Giới hạn thủy động lực học của động lực Kawasaki với trường hỗn độn không bị chặn

Nguyễn Đặng Thiên Thư

Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam.
Tác giả liên hệ chính. Email: nguyendangthienthu@qnu.edu.vn

TÓM TẮT

Vào năm 2003, Faggionato A. và Martinelli F. đã chứng minh được rằng giới hạn thủy động lực học của một quá trình loại trừ đơn giản dưới tác động của một trường hỗn độn bị chặn có thể được mô tả bởi một bài toán Cauchy. Trong bài báo này, ta chỉ ra rằng một kết luận tương tự cũng đúng nếu trường hỗn độn không bị chặn và phân phối của nó thoả mãn một điều kiện nào đó.

Từ khóa: *Giới hạn thủy động lực học, Hệ không bị chặn, Quá trình loại trừ.*

Hydrodynamic limit of the Kawasaki dynamics with unbounded disorder

Nguyễn Đặng Thiên Thư*

Department of Mathematics and Statistics, Quy Nhon University, Vietnam.

ABSTRACT

In 2003, Faggionato A. and Martinelli F. proved that the hydrodynamic behavior of a simple exclusion process under the influence of a bounded disorder field can be described by a Cauchy problem. In this paper, we show that a similar conclusion also holds true if the disorder field is unbounded and satisfies a certain condition on its distribution.

Keywords: *Hydrodynamic limits, Disordered systems, Exclusion processes.*

1. INTRODUCTION AND MAIN RESULT

A lot of techniques have been developed so far in order to investigate the hydrodynamic limit of an interacting particle system in which each particle moves on an integer lattice. An interacting particle system is said to have a hydrodynamic limit if for which there exists a time and space rescaling in which the conserved quantities evolve according to a certain partial differential equation. This partial differential equation is called the hydrodynamic equation corresponding with the system. The simplest and most widely studied interacting particle system is the simple exclusion process, where a particle sitting on a site x of the d-dimensional torus $\mathbf{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with unit volume, waits an exponential time and attempts to jump to the nearest neighbor site y together with the exclusion rule that forbids the jumps to occupied sites. Furthermore, it is assumed that the total number of particles is the unique quantity conserved by the time evolution. We denote by $\mathbf{T}^d_N := \mathbb{Z}^d/N\mathbb{Z}^d$ the corresponding microscopic space and by π_t^N the empirical measure on $\mathbf{T}^d$ obtained by assigning to each particle

* *Corresponding author. Email: nguyendangthienthu@qnu.edu.vn*

a mass N^{-d},

$$\pi_t^N(d\theta) = \pi_t^N(\eta, d\theta) = \underset{x\in\mathbf{T}_N^d}{Av}\, \eta_x(tN^2)\delta_{x/N}(d\theta)$$

where $\underset{x\in\mathbf{T}_N^d}{Av}$ stands for the spatial average and η_x is the number of particles at site x. Then the dynamics is determined by the diffusively rescaled Markov generator $N^2\mathcal{L}_N$,

$$\mathcal{L}_N f(\eta) = \sum_{e\in\mathcal{E}} \sum_{x\in\mathbf{T}_N^d} c_{x,x+e}(\eta)[f(\eta^{x,x+e}) - f(\eta)],$$

where $\mathcal{E}$ is the canonical basis of the d-dimensional lattice $\mathbb{Z}^d$.

In the above expression, the configuration $\eta^{x,y}$ is obtained from η by exchanging the values η_x and η_y, the positive and bounded transition rate $c_{x,y}(\eta)$ is not only translation invariant in the sense that $c_{x+z,y+z}(\eta) = c_{z,y}(\tau_z\eta)$, where $\tau_z\eta$ is the particle configuration translated by z, but also satisfies the identity $c_{x,y}(\eta) = c_{y,x}(\eta)$.

Given a symmetric, finite range and translation invariant transition probability $p(x, y)$ on $\mathbb{Z}^d$, i.e. $p(x, y) = p(0, y-x) =: p(y-x)$, the set $\{x : p(x) > 0\}$ is finite and $p(x) = p(-x)$. If $c_{x,x+e}(\eta) := \sum_{y\in\mathbb{Z}^d} p(e+yN), \forall x \in \mathbf{T}_N^d, \forall e \in \mathcal{E}$ then such a system is called a symmetric simple exclusion process. We denote by $\mathcal{M}_2$ the set of probability measures on $\mathbf{T}^d$, which are absolutely continuous with respect to Lebesgue measure with density bounded by 1. Now, let a sequence of probability measures $\{\mu^N\}_{N\geq 1}$ on $\Omega_N = \{0,1\}^{\mathbf{T}_N^d}$ be associated to a function ρ_0 on $\mathbf{T}^d$ in the sense that under μ^N, the sequence of empirical measures π_0^N on $\mathbf{T}^d$ converges in probability to $\rho_0(\theta)d\theta \in \mathcal{M}_2$. Then, it is shown that after a suitable space and time rescaling, the corresponding sequence $\{\mathbb{P}_t^{\mu^N}\}_{N\geq 1}$ of the distributions at time t of a symmetric simple exclusion process with the initial measures μ^N, is associated to the density of particles $\rho(t, \cdot)$ which is the unique weak solution of the heat equation

$$\partial_t\rho(t, \theta) = \frac{1}{2}\Delta\rho(t, \theta), \;\; \rho(0, \theta) = \rho_0(\theta). \quad (1)$$

More generally, the hydrodynamic behavior of the gradient exclusion process and the nongradient exclusion process are also obtained. For a more comprehensive view, we recommend[2] Section 7.

