Một số kết quả của nguyên lý cực đại trên đĩa đơn vị

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TÓM TẮT

Trong bài báo này, chúng tôi áp dụng nguyên lý cực đại cho hàm điều hòa dưới trên mặt phẳng phức để chứng minh một số kết quả liên quan tới các hàm chỉnh hình và hàm điều hòa dưới xác định trong đĩa đơn vị trên mặt phẳng phức.
Từ khóa: *nguyên lý cực đại, hàm điều hòa dưới, hàm chỉnh hình, lý thuyết thế vị, giải tích phức.*

Some results of the maximum principle on the unit disc

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ABSTRACT

In this note, we apply the maximum principle of subharmonic functions on the complex plane to prove some results related to the holomorphic functions and the subharmonic functions on unit disc in complex plane.

Key words: *maximum principle, subharmonic function, holomorphic function, potential theory, complex analysis.*

1. Introduction

In potential theory, the subharmonic functions are usually defined on the open set in $\mathbb{R}^n$ (see[1]). This is an advantage to use analytic tools of many variable functions. However, it does not take advantages of the complex number and complex variable function theory. On the other hand, it is hard to extend to the pluripotential theory (see[2],[3]). The Theorem 2 gives the relation between the holomorphic functins and subharmonic functions. This allows using the complex analytic tools when we study the subharmonic functions on the complex plane.

The maximum principle of subharmonic functions is an interesting topic in potential theory. This principle is established by Phragmén and Lindelöf in[4]. The potential theory is a branch of complex analysis that is concentrated to study in the near decades and quite new in Viet Nam. The maximum principle is established and proved depend on the topology on the extended complex plane (Theorem 3). Because the extended complex plane $\mathbb{C}_\infty$ is homeomorphic with the Riemann sphere in the metric space $\mathbb{R}^3$, so the extended complex plane $\mathbb{C}_\infty$ is a compact set. This has made the proof of the maximum principle quite simply.

The main aim of this paper is to use the maximum principle to prove some results of the holomorphic function and subharmonic functions on the unit disc in the complex plane (Theorem 6 and Theorem 8).

2. Preliminaries

We denote $\mathbb{C}$ to be the set of all complex numbers (or the complex plane). And $\mathbb{C}_\infty$ is the extended complex plane that is homeomorphic with the Riemann sphere in the metric space $\mathbb{R}^3$ (see[5]). Because the Riemann sphere is a compact set in $\mathbb{R}^3$, $\mathbb{C}_\infty$ is a compact set.

In this note, we call the domain to be an open and connected set in $\mathbb{C}$ or $\mathbb{C}_\infty$. Let D be a domain then the closure $\overline{D}$ always takes in $\mathbb{C}_\infty$. So if D is an unbonded domain in $\mathbb{C}$ then $\infty \in D$ and in $\mathbb{C}_\infty$, $\overline{D}$ is a compact set. We also denote $\Delta(\omega, \rho)$ to be a disc in $\mathbb{C}$, that is

$$\Delta(\omega, \rho) := \{z \in \mathbb{C} : |z - \omega| < \rho\}.$$

Definition 1 (see[1],[2],[3])**.** Let U be an open set in $\mathbb{C}$. The function $u : U \to [-\infty, \infty)$ is called subharmonic if it is an upper semicontinuous function and satisfies the local submean inequality, that is for all $w \in U$ there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta, \quad (0 \leq r < \rho). \quad (1)$$

The function $v : U \to (-\infty, \infty]$ is superharmonic if the function $-v$ is subharmonic.

We denote $\mathrm{SH}(U)$ be the set of all subharmonic functions on U. The submean inequality (1) is local, i.e the number ρ depends on w. So the subharmonicity also has local property, that is if $(U_\alpha)_{\alpha \in I}$ is a open cover of U then the function u is subharmonic function on U iff it is a subharmonic function on every U_α.

The following result is the relation between the holomorphic function and the subharmonic function.

