mathrm{d}x = \int_\Omega f\phi\mathrm{d}x, \ \forall \phi \in C^\infty(\overline{\Omega}). \tag{11}$$

In this section, we are interested in studying the behavior of solutions u_p as $p \to \infty$. More precisely, we will show that the sequence $\{u_p\}$ converges weakly in Sobolev spaces to a 1-Lipschitz function u_∞ as $p \to \infty$.

3.1. Bound of the gradients

Our aim is to prove that the L^p-norm of the gradients ∇u_p is bounded independently of all $p \ge n+1$.

Lemma 3. *Let u_p be a unique solution to* (9) *with $\int_\Omega u_p\mathrm{d}x = 0$. Then there exists a positive constant C independent of $p \ge n+1$ such that*

$$\|\nabla u_p\|_{L^p(\Omega)} \le C^{\frac{1}{p-1}} \text{ for all } p \ge n+1. \tag{12}$$

Proof. As a consequence of Theorem 1, there exists a positive constant C_Ω independent of $p \ge n+1$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega)} \tag{13}$$

for all $p \ge n+1$ and all Sobolev functions $u \in W^{1,p}(\Omega)$ of mean value zero, i.e., $\int_\Omega u\mathrm{d}x = 0$. In particular, applying for $u = u_p$ we get

$$\|u_p\|_{L^\infty(\Omega)} \le C_\Omega\|\nabla u_p\|_{L^p(\Omega)}, \text{ for all } p \ge n+1. \tag{14}$$

On the other hand, using (11) with $\phi = u_p$, Hölder's inequality and (14), we obtain

$$\begin{aligned} \int_\Omega |\nabla u_p|^p\,\mathrm{d}x &= \int_\Omega fu_p\,\mathrm{d}x \le \|f\|_{L^1(\Omega)}\|u_p\|_{L^\infty(\Omega)} \\ &\le C_\Omega\|f\|_{L^1(\Omega)}\|\nabla u_p\|_{L^p(\Omega)}. \end{aligned}$$

It follows that

$$\|\nabla u_p\|_{L^p(\Omega)} \le C^{\frac{1}{p-1}} \text{ for all } p \ge n+1,$$

with $C := C_\Omega \|f\|_{L^1(\Omega)}$ being independent of all $p \ge n+1$. □

3.2. Weak convergence

As a consequence of the previous bound of the gradients, we obtain uniform convergence of u_p and weak convergence of the gradients ∇u_p.

Proposition 4. *Let u_p be a unique solution to (9) with $\int_\Omega u_p \mathrm{d}x = 0$. Then, up to a subsequence, u_p converges uniformly on $\overline{\Omega}$ to a limit function $u_\infty \in W^{1,\infty}(\Omega)$ and $\nabla u_p \rightharpoonup \nabla u_\infty$ weakly in $L^m(\Omega)$ as $p \to \infty$ for any $1 \le m < \infty$. Moreover, the limit function u_∞ is 1-Lipschitz, that is,*

$$|\nabla u_\infty(x)| \le 1 \text{ for a.e. in } \Omega.$$

Proof. Fix any $m > n$. Let $p^* = \dfrac{p}{m}$. Using Hölder's inequality with p^* and q^* satisfying $\dfrac{1}{p^*} + \dfrac{1}{q^*} = 1$ and Lemma 3, we obtain

$$\begin{aligned} \|\nabla u_p\|_{L^m(\Omega)} & \left(\int_\Omega |\nabla u_p|^m \mathrm{d}x \right)^{\frac{1}{m}} \\ & \le \left(\int_\Omega |\nabla u_p|^{mp^*} \mathrm{d}x \right)^{\frac{1}{mp^*}} \left(\int_\Omega \mathrm{d}x \right)^{\frac{1}{mq^*}} \\ & = \|\nabla u_p\|_{L^p(\Omega)} |\Omega|^{\frac{1}{m} - \frac{1}{p}} \\ & \le |\Omega|^{\frac{1}{m} - \frac{1}{p}} C^{\frac{1}{p-1}} \end{aligned} \tag{15}$$

for all $p \ge \max\{n+1, m\}$, where C is a constant independent of p from Lemma 3. Observe that $|\Omega|^{\frac{1}{m} - \frac{1}{p}} C^{\frac{1}{p-1}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, the sequence of gradients ∇u_p is bounded in $L^m(\Omega)$ and so is $\{u_p\}$ in $W^{1,m}(\Omega)$ (u_p is of mean value zero). Therefore, up to a subsequence, u_p converges uniformly on $\overline{\Omega}$ to a limit function u_∞ and $\nabla u_p \rightharpoonup \nabla u_\infty$ weakly in $L^m(\Omega)$ as $p \to \infty$. Obviously, the weak convergence of ∇u_p also holds true in $L^m(\Omega)$ for any $1 \le m \le n$. Finally, taking the limit as $p \to \infty$ in (15), we arrive to

$$\|\nabla u_\infty\|_{L^m(\Omega)} \le |\Omega|^{\frac{1}{m}}. \tag{16}$$

Letting $m \to \infty$ we obtain $\|\nabla u_\infty\|_{L^\infty(\Omega)} \le 1$, which completes the proof. □

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