In[1], Faggionato A. and Martinelli F. study the hydrodynamic behavior of a simple exclusion process under the influence of an external random field. The disorder field α is given by independent identically distributed random variables with $|\alpha_x| \leq B, \forall x \in \mathbb{Z}^d$. Let us describe the dynamics. Given a disorder configuration α and a subset Λ of $\mathbb{Z}^d$, they define the grand canonical Gibbs measure $\mu_\Lambda^{\alpha,\lambda}$ on $\{0,1\}^\Lambda$ associated with the chemical potential $\lambda \in \mathbb{R}$ as the product measure

$$\mu_\Lambda^{\alpha,\lambda}(\eta) = \prod_{x\in\Lambda} \frac{e^{(\alpha_x+\lambda)\eta_x}}{1 + e^{\alpha_x+\lambda}}$$

and the corresponding canonical measure $\nu_{\Lambda,\rho}^\alpha$ with density ρ as

$$\nu_{\Lambda,\rho}^\alpha(\cdot) = \mu_\Lambda^{\alpha,\lambda}(\cdot|m_\Lambda = \rho),$$

where the particle density $m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \eta_x$.

The positive and bounded transition rate $c_{x,y}^\alpha(\eta)$ depending on the disorder configuration (α_x, α_y) is not only translation invariant, i.e.

$$c_{x,y}^{\tau_z\alpha}(\tau_z\eta) = c_{x+z,y+z}^\alpha(\eta), \forall z \in \mathbb{Z}^d,$$

where $\tau_z\alpha, \tau_z\eta$ is the disorder and particle configuration translated by z, but also satisfies the identity $c^\alpha_{x,y}(\eta) = c^\alpha_{y,x}(\eta)$. Moreover, it satisfies the detailed balance condition with respect to the Gibbs measure $\mu^{\alpha,\lambda}_\Lambda$, i.e.

$$c^\alpha_{x,y}(\eta) = c^\alpha_{x,y}(\eta^{x,y})e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}.$$

Then, the lattice gas with Kawasaki dynamics is a continuous time Markov chain determined by the diffusively rescaled Markov generator $N^2\mathcal{L}^\alpha_N$, where $\mathcal{L}^\alpha_N := \mathcal{L}^\alpha_{\mathbf{T}^d_N}$ and for all $\Lambda \subset \mathbb{Z}^d$,

$$\mathcal{L}^\alpha_\Lambda f(\eta) = \sum_{e\in\mathcal{E}}\sum_{x\in\Lambda} c^\alpha_{x,x+e}(\eta)[f(\eta^{x,x+e}) - f(\eta)].$$

Since the present system does not satisfy the gradient condition, except the trivial case when the disorder field is constant, they apply the classical approach, i.e. looking for a generalized Fick's law to establish the hydrodynamic limit of a disordered nongradient system but with a slight modification. Namely, they are interested in the limit

$$\lim_{l\uparrow\infty}(2l)^{-d}\mathbb{E}[\mu^{\alpha,\lambda_0(\rho)} \Big(\sum_{|x|\le l-\sqrt{l}} \tau_x f, (-\mathcal{L}^\alpha_{\Lambda_l})^{-1}\sum_{|x|\le l-\sqrt{l}} \tau_x g\Big)],$$

where $\mathbb{E}$ stands for the corresponding expectation with respect to the product measure defined on the disorder configuration space and $\lambda_0(\rho)$ is the annealed chemical potential such that $\mathbb{E}[\mu^{\alpha,\lambda_0(\rho)}(\eta_0)] = \rho$.

By the theory of closed and exact forms generalized from the ones for the nondisordered and nongradient system, it can be proved that the above limit exists and defines the semi-inner product $V_\rho(f,g)$. On the other hand, the subspace $\{\sum_{e\in\mathcal{E}} a_e j^\alpha_{0,e} + \mathcal{L}^\alpha_{\mathbb{Z}^d} g : a \in \mathbb{R}^d, g \in \mathbb{G}\}$ is dense in $\mathcal{G}$ endowed with the semi-inner product V_ρ, c.f.[1] Section 7.1, where the instantaneous current $j^\alpha_{x,y}(\eta) = c^\alpha_{x,y}(\eta)(\eta_x - \eta_y)$ defined as the difference between the rate at which a particle jumps from x to y and the rate at which a particle jumps from y to x, $\mathbb{G}$ stands for the space of local and bounded functions on $[-B,B]^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$, and $\mathcal{G}$ is the space of mean-zero functions $g \in \mathbb{G}$ with respect to all canonical measures on some cube. Nevertheless, $\eta_e - \eta_0 \notin \mathcal{G}$ because of the disorder field, hence $\{\eta_e - \eta_0\}_{e\in\mathcal{E}}$ no longer forms a basis of $\mathcal{L}^\alpha\mathbb{G}^\perp$. Therefore, they try to consider the difference

