Application of Intuitionistic Fuzzy Preference Relation and Intuitionistic Fuzzy Weighted Vector in Feature Ranking

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ABSTRACT

This paper proposes interval weight vector and applies it to feature ranking problem. This is an extension and follow-up to the paper[1] in FEDCIS 2022. We would like to compare two methods of using real weight vector and interval weight vector. Then, we indicate the advantages of the new method and give examples to illustrate the application of the methods.
Key words: *Intuitionistic fuzzy value, Intuitionistic fuzzy preference relation, Feature ranking, Intuitionistic fuzzy weighted vector, Interval weight vector.*

1. Introduction

To express the opinions of a decision maker (DM) about each pair of alternatives, fuzzy set theory (Zadeh, 1965)[2] or intuitionistic fuzzy set theory (Atanassov, 1986)[3] is recommended. With fuzzy set theory, decision makers can only express the satisfaction degree and the dissatisfaction degree. Intuitionistic fuzzy values reflect the decision maker's opinion more comprehensively, express not only the satisfaction degree and the dissatisfaction degree but also the hesitation degree of DM's opinion. An intuitionistic fuzzy preference relation (IFPR) is a matrix of which every element represents the opinions of the decision makers about each pair of alternatives. There are IFPRs satisfying the additive consistency or multiplicative consistency. With every IFPR, let's find out an interval Intuitionistic fuzzy (IF) weighted vector representing the rank of the DM's opinion to the alternatives.

In[1], the authors propose the RAFAR method to rank alternatives in multi-choices decision making problems. The main idea of RAFAR is building an IFPR matrix from datas, establishing a linear optimal model and ranking alternatives from this problem. The algorithm uses the decision making method in Intuitionistic Fuzzy Theory. The work[1] also supplies an experiment and compares with some other methods. The experimental results are very positive. However, it reveals cetain shortcomings.

In this paper, we mention about additive consistent IFPR and multiplicative consistent IFPR, propose features ranking methods based on interval IFV weighted vectors. We then give an example to illustrate and verify the results.

The paper is organized as follows: The second part is the basic notions, recalling basic definitions and results available of Fuzzy theory and Intuitionistic Fuzzy theory. Next, we present Intuitionistic Fuzzy Preference relation and ranking algorithms. In section four, the paper shows an example and applications of the interval weight vector method and indicates the advantage of this method compared to the RAFAR in[1].

2. Basic notions

2.1. Intuitionistic fuzzy sets and Intuitionistic fuzzy matrices

Fuzzy set (FS) theory which was introduced by Zadeh[2] in 1965 just considers the problems with the degree of membership and non-membership without mentioning the degree of hesitation of no decision-making. The Atanassov's intuitionistic fuzzy set (IFS) theory[3] considers fully expressing affirmation, negation and hesitation of decision-makers. Therefore, with real-life situations, IFS theory solves the problems more successfully than FS theory. In this part, some basic notions related to IFS are recalled.

Definition 1 (IFS[3])**.** Let $X = \{x_1, x_2, ..., x_m\}$ be a finite universal set. *An intuitionistic fuzzy set in* X is a set $A = \{(x_i, \mu_A(x_i), \nu_A(x_i)) : x_i \in X\}$, where $\mu_A(x_i), \nu_A(x_i) \in [0,1]$ and $0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1$ for any $x_i \in X$. The functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are called the *membership* and *non-membership* functions, respectively.

The value $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i)$ is called the intuitionistic index of the element x_i in the set A. It describes a degree of hesitation (or uncertainty) whether x_i is in A or not. For any $x_i \in X$, we have $0 \leq \pi_A(x_i) \leq 1$.

The class of IFS in a universe X is denoted by $\mathcal{IFS}(X)$.

Relations (between two sets X and Y) in traditional set theory are defined as subsets of the Cartesian product $X \times Y$. It is quite natural to define intuitionistic fuzzy relations as IFSs in $X \times Y$. If $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$, then any intuitionistic fuzzy relation in $X \times Y$ can be represented by an $m \times n$ matrix $R = (\rho_{ij})_{m\times n}$, where $\rho_{ij} = (\mu_{ij}, \nu_{ij}) \in \mathcal{F}$ is the IFV describing the membership and non-membership of (x_i, y_j) to this relation.