Theorem 2. *Let f be a holomorphic function on open set U in $\mathbb{C}$. Then $\log|f|$ be a subharmonic function on U.*

Proof. See Proposition 1.2.23 in[2]. $\square$

Theorem 3 (The maximum principle). *Let u be a subharmonic function on the domain D in $\mathbb{C}$. Then we have*
a. If u has global extremum on D then u is constant on D.
b. If $\limsup_{z\to\xi} u(z) \le 0$ for all $\xi \in \partial D$ then $u \le 0$ on D.

Proof. a. Suppose that u has global extremum value M on D, i.e there exist $z_0 \in D$ such that

$$u(z) \le M, \forall z \in D \text{ và } u(z_0) = M.$$

Set

$$A = \{z \in D : u(z) < M\}$$

and

$$B = \{z \in D : u(z) = M\}.$$

Then by the semicontinuous of u, we infer that A is open. We prove that B also is open. Indeed, take $\omega \in B$, by Definition 1 there exist $\rho > 0$ such that

$$M = u(\omega) \le \frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it})dt \le M$$

foll all $0 \le r < \rho$. Infer that

$$\frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it})dt = M, \quad \forall 0 \le r < \rho.$$

Since $u(\omega + re^{it}) \le M$ for all $r \in [0, \rho)$ and for all $t \in [0, 2\pi)$ so we have $u(\omega + re^{it}) = M, \forall 0 \le r < \rho$ and $\forall 0 \le t < 2\pi$. So $\Delta(\omega, \rho) \subset B$ and so B is open. So we have A and B be an open partition of D. Since D is a connected set, we infer either $A = D$ or $B = D$. Because $B \ne \emptyset$ ($z_0 \in B$) so $B = D$. So we conclude that $u = M$ on D.
b. We extend the function u to the boundary ∂D by set

$$u(\xi) := \limsup_{z\to\xi} u(z) \quad (\xi \in \partial D).$$

Then u is the semicontinous function on $\overline{D}$. Since $\overline{D}$ is a compact set so u has maximum at some $\omega \in \overline{D}$. If $\omega \in \partial D$ then by assumption we have $u(\omega) \le 0$ and so $u \le 0$ on D. If $\omega \in D$ then by the part a., u is constant on D and so on $\overline{D}$. This infers that $u \le 0$ on D. $\square$

Remark 4. In Theorem 3(a), if u has the local extremum or the global minimum on D then the conclusion is failed. Example: Let $u(z) = \max(Rez, 0)$ on $\mathbb{C}$. Then u is the subharmonic function on $\mathbb{C}$. Moreover, u has the local extremum and the global minimum on $\mathbb{C}$, but u is not a constant on $\mathbb{C}$.

3. Main results

In this section, we apply the maximum principle to prove some results for the functions on the unit disc. First, we have the lemma.

Lemma 5. *Let u be a subharmonic function on $\Delta(0,1)$ such that $u < 0$. Then for all $\xi \in \partial\Delta(0,1)$ we have*

$$\limsup_{r\to 1^-} \frac{u(r\xi)}{1-r} < 0.$$

Proof. Set $v(z) = u(z) + c\log|z|$ (here c is a positive constant) on $A = \{\frac{1}{2} < |z| < 1\}$. Then we have
• The function v is a subharmonic function on A (by Theorem 2).
• For all $|\xi| = 1$ we have $\limsup_{z\to\xi} v(z) \le 0$.
To applying the maximum principle (Theorem 3) to the function v on A, we need to find c such that for all $|\xi| = \frac{1}{2}$ we have

$$\limsup_{z\in A, z\to\xi} v(z) \le 0.$$

Set $\lambda = \sup\{u(\xi) : |\xi| = \frac{1}{2}\}$. We infer that $\lambda < 0$. We have

$$\limsup_{z\in A, z\to\xi} v(z) \le \lambda + c\log\frac{1}{2} \le 0.$$

From this inequality we have $c \ge \frac{\lambda}{\log 2}$.
Now, with $c \ge \frac{\lambda}{\log 2}$, applying Theorem 3 to the function v we infer

$$v(z) \le 0 \Leftrightarrow u(z) \le -c\log|z|, \quad \forall \frac{1}{2} < |z| < 1.$$