$$\eta_e - \eta_0 - \nu^\alpha_{\Lambda_n,m_n(\eta)}(\eta_e - \eta_0).$$

Similar to the one considered in[4], it can be proved that for each $e \in \mathcal{E}$, the sequence of these differences is Cauchy in $\overline{\mathcal{G}}$ with the semi-norm $V^{1/2}_\rho$ and that the limit points of these sequences with e varying in $\mathcal{E}$ form a basis of $\mathcal{L}^\alpha\mathbb{G}^\perp$. Then the current $j^\alpha_{x,x+e}$ can be written as the sum of some negligible fluctuation $\tau_x\mathcal{L}^\alpha g$ and

$$-\sum_{e'\in\mathcal{E}} D_{e,e'}(\rho)\tau_x(\eta_{e'} - \eta_0 - \nu^\alpha_{\Lambda_n,m_n(\eta)}(\eta_{e'} - \eta_0))$$

for some $d \times d$ matrix $D(\rho)$.

In order to get a generalized Fick's law, it remains to show the negligibility of the term $\tau_x(\nu^\alpha_{\Lambda_n,m_n(\eta)}(\eta_e - \eta_0)), \forall x \in \mathbf{T}^d_N$. Unfortunately, in the presence of the disorder field, $||\nu^\alpha_{\Lambda_n,m_n(\eta)}(\eta_e - \eta_0)||_\infty = O(1), \forall n$.Therefore, Faggionato and Martinelli considered the gradient of the density in two large adjacent cubes in the hope that the fluctuations are small. This problem has already solved by Faggionato and Martinelli as long as $d \ge 3$ in[1].

They finally arrive at the main conclusion on the hydrodynamic limit of a system with the bounded disorder. Namely, in dimension larger than 2, for almost every disorder configuration, by rescaling space and time diffusively, the hydrodynamic equation is the nonlinear parabolic equation

$$\partial_t \rho(t,\theta) = \nabla(D(\rho(t,\theta))\nabla\rho(t,\theta)),$$

$$\rho(0,\theta) = \rho_0(\theta),$$

where the deterministic diffusion matrix D can be described by the following variational characterization, for any $a \in \mathbb{R}^d$,

$$(a, D(\rho)a) = \frac{1}{2\chi(\rho)} \inf_{g\in\mathbb{G}} \sum_{e\in\mathcal{E}} \mathbb{E}\Big[\mu^{\alpha,\lambda_0(\rho)} \Big(c^{\alpha}_{0,e}\Big(a_e(\eta_e - \eta_0) + \nabla_{0,e}\Big(\sum_{x\in\mathbb{Z}^d} \tau_x g\Big)\Big)^2\Big)\Big]$$

where $\chi(\rho) = \mathbb{E}[\mu^{\alpha,\lambda_0(\rho)}(\eta_0) - \mu^{\alpha,\lambda_0(\rho)}(\eta_0)^2]$ and $\nabla_{x,y} g(\alpha,\eta) = g(\alpha,\eta^{x,y}) - g(\alpha,\eta)$. Furthermore, Faggionato and Martinelli also prove that $D(\cdot)$ is positive, bounded and continuous in $(0,1)$. That hydrodynamic equation is obtained under the assumption that $D(\rho)$ has a continuous extension in the closed interval $[0,1]$. A few years later on, in[3], Quastel proved that this diffusion matrix D is actually continuous in the closed interval $[0,1]$.

The above conclusion on the hydrodynamic behavior of a disordered system has been obtained as long as the disorder field is bounded. A natural question posed here is that whether that conclusion holds true if the random field is unbounded. In this paper, we will indicate that a similar conclusion also holds true if the disorder field is unbounded and satisfies a certain condition on its distribution. Let us consider the disorder field given by independent identically distributed random variables satisfying the technical condition that there exists some $u > 0$ such that for all $x \in \mathbb{Z}^d, \mathbb{E}[e^{u|\alpha_x|}] < +\infty$. Then, under the assumption that the diffusion matrix D has a continuous extension in the closed interval $[0,1]$, in dimension larger than 2, for almost every disorder configuration, by rescaling space and time diffusively, the macroscopic evolution of the system with this new disorder field is described by the nonlinear parabolic equation as above.

More precisely, let us present the main result as follows.

Theorem 1.1. *Let $d \geq 3, T > 0$. Let the disorder field be given by iid random variables satisfying the technical condition*

$$\exists u > 0 \text{ such that } \forall x \in \mathbb{Z}^d, \mathbb{E}[e^{u|\alpha_x|}] < +\infty. \tag{2}$$

Assume that the diffusion matrix D defined as in[1] Theorem 2.1 for $\rho \in (0,1)$ has a continuous extension in the closed interval $[0,1]$. Consider a sequence of probability measures $\{\mu^N\}_{N\geq 1}$ on Ω_N associated to the macroscopic profile $\rho_0(\theta)d\theta \in \mathcal{M}_2$. Then for almost every disorder configuration α, any $t \in [0,T]$, the sequence of probability measures $\{\mathbb{P}^{\alpha,\mu^N}_t\}_{N\geq 1}$ is associated to the macroscopic profile $\rho(t,\theta)d\theta \in \mathcal{M}_2$ whose density is the unique weak solution $\rho \in C([0,T],\mathcal{M}_2)$ of the Cauchy problem

$$\begin{cases} \partial_t\rho(t,\theta) &= \nabla(D(\rho(t,\theta))\nabla\rho(t,\theta)) \\ \rho(0,\theta) &= \rho_0(\theta). \end{cases} \tag{3}$$

and satisfying the energy estimate

$$\int_0^T \int_{\mathbf{T}^d} |\nabla\rho(t,\theta)|^2 d\theta dt < \infty.$$

This result follows from repeating the proof for the case of bounded random field and replacing the arguments where the boundedness is used by the different ones.