Definition 2 (Intuitionistic fuzzy matrices - IFM)**.** Any matrix P of order $m \times n$ with values from $\mathcal{F} = \{(a_1, a_2) \in [0,1]^2 : a_1 + a_2 \leq 1\}$ is called Intuitionistic Fuzzy Matrices. An IFM is said to be square intuitionistic fuzzy matrix (SIFM) if the number of rows is equal to the number of columns. Moreover:

1. An identity IFM $\mathbb{I}$ of order n is the square intuitionistic fuzzy matrix (SIFM) of order n with all diagonal entries $(1,0)$ and non-diagonal entries $(0,1)$.
2. A null intuitionistic fuzzy matrix (IFM) $\mathbb{O}$ of order n is the square intuitionistic fuzzy matrix (SIFM) of order n with all entries $(0,1)$.

The concepts of intuitionistic fuzzy relation and intuitionistic fuzzy matrix (IFM) have been studied by many authors[4,5,6]. IFM is a generalization of Fuzzy Matrix and has been useful in dealing with decision-making, clustering analysis, relational equations, etc.

2.2. Fuzzy Preference Relation

Let $X = \{x_1, \cdots, x_n\}, n \geq 3$ be a finite set of objects or alternatives. Decision makers (DMs) compare each pair of alternatives so as to express their opinions or preferences on such a set.

Recall that any set $R \subset X \times X$ is called *a relation* on X. The relation $R \subset X \times X$ is called *an (partial) order on* X if it is reflexive, antisymmetric and transitive. In addition, R is *a linear order* if for any $x, y \in X$, either $(x,y) \in R$ or $(y,x) \in R$. We can extend the concept of linear order into *fuzzy prefenerence relation* and *intuitionistic fuzzy preference relation* as follows:

Definition 3 (fuzzy preference relation - FPR)**.** A fuzzy preference relation on X is a fuzzy set on $X \times X$, which is characterized by a membership function $\mu_P : X \times X \to [0,1]$. If we denote $p_{ij} = \mu_P(x_i, x_j)$, then the fuzzy preference relation can be represented by the $n \times n$ matrix $P = (p_{ij})_{i,j=\overline{1;n}}$, satisfying the additive reciprocal conditions, i.e.:

$$p_{ij} + p_{ji} = 1 \text{ and } p_{ii} = 1/2.$$

for all $i, j = 1, \cdots, n$.

The value $p_{ij} = \mu_P(x_i, x_j) \in [0,1]$ is interpreted as the preference degree of x_i over x_j. If $p_{ij} = 1/2$, then we say that there is no difference between x_i and x_j, $p_{ij} = 1$ indicates that x_i is absolutely better than x_j (traditional preference relation), $p_{ij} > 1/2$ indicates that x_i is preferable to x_j. Moreover, the transitive property of a FPR can be expressed by either additive or multiplicative consistency:

Definition 4 (Additive and multiplicative consitent FPR (Tanino, 1984)[7])**.** Let $P = (p_{ij})$ be a fuzzy preference relation.

- P is called *additively consistent* if
$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2}$$
for all $i, j, k = 1, \cdots, n$.
- P is called *multiplicatively consistent* if
$$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ji} \cdot p_{ik} \cdot p_{kj}$$
for all $i, j, k = 1, \cdots, n$.

Notice that if a fuzzy preference relation $P = (p_{ij})$ is additively consistent and if both $p_{ij} > 1/2$ (x_i is preferable to x_j) and $p_{jk} > 1/2$ (x_j is preferable to x_k) then $p_{ki} < 1/2$, which implies that $p_{ik} > 1/2$ (x_i is preferable to x_k). This means the additively consistent FPRs are also transitive. This fact is also true for multiplicative consistency.

3. Intuitionistics Fuzzy Preference Relation and Ranking Algorithms

Usually, the intuitionistic fuzzy preference relation expresses the opinions of the decision makers about each pair of choices (alternatives), but we would like to convert this relation into a linear order (a ranking list).

Definition 5 (Intuitionistic Fuzzy Preference Relation - IFPR)**.** An intuitionistic fuzzy preference relation B on $X = \{x_1, \cdots, x_n\}$ is defined as a matrix $B = (b_{ij})_{n\times n}$, where $b_{ij} = (\mu_{ij}, \nu_{ij})$ for all $i, j = 1, 2, \cdots, n$ is an intuitionistic fuzzy value, composed by the certainty degree μ_{ij} to which x_i is preferred to x_j and the certainty degree ν_{ij} to which x_i is non-preferred to x_j, and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ is interpreted as the hesitation degree to which x_i is preferred to x_j. Moreover, μ_{ij} and ν_{ij} satisfy the following conditions:

$$\mu_{ij} + \nu_{ij} \leq 1, \quad \mu_{ij} = \nu_{ji}, \quad \mu_{ii} = \nu_{ii} = 0.5$$

for all $i, j = 1, 2, \cdots, n$.