Then for all $|\xi| = 1$ we have

$$\limsup_{r\to 1^-} \frac{u(r\xi)}{1-r} \le \limsup_{r\to 1^-}(-c)\frac{\log r}{1-r} = c.$$

From the estimations above, if we choose the constant c such that $\frac{\lambda}{\log 2} \le c < 0$ then we have the conclude in the theorem. $\square$

Theorem 6. *Set $\Delta = \Delta(0,1)$. Let $f : \Delta \longrightarrow \Delta$ be a holomorphic function such that*

$$f(z) = z + o(|1-z|^3) \quad \textit{when } z \to 1.$$

a. Let $\phi(z) = \frac{1+z}{1-z}$ and $u(z) = Re(\phi(z) - \phi(f(z)))$. Prove that

$$\limsup_{z\to\xi} u(z) \le 0 \quad \forall \xi \in \partial\Delta \setminus \{1\},$$

and $u(z) = o(|1-z|)$ when $z \to 1$.
b. Prove that $u \le 0$ on Δ.
c. Prove that $u \equiv 0$ on Δ.
d. Prove that $f(z) \equiv z$ on Δ.

Proof. a. We have

$$\text{Re}\phi(z) = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}}\right) = \frac{1-|z|^2}{|1-z|^2}.$$

This infers that for every $\xi \in \partial\Delta \setminus \{1\}$ we have
• $\limsup_{z\to\xi} \mathrm{Re}\phi(z) = 0$.
• For all $z \in \Delta$ then $\mathrm{Re}\phi(z) > 0$. So we infer $\mathrm{Re}\phi(f(z)) > 0$.
Now, for all $\xi \in \partial\Delta \setminus \{1\}$ we have

$$\limsup_{z\to\xi} u(z) \le \limsup_{z\to\xi} \phi(z) = 0.$$

In case $z \to 1$ we have

$$\begin{aligned}\phi(z) - \phi(f(z)) &= \frac{1+z}{1-z} - \frac{1+z+o(|1-z|^3)}{1-z-o(|1-z|^3)}\\ &= \frac{-(1+z)o(|1-z|^3) - (1-z)o(|1-z|^3)}{(1-z)(1-z-o(|1-z|^3))}\\ &= \frac{-2.o(|1-z|^3)}{(1-z)(1-z-o(|1-z|^3))} = o(|1-z|).\end{aligned}$$

From this we infer

$$u(z) = \mathrm{Re}(\phi(z) - \phi(f(z))) = o(|1-z|) \quad \text{when } z \to 1.$$

b. From the above formula, we infer that u is a subharmonic function on Δ. By (a.) we infer that

$$\limsup_{z\to\xi} u(z) \le 0 \quad \text{for all } \xi \in \partial\Delta.$$

By the maximum principle (Theorem 3), we derive $u \le 0$ on Δ.
c. By (b.) we have $u \le o$ on Δ.
If $u < 0$ on Δ then by Lemma 5, for all $\xi \in \partial\Delta$ we have

$$\limsup_{r\to 1^-} \frac{u(r\xi)}{1-r} < 0. \quad (*)$$

When $\xi = 1$, by (a.) we have

$$u(r) = o(|1-r|) \quad \text{when } r \to 1^-.$$

This infer that

$$\limsup_{r\to 1^-} \frac{u(r)}{1-r} = \limsup_{r\to 1^-} \frac{o(|1-r|)}{1-r} = 0.$$

This is an contradiction with $(*)$.
So $u \equiv 0$ on Δ.
d. By (c.) we have

$$\mathrm{Re}\frac{1+z}{1-z} = \mathrm{Re}\frac{1+f(z)}{1-f(z)} \quad \text{on } \Delta.$$

This derive the function $g(z) := \frac{1+z}{1-z} - \frac{1+f(z)}{1-f(z)}$ is holomorphic on Δ that has real part equal zero. By the Cauchy - Riemann condition (Theorem 2 in[3]), the imaginary part of $g(z)$ is constant. So we have $g(z) = ai$ here a be complex number.
On the other hand, by (a.), we have

$$g(z) = o(|1-z|) \quad \text{when } z \to 1.$$

This infer that $\lim_{z\to 1} g(z) = 0$ or $ai = 0$. So we have $a = 0$, i.e $g \equiv 0$ on Δ.
So for all $z \in \Delta$ we have

$$\frac{1+z}{1-z} = \frac{1+f(z)}{1-f(z)} \Leftrightarrow \frac{2}{1-z} = \frac{2}{1-f(z)} \Leftrightarrow f(z) = z.$$