For this purpose, it is enough to deal with all the estimates involved in the boundedness of the random field we have considered above. Namely, they are the estimate on the entropy $H(\mathbb{P}^{\mu^N}|\mathbb{P}^{\mu_N})$, the equilibrium bounds and the results applying them as well as the description of $(a, D(0)a), (a, D(1)a)$ as the limit of $(a, D(\rho)a)$ when $\rho \downarrow 0$ and $\rho \uparrow 1$, respectively. We will carry them out with more details in the following three sections.

2. ESTIMATE ON THE ENTROPY

As mentioned in[1] Section 4, if $|\alpha_x| \leq B, \forall x \in \mathbb{Z}^d$ then there exists some constant $C(B) > 0$ such that for every disorder configuration α and any $N \geq 1$,

$$\mu_N(\eta) = \prod_{x \in T_N^d} \frac{e^{\alpha_x \eta_x}}{1 + e^{\alpha_x}} \geqslant e^{-C(B)N^d}, \ \forall \eta \in \Omega_N$$

and the estimate on the entropy $H(\mathbb{P}^{\mu^N}|\mathbb{P}^{\mu_N})$ will thus follow, i.e. $\exists C > 0$ such that $H(\mathbb{P}^{\mu^N}|\mathbb{P}^{\mu_N}) \leq CN^d$.

Let us now show that for the present model, i.e. the system with unbounded random field satisfying the technical condition (2), we also obtain the above bound on $\mu_N(\eta)$ for almost any disorder configuration and any N large enough.

Lemma 2.1. *There exists a constant $C > 0$ such that for almost any disorder configuration α and any N large enough,*

$$\mu_N(\eta) = \prod_{x \in T_N^d} \frac{e^{\alpha_x \eta_x}}{1 + e^{\alpha_x}} \geq e^{-CN^d}, \quad \forall \eta \in \Omega_N. \tag{4}$$

Proof. By the technical condition (2), there exists $u_0 > 0$ such that $\mathbb{E}[e^{u_0|\alpha_x|}] \leq C'$. We set

$$\mathcal{Q}_N := \left\{ \alpha : \prod_{x \in T_N^d} \frac{e^{-|\alpha_x|}}{1 + e^{\alpha_x}} \geq e^{-CN^d} \right\}$$

where $C := K + \ln 2$ for some $K > \dfrac{2 \ln C'}{u_0}$. Then, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{Q}_N) &\leq \mathbb{P}\left(\prod_{x \in T_N^d} \frac{1}{2e^{2|\alpha_x|}} \geq e^{-CN^d} \right) \\
&\leq \mathbb{P}\left(\prod_{x \in T_N^d} e^{2|\alpha_x|} \geq e^{KN^d} \right) \\
&\leq \inf_{u \geq 0} e^{-\frac{u}{2}KN^d} \mathbb{E}\left[\left(\prod_{x \in T_N^d} e^{2|\alpha_x|} \right)^{\frac{u}{2}} \right] \\
&= \inf_{u \geq 0} \left(e^{-\frac{u}{2}K} \mathbb{E}[e^{u|\alpha_x|}] \right)^{N^d} \\
&\leq \left(e^{-\frac{u_0}{2}K} \mathbb{E}[e^{u_0|\alpha_x|}] \right)^{N^d} \\
&\leq e^{(\ln C' - \frac{u_0}{2}K)N^d}.
\end{aligned}$$

Hence,

$$\sum_{N \geq 1} \mathbb{P}(\mathcal{Q}_N) \leq \sum_{N \geq 1} e^{(\ln C' - \frac{u_0}{2}K)N^d} < +\infty.$$

Then by Borel-Cantelli lemma, for almost every disorder configuration α, for all $N \geq N_0(\alpha), \alpha \notin \mathcal{Q}_N$. Moreover, due to the fact that $-|\alpha_x| \leq \alpha_x \eta_x, \forall \eta \in \Omega_N$, the assertion of the lemma follows. $\square$

3. GENERALIZED EQUILIBRIUM BOUNDS

In the proof of[1] Lemma A.2, all the estimates originate from bounding $\mu^\lambda(\eta_x)$ at a single point x, i.e.

$$\frac{e^{-2B}e^\lambda}{1+e^\lambda} \le \mu^\lambda(\eta_x) = \frac{e^{\alpha_x+\lambda}}{1+e^{\alpha_x+\lambda}} \le \frac{e^{2B}e^\lambda}{1+e^\lambda} \tag{5}$$

if $|\alpha_x| \le B, \forall x \in \mathbb{Z}^d$. It thus implies the estimate

$$C^{-1}\mu^\lambda(m_\Lambda) \le \mu^\lambda(m_\Delta) \le C\mu^\lambda(m_\Lambda) \tag{6}$$

for all $\Delta \subset \Lambda$ and every disorder configuration.