For any IFPR $B = (b_{ij})_{n\times n}$, where $b_{ij} = (\mu_{ij}, \nu_{ij})$ for all $i, j = 1, 2, \cdots, n$, we can define two FPRs $L_B = (\lambda_{ij})_{n\times n}$ and $U_B = (\upsilon_{ij})_{n\times n}$ as follows:

$$\lambda_{ij} = \begin{cases} \mu_{ij} & \text{if } i < j \\ \frac{1}{2} & \text{if } i = j \\ 1 - \mu_{ji} & \text{if } i > j \end{cases} \quad \upsilon_{ij} = \begin{cases} 1 - \nu_{ij} & \text{if } i < j \\ \frac{1}{2} & \text{if } i = j \\ \nu_{ji} & \text{if } i > j \end{cases} \tag{1}$$

We call these matrices the *lower bound* and *upper bound* of B. Both L_B and U_B satisfy the conditions about FPR and the IFPR B can be interpreted as a collection of FPR R bounded by L_B and U_B, i.e.

$$L_B \leq B \leq U_B$$

where the relation $A \leq B$ between matrices $A = (a_{ij})$ and $B = (b_{ij})$ means $a_{ij} \leq b_{ij}$ for all possible indexes i and j.

3.1. Model for deriving weight vector

The first idea is to assign a weight w_i to the i-th alternative so that the higher weight means the more preferred choice. Without lost of generality, we can assume that the weight vector can be determined in form of *a probability vector*, i.e. a vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ such that $w_i \in [0, 1]$ for $i = 1, \cdots, n$ and $\sum_{i=1}^{n} w_i = 1$.

Lemma 6. *For any probability vector* $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$*,*

- *the matrix* $\mathbf{A}(\mathbf{w}) = (a_{ij})$*, where*

$$a_{ij} = \frac{w_i - w_j + 1}{2}$$

  is an additively consistent FPR

- *the matrix* $\mathbf{P}(\mathbf{w}) = (p_{ij})$*, where*

$$p_{ij} = \frac{w_i}{w_i + w_j}$$

  is a multiplicative consistent FPR

Definition 7 (Additive consistent preference relation)**.** Let B be an IFPR and L_B, U_B be the lower and upper bounds of B,

- B is called *the additive consistent IFPR* if there exists a probability vector $\mathbf{w}$ such that

$$L_B \leq \mathbf{A}(\mathbf{w}) \leq U_B$$

- B is called *the multiplicative consistent IFPR* if there exists a probability vector $\mathbf{w}$ such that

$$L_B \leq \mathbf{P}(\mathbf{w}) \leq U_B$$

Theorem 8. *A matrix* $B = ((\mu_{ij}, \nu_{ij}))_{n\times n}$ *is an additive consistent preference relation on* $X = \{x_1, \cdots, x_n\}$ *if and only if there exists a probability vector* $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ *satisfying the conditions:*

$$\mu_{ij} \leq 0.5(w_i - w_j + 1) \leq 1 - \nu_{ij} \tag{2}$$

for all $1 \leq i < j \leq n$*.*

Theorem 9. *A matrix* $B = ((\mu_{ij}, \nu_{ij}))_{n\times n}$ *is an multiplicative consistent preference relation on* $X = \{x_1, \cdots, x_n\}$ *if and only if there exists a probability vector* $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ *satisfying the conditions:*

$$\mu_{ij} \leq \frac{w_i}{w_i + w_j} \leq 1 - \nu_{ij} \text{ for all } 1 \leq i < j \leq n. \tag{3}$$

The concept of additive consistency and multiplicative consistency for the IFPR can be represented as follows:

In this case the condition in Eq. (2) can be relaxed by introducing the non-negative deviation variables l_{ij} and r_{ij} for $1 \leq i < j \leq n$ such that

$$\mu_{ij} - l_{ij} \leq 0.5(w_i - w_j + 1) \leq 1 - \nu_{ij} + r_{ij} \tag{4}$$

for all $1 \leq i < j \leq n$. As the deviation variables l_{ij} and r_{ij} become smaller, B becomes closer to an additive consistent intuitionistic fuzzy preference relation. Therefore, in order to find the smallest deviation variables the linear optimization model can be developed as follows[8],[9]:

Model (A1):

$$\delta = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (l_{ij} + r_{ij})$$

$$s.t. \begin{cases} \left.\begin{array}{l} 0.5(w_i - w_j + 1) + l_{ij} \geq \mu_{ij} \\ 0.5(w_i - w_j + 1) - r_{ij} \leq 1 - \nu_{ij} \\ l_{ij}, r_{ij} \geq 0 \end{array}\right\} \quad (*) \\ w_i \geq 0 \quad \text{for } i = 1, \cdots, n, \\ \sum_{i=1}^{n} w_i = 1 \end{cases}$$

where conditions (*) hold for all $1 \leq i < j \leq n$.