□

Remark 7. In Theorem 6, if we suppose that

$$f(z) = z + O(|1-z|^3) \quad \text{when } z \to 1$$

then the conclude in (d.) is failed.
Indeed, considering $f(z) = z + \lambda(1-z)^3$, here $\lambda > 0$ enough small. Then with $|z| = 1$ we have

$$\begin{aligned}|f(z)|^2 &= (z + \lambda(1-z)^3)(\bar{z} + \lambda\overline{(1-z)^3})\\ &= 1 - 2\lambda\mathrm{Re}(\bar{z}.(1-z)^3) + \lambda^2(1-z)^3\overline{(1-z)^3}\\ &= 1 + 2\lambda[4\mathrm{Re}z - 3 - \mathrm{Re}z^2] + 8\lambda^2(1-\mathrm{Re}z)^3.\end{aligned}$$

Set $z = \cos t + i\sin t$ here $0 \le t \le 2\pi$. Then to prove that $|f(z)|^2 \le 1$ we need the following

$$2\lambda[4\mathrm{Re}z - 3 - \mathrm{Re}z^2] + 8\lambda^2(1-\mathrm{Re}z)^3 \le 0 \ \forall |z| = 1.$$

This is equivalent

$$\begin{aligned}&4\cos t - 3 - \cos 2t + 4\lambda(1-\cos t)^3 \le 0 \quad \forall 0 \le t \le 2\pi\\ &\Leftrightarrow (1-\cos t)^2(-2 + 4\lambda(1-\cos t)) \le 0 \quad \forall 0 \le t \le 2\pi.\end{aligned}$$

This is true if we choose $0 < \lambda < \frac{1}{4}$.
So the function $f : \Delta \longrightarrow \Delta$ is holomorphic and satisfies $f(z) = z + O(|1-z|^3)$. But f is not identical function.

Theorem 8. *Let u be a subharmonic function on $\Delta(0,1)$ such that*

$$u(z) \le -\log|Imz| \quad (|z| < 1).$$

Then prove that

$$u(z) \le -\log\left|\frac{1-z^2}{2}\right| \quad (|z| < 1).$$

Proof. With $0 < r < 1$ we consider the function following

$$v(z) = u(z) + \log\left|\frac{r^2 - z^2}{2r}\right|, \quad z \in \Delta(0,r).$$

Then by Theorem 2, v is a subharmonic function on $\Delta(0,r)$. Take $\xi \in \partial\Delta(0,r)$. We consider two cases as follow.
• If $\xi \ne r$ and $\xi = r(\cos t + i\sin t)$ then we have

$$\begin{aligned}\limsup_{z\to\xi} v(z) &= \limsup_{z\to\xi}\left(u(z) + \log\left|\frac{r^2-z^2}{2r}\right|\right)\\ &\le \limsup_{z\to\xi} \log\left|\frac{r^2-z^2}{2r|\mathrm{Im}z|}\right|\\ &= \limsup_{z\to\xi} \log\frac{|r^2-\xi^2|}{2r|\mathrm{Im}\xi|} = 0.\end{aligned}$$

• If $\xi = r$ then we have

$$\limsup_{z\to r} v(z) = \limsup_{z\to r}\left(u(z) + \log\left|\frac{r^2-z^2}{2r}\right|\right) = -\infty.$$

(this because u is bounded on $\overline{\Delta}(0,r)$). By applying the maximum principle (Theorem 3) to function v on $\Delta(0,r)$ we infer

$$u(z) \le -\log\left|\frac{r^2-z^2}{2r}\right|, \quad \forall z \in \Delta(0,r).$$

Let $r \to 1^-$ we get

$$u(z) \le -\log\left|\frac{1-z^2}{2}\right| \quad \forall z \in \Delta(0,1).$$

□

Acknowledgment

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