Now, if the disorder field is unbounded then we no longer obtain an estimate as (5). However, the bound (6) still holds true but for almost all disorder configurations and all finite subsets $\Lambda \subset \mathbb{Z}^d$ large enough by applying the Large deviation estimate ([1] Lemma A.1). More precisely,[1] Lemma A.2 is substituted by a more generalized one as follows.

Lemma 3.1. *Given a nonempty finite subset $\Lambda \subset \mathbb{Z}^d$ of cardinality L and $\lambda \in \mathbb{R}$. We set $A := \mathbb{E}\left(\frac{e^{\alpha_0+\lambda}}{1+e^{\alpha_0+\lambda}}\right), \rho := \mu^\lambda(m_\Lambda)$ and $a_\rho := \min(\rho, 1-\rho)$. Then, there exists a constant $C > 0$ such that for almost any disorder configuration α, any L large enough, any subset $\Delta \subset \Lambda$ satisfying $KL \le |\Delta| \le L$, where $\frac{16\ln 2}{(C-2)^2A^2} < K < \frac{1}{2}$,*

a) $C^{-1}|\Delta|\rho \le \mu^\lambda(N_\Delta) \le C|\Delta|\rho$,

b) $C^{-1}|\Delta|(1-\rho) \le \mu^\lambda(|\Delta| - N_\Delta) \le C|\Delta|(1-\rho)$,

c) $C^{-1}|\Delta|a_\rho \le \mu^\lambda(N_\Delta; N_\Delta) \le C|\Delta|a_\rho$,

d) $|\mu^\lambda(f; N_{\Delta_f})| \le C\|f\|_\infty \min(|\Delta_f|a_{\overline{\rho}}, \sqrt{|\Delta_f|a_{\overline{\rho}}})$,

for any function f with Δ_f large enough, where $\overline{\rho} := \mu^\lambda(m_{\Delta_f})$.

Proof. Due to[1] Lemma A.1 applied to $f(\alpha) = \frac{e^{\alpha_0+\lambda}}{1+e^{\alpha_0+\lambda}} - A$, for any $\delta > 0$ and any nonempty finite subset $\Lambda \subset \mathbb{Z}^d$, we obtain

$$\mathbb{P}(|\mu^\lambda(m_\Lambda) - A| \ge \delta) \le 2e^{-\frac{1}{4}\delta^2|\Lambda|}. \tag{7}$$

We set

$$\mathcal{S} := \{\Delta : \Delta \subset \Lambda \text{ with } KL \le |\Delta| \le L\},$$
$$\mathcal{Q}_L := \left\{\exists \Delta \in \mathcal{S} : \mu^\lambda(m_\Delta) > C\mu^\lambda(m_\Lambda)\right\},$$

for a constant $C > 2 + \frac{4\sqrt{2\ln 2}}{A}$ and any $\frac{16\ln 2}{(C-2)^2A^2} < K < \frac{1}{2}$. Therefore, by (7), we get

$$\begin{aligned}
\sum_{L\ge 0}\mathbb{P}(\mathcal{Q}_L) &\le \sum_{L\ge 0}\sum_{\Delta\in\mathcal{S}}\mathbb{P}(\mu^\lambda(m_\Delta) > C\mu^\lambda(m_\Lambda)) \\
&\le \sum_{L\ge 0} 2^L[\mathbb{P}(\mu^\lambda(m_\Delta) \ge \frac{C}{2}A) + \mathbb{P}(\mu^\lambda(m_\Lambda) \le \frac{A}{2})] \\
&\le \sum_{L\ge 0} 2^L[\mathbb{P}(\mu^\lambda(m_\Delta) - A \ge (\frac{C}{2} - 1)A) \\
&\qquad + \mathbb{P}(\mu^\lambda(m_\Lambda) - A \le -\frac{A}{2})] \\
&\le \sum_{L\ge 0} 2^L(e^{-\frac{1}{16}(C-a)^2A^2KL} + e^{-\frac{1}{16}A^2L}) < +\infty.
\end{aligned}$$

Then, by Borel-Cantelli lemma, for almost all disorder configurations α, there exists $L_0(\alpha)$ such that for all $L \ge L_0(\alpha), \alpha \notin \mathcal{Q}_L$. It implies the upper bound of a). For the lower bound of a), we do the same argument. Then, the estimate b) easily follows from a).

Let us now verify the estimate c). We obtain the upper bound from using the fact that $\mu^\lambda(N_\Delta; N_\Delta) \le \mu^\lambda(N_\Delta)$ and applying a). For the lower bound, we assume $\rho \in [0, \frac{1}{2}]$ and consider the set $W := \left\{x \in \Lambda : \mu^\lambda(\eta_x) \le \frac{1}{2}\right\}$.

⋆ If $|W| \geq \frac{1}{2}|\Lambda|$ then $W \in \mathcal{S}$. Hence, we have

$$\begin{aligned}\mu^\lambda(N_\Lambda; N_\Lambda) &= \sum_{x\in\Lambda} \mu^\lambda(\eta_x;\eta_x) \geq \sum_{x\in W} \mu^\lambda(\eta_x;\eta_x)\\ &= \sum_{x\in W} \mu^\lambda(\eta_x)(1-\mu^\lambda(\eta_x))\\ &\geq \frac{1}{2}\sum_{x\in W}\mu^\lambda(\eta_x) = \frac{1}{2}\mu^\lambda(N_W)\\ &\geq \frac{1}{2}C^{-1}|W|\rho \geq C^{-1}|\Lambda|\rho.\end{aligned}$$

⋆ If $|W| < \frac{1}{2}|\Lambda|$ then $|W^c| \geq \frac{1}{2}|\Lambda|$, i.e. $W^c \in \mathcal{S}$. Hence, we have

$$\begin{aligned}\mu^\lambda(N_\Lambda; N_\Lambda) &= \sum_{x\in\Lambda} \mu^\lambda(\eta_x;\eta_x) \geq \sum_{x\in W^c} \mu^\lambda(\eta_x;\eta_x)\\ &= \sum_{x\in W^c} \mu^\lambda(\eta_x)(1-\mu^\lambda(\eta_x))\\ &\geq \frac{1}{2}\sum_{x\in W^c}\mu^\lambda(1-\eta_x)\\ &= \frac{1}{2}\mu^\lambda(|W^c| - N_{W^c})\\ &\geq \frac{1}{2}C^{-1}|W^c|(1-\rho) \geq C^{-1}|\Lambda|\rho.\end{aligned}$$

From two cases above, we get

$$\mu^\lambda(N_\Lambda; N_\Lambda) \geq C^{-1}|\Lambda|a_\rho. \tag{8}$$

Now, given $\Delta \in \mathcal{S}$, applying (8) gives us $\mu^\lambda(N_\Delta; N_\Delta) \geq C^{-1}|\Delta|\min(\rho', 1-\rho')$, where $\rho' := \mu^\lambda(m_\Delta)$. This estimate together with a), b) then imply the lower bound of c).

The estimate d) then follows from a) and c) similar to the one as presented in[1] Lemma A.2. □

Lemma 3.2. *There exists a constant $C > 0$ such that for almost any disorder configuration α, any function f with support Δ_f large enough and any subset $\Lambda \subset \mathbb{Z}^d$ large enough, for all $\lambda, \lambda' \in \mathbb{R}$,*

$$\begin{aligned}&|\mu^{\lambda'}(f) - \mu^\lambda(f)|\\ &\quad\leq C\|f\|_\infty |\Delta_f||\mu^{\lambda'}(m_{\Delta_f}) - \mu^\lambda(m_{\Delta_f})|,\end{aligned} \tag{9}$$

$$\begin{aligned}&|\mu^{\lambda'}(m_\Lambda; N_\Lambda) - \mu^\lambda(m_\Lambda; N_\Lambda)|\\ &\quad\leq \frac{2|\Lambda'|}{|\Lambda|}|\mu^{\lambda'}(m_{\Lambda'}) - \mu^\lambda(m_{\Lambda'})|, \forall \Lambda' \subset \Lambda.\end{aligned} \tag{10}$$

For any $\rho, \rho' \in (0,1)$,

$$|\mu^{\lambda_0(\rho')}(\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0)| \leq C|\rho' - \rho|, \tag{11}$$

$$|\mu^{\lambda_0(\rho')}(\eta_0;\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0;\eta_0)| \leq C|\rho' - \rho|, \tag{12}$$

$$|\lambda_\Lambda(\rho) - \lambda_0(\rho)| \leq \frac{C}{\rho(1-\rho)}|\rho - \mu^{\lambda_0(\rho)}(m_\Lambda)|. \tag{13}$$

Proof. Let us first prove (9). By setting $\rho := \mu^\lambda(m_{\Delta_f}), \rho' := \mu^{\lambda'}(m_{\Delta_f})$, we can write

$$\begin{aligned}|\mu^{\lambda'}(f) - \mu^\lambda(f)| &= \Big|\int_\rho^{\rho'} \frac{d}{ds}\mu^{\lambda_{\Delta_f}(s)}(f)ds\Big|\\ &\leq \int_\rho^{\rho'} |\mu^{\lambda_{\Delta_f}(s)}(f; N_{\Delta_f})\lambda'_{\Delta_f}(s)|ds.\end{aligned} \tag{14}$$

By Lemma 3.1, for any function f with support large enough,

$$\begin{aligned}&|\mu^{\lambda_{\Delta_f}(s)}(f; N_{\Delta_f})|\\ &\leq C\|f\|_\infty|\Delta_f|\\ &\quad\times \min(\mu^{\lambda_{\Delta_f}(s)}(m_{\Delta_f}), 1 - \mu^{\lambda_{\Delta_f}(s)}(m_{\Delta_f}))\\ &= C\|f\|_\infty|\Delta_f|\min(s, 1-s).\end{aligned}$$

Moreover,

$$\begin{aligned}\lambda'_{\Delta_f}(s) &= \frac{1}{\mu^{\lambda_{\Delta_f}(s)}(m_{\Delta_f}; N_{\Delta_f})}\\ &\leq \frac{C}{\min(\mu^{\lambda_{\Delta_f}(s)}(m_{\Delta_f}), 1 - \mu^{\lambda_{\Delta_f}(s)}(m_{\Delta_f}))}\\ &= \frac{C}{\min(s, 1-s)}.\end{aligned}$$

Therefore, for any function f with support large enough,

$$|\mu^{\lambda_{\Delta_f}(s)}(f;N_{\Delta_f})\lambda'_{\Delta_f}(s)| \le C\|f\|_\infty|\Delta_f|.$$

Hence, we obtain (9).

Now we verify (10). Let us observe that

$$\begin{aligned}
&|\mu(N_\Lambda;N_\Lambda;N_\Lambda)| \le 2\sum_{x\in\Lambda}|\mu(\eta_x;\eta_x;\eta_x)| \\
&\le 2\sum_{x\in\Lambda}\mu(\eta_x)(1-\mu(\eta_x))|2\mu(\eta_x)-1| \\
&\le 2\sum_{x\in\Lambda}\mu(\eta_x)(1-\mu(\eta_x)) \\
&= 2\sum_{x\in\Lambda}\mu(\eta_x;\eta_x) = 2\mu(N_\Lambda;N_\Lambda). \qquad (15)
\end{aligned}$$

We set $\tilde\rho := \mu^\lambda(m_{\Lambda'}), \tilde\rho' := \mu^{\lambda'}(m_{\Lambda'})$. By this setting, we have $\lambda_{\Lambda'}(\tilde\rho) = \lambda, \lambda_{\Lambda'}(\tilde\rho') = \lambda'$ and we can write

$$\begin{aligned}
&|\mu^{\lambda'}(m_\Lambda;N_\Lambda) - \mu^\lambda(m_\Lambda;N_\Lambda)| \\
&= \Big|\int_{\tilde\rho}^{\tilde\rho'}\frac{d}{ds}\mu^{\lambda_{\Lambda'}(s)}(m_\Lambda;N_\Lambda)ds\Big| \\
&\le \int_{\tilde\rho}^{\tilde\rho'}|\mu^{\lambda_{\Lambda'}(s)}(m_\Lambda;N_\Lambda;N_\Lambda)\lambda_{\Lambda'}(s)|ds \\
&\le \int_{\tilde\rho}^{\tilde\rho'}\frac{2}{|\Lambda|}\mu^{\lambda_{\Lambda'}(s)}(N_\Lambda;N_\Lambda)\lambda_{\Lambda'}(s)ds \\
&\le \int_{\tilde\rho}^{\tilde\rho'}\frac{2}{|\Lambda|}\frac{\mu^{\lambda_{\Lambda'}(s)}(N_{\Lambda'};N_{\Lambda'})}{\mu^{\lambda_{\Lambda'}(s)}(m_{\Lambda'};m_{\Lambda'})}ds \\
&\le \frac{2|\Lambda'|}{|\Lambda|}|\tilde\rho' - \tilde\rho|.
\end{aligned}$$

and we arrive at (10).

Let us next consider the estimate (11). With no restriction, we assume that $0 < \rho < \rho' < 1$. Since $\lim_{\tilde B\uparrow\infty}\mathbb{P}(|\alpha_0| \le \tilde B) = 1$, there exists some constant $\tilde B > 0$ such that $\mathbb{P}(|\alpha_0| \le \tilde B) \ge \frac{1}{2}$ and we have

$$\begin{aligned}
\chi(\rho) &= \mathbb{E}\left[\mu^{\lambda_0(\rho)}(\eta_0;\eta_0)\right] \\
&= \mathbb{E}\left[\frac{e^{\alpha_0+\lambda_0(\rho)}}{\left(1+e^{\alpha_0+\lambda_0(\rho)}\right)^2}\right] \\
&= \mathbb{E}\left[\mathbb{I}_{\{|\alpha_0|\leqslant\tilde B\}}\frac{e^{\alpha_0+\lambda_0(\rho)}}{\left(1+e^{\alpha_0+\lambda_0(\rho)}\right)^2}\right] \\
&\quad + \mathbb{E}\left[\mathbb{I}_{\{|\alpha_0|>\tilde B\}}\frac{e^{\alpha_0+\lambda_0(\rho)}}{\left(1+e^{\alpha_0+\lambda_0(\rho)}\right)^2}\right] \\
&\ge \mathbb{P}(|\alpha_0| \le \widetilde B)\min\left(h(-\widetilde B), h(\widetilde B)\right),
\end{aligned}$$

where $h(\beta) = \frac{e^{\beta+\lambda_0(\rho)}}{\left(1+e^{\beta+\lambda_0(\rho)}\right)^2}$. Therefore,

$$\begin{aligned}
&|\mu^{\lambda_0(\rho')}(\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0)| \\
&=\Big|\int_\rho^{\rho'}\frac{d}{ds}\mu^{\lambda_0(s)}(\eta_0)ds\Big| \le \int_\rho^{\rho'}\frac{\mu^{\lambda_0(s)}(\eta_0)}{\chi(s)}ds \\
&\le C(\rho'-\rho),
\end{aligned}$$

and this implies (11).

On the other hand, (12) is a simple consequence of (11). In fact, we have

$$\begin{aligned}
&|\mu^{\lambda_0(\rho')}(\eta_0;\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0;\eta_0)| \\
&\le |\mu^{\lambda_0(\rho')}(\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0)| \\
&\qquad + |\mu^{\lambda_0(\rho')}(\eta_0)^2 - \mu^{\lambda_0(\rho)}(\eta_0)^2| \\
&\le C|\mu^{\lambda_0(\rho')}(\eta_0) - \mu^{\lambda_0(\rho)}(\eta_0)| \\
&\le C|\rho'-\rho|.
\end{aligned}$$

□

4. ON THE CONTINUITY OF THE DIFFUSION MATRIX AT THE END POINTS 0 AND 1

As we can see in the proof of[3] Theorem 5, Section 11, the assumption on the boundedness of the disorder field is essential to describe $(a, D(0)a)$ and $(a, D(1)a)$ as the limit of $(a, D(\rho)a)$ when $\rho \downarrow 0$ and $\rho \uparrow 0$, respectively. More concretely, let us go through that proof. By the boundedness of the random field, there exists some constant $C_3 > 0$ such that

$$p_x := \frac{e^{\alpha_x+\lambda_0(\rho)}}{1+e^{\alpha_x+\lambda_0(\rho)}} \geq C_3\rho. \qquad (16)$$

This assertion helps the author to verify the following inequality

$$\Psi(\rho) \geq C_1\rho^2\mathbb{E}[\widehat{g}^2_{0,e}].$$

Unfortunately, when the random field is unbounded, that assertion does not hold true. Indeed, if there were some constant $C_3 > 0$ such that we have the estimate (16), then by taking the limit $\alpha_x \downarrow -\infty$ in both sides of (16), we would get a wrong bound $C_3 \leq 0$.

Moreover, as we keep looking at that proof, we can see that the assumption on the boundedness of the disorder field is also used to prove the equality (11.35) as follows

$$\lim_{\rho\downarrow 0} \lambda_0'(\rho) \inf_{U(\alpha)} \mathbb{E}\Big[\sum_{e\in\mathcal{E}}(\mu^{\lambda_0(\rho)}(\eta_e-\eta_0)^2 + C\rho^2) \times (\beta_e+\tau_{-e}U-U)^2\Big]$$
$$= \frac{\inf_{U(\alpha)}\mathbb{E}\Big[\sum_{e\in\mathcal{E}}(e^{\alpha_0}+e^{\alpha_e})(\beta_e+\tau_{-e}U-U)^2\Big]}{\mathbb{E}[e^{\alpha_0}]}.$$

Namely, one has

$$\lambda_0'(\rho) = \frac{1}{\chi(\rho)} = \frac{1}{\mathbb{E}\left[\dfrac{e^{\alpha_0+\lambda_0(\rho)}}{(1+e^{\alpha_0+\lambda_0(\rho)})^2}\right]},$$

and

$$\mu^{\lambda_0(\rho)}(\eta_e-\eta_0)^2 = \frac{e^{\lambda_0(\rho)}(e^{\alpha_0}+e^{\alpha_e})}{(1+e^{\alpha_0+\lambda_0(\rho)})(1+e^{\alpha_e+\lambda_0(\rho)})}.$$

If $-B \leq \alpha_x \leq B$ then

$$\frac{e^{\lambda_0(\rho)}(e^{\alpha_0}+e^{\alpha_e})}{(1+e^{B+\lambda_0(\rho)})^2} \leq \mu^{\lambda_0(\rho)}(\eta_e-\eta_0)^2 \leq \frac{e^{\lambda_0(\rho)}(e^{\alpha_0}+e^{\alpha_e})}{(1+e^{-B+\lambda_0(\rho)})^2}.$$

Hence, the equality (11.35) follows from checking that as $\rho \to 0, \lambda_0(\rho) \sim \ln\rho - \ln z$ and $\lambda_0'(\rho) \sim \rho$.

We now claim that it is not necessary to require the boundedness of the disorder field in order to obtain the equality (11.35). In fact, due to the Monotone Convergence Theorem, we have the following two estimates

$$\lim_{\lambda\downarrow-\infty} \frac{1}{\mathbb{E}\left[\dfrac{e^{\alpha_0}}{(1+e^{\alpha_0+\lambda})^2}\right]} = \frac{1}{\mathbb{E}[e^{\alpha_0}]},$$

$$\lim_{\lambda\downarrow-\infty} \mathbb{E}\left[\frac{(e^{\alpha_0}+e^{\alpha_e})(\beta_e+\tau_{-e}U-U)^2}{(1+e^{\alpha_0+\lambda})(1+e^{\alpha_e+\lambda})}\right] = \mathbb{E}[(e^{\alpha_0}+e^{\alpha_e})(\beta_e+\tau_{-e}U-U)^2].$$

They immediately imply the equality (11.35).

Therefore, based on the previous observations, in order to get the hydrodynamic behavior of our system with unbounded disorder as the Cauchy problem (3), we just add another assumption that the diffusion matrix $D(\rho)$ has a continuous extension in the closed interval $[0, 1]$, i.e. we arrive at the assertion of Theorem 1.1.

ACKNOWLEDGMENTS

This study is conducted within the framework of science and technology projects at institutional level of Quy Nhon University under the project code T2022.746.02.

REFERENCES

1. Faggionato, A., Martinelli, F. Hydrodynamic limit of a disordered lattice gas, *Probability Theory Related Fields*, **2003**, *127*, 535–608.

2. Kipnis, C., Landim, C. Scaling limits of interacting particle systems, Grundlehren der mathematischen Wissenschaften 320, *Springer Verlag*, **1999**.

3. Quastel, J., Yau, H.T. Bulk diffusion in a system with site disorder, *The Annals of Probability*, **2006**, *34*(5), 1990–2036.

4. Varadhan, S.R.S., Yau, H.T. Diffusive limit of lattice gases with mixing conditions, *Asian J. Math.*, **1997**, *1*(4), 623–678.