Checking for the multiplicative consistency is quite similar to the additive consistency. In this case, we can establish the optimization model (**M1**). In contrast to model (**A1**), this model is nonlinear.

Model (M1):

$$\delta = \min \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(l_{ij}+r_{ij})$$

$$s.t. \quad \begin{cases} \left.\begin{array}{l} \dfrac{w_i}{w_i+w_j} + l_{ij} \geq \mu_{ij} \\ \dfrac{w_i}{w_i+w_j} - r_{ij} \leq 1-\nu_{ij} \\ l_{ij}, r_{ij} \geq 0 \end{array}\right\} \quad (*) \\ w_i \geq 0 \quad \text{for } i=1,\cdots,n, \\ \sum_{i=1}^{n} w_i = 1 \end{cases}$$

where (*) hold for all $1 \leq i < j \leq n$.

3.2. Model for deriving interval weight vector

The drawback of the real weight vector method presented in the previous section is based on the fact that sometimes the optimization problems (models **A1** and **M1**) return similar or even exactly the same values to different weights w_i. In such case, the corresponding alternatives will be ranked in a random order. This is often the case when the optimization problems (**A1** and **M1**) have no unique solutions.

Another idea for ranking problem is based on fuzzification of the real weights $w_1, ..., w_n$ which are assigned to the alternatives $x_1, ..., x_n$. Thus, instead of probability vector, we are looking for *interval weight vector*:

$$\mathbf{w} = ([l_1, r_1], \cdots, [l_n, r_n])^T,$$

where $0 \leq l_i \leq r_i \leq 1$ for $i = 1, \cdots, n$. The interval $[l_i, r_i]$ can be interpreted as the set of all possible values that can be assigned to the alternative x_i for $i \in \{1, 2, \cdots, n\}$. It is easy to notice that in this case, the pair $(l_i, 1 - r_i)$ can be interpreted as an intuitionistic fuzzy value corresponding to the real weight w_i in the model described in Section 3.1.

In this section, we recall two methods for deriving interval weight vector for a given IFPR:

The first method[9] is a continuation of the models **A1** and **M1**. Let δ^o be the optimal value and let l_{ij}^o and r_{ij}^o for $1 \leq i < j \leq n$ be optimal deviation values of the optimization model (**A1**). One can see that if $\delta^o = 0$ then B is an additive consistent intuitionistic fuzzy preference relation. Otherwise, we can improve the additive consistency of B by defining the new intuitionistic fuzzy preference relation $\mathring{B} = ((\mathring{\mu}_{ij}, \mathring{\nu}_{ij}))_{n\times n}$, where

$$\mathring{\mu}_{ij} = \begin{cases} \mu_{ij} - l_{ij}^o & \text{if } i<j \\ 0.5 & \text{if } i=j \\ \mathring{\nu}_{ij} & \text{if } i>j \end{cases} \qquad \mathring{\nu}_{ij} = \begin{cases} \nu_{ij} - r_{ij}^o & \text{if } i<j \\ \nu_{ij} = 0.5 & \text{if } i=j \\ \mathring{\mu}_{ij} & \text{if } i>j \end{cases}$$

Based on matrix $\mathring{B}$ we can calculate the priority weight vector $\mathbf{w} = (w_1, \ldots, w_n)^T$ by establishing the weight intervals $[w_k^-, w_k^+]$ for each $k = 1, \cdots, n$. In order to do that, we solve the following optimization models:

Model (A2): for each $k = 1, 2, \cdots, n$:

$$(l_k, r_k) = (\min w_k, \max w_k)$$

$$s.t. \quad \begin{cases} \left.\begin{array}{l} 0.5(w_i - w_j + 1) \geq \mathring{\mu}_{ij} \\ 0.5(w_i - w_j + 1) \leq 1 - \mathring{\nu}_{ij} \end{array}\right\} \quad (*) \\ w_i \geq 0 \quad \text{for } i=1,\cdots,n, \\ \sum_{j=1}^{n} w_j = 1. \end{cases}$$

where (*) hold for all $1 \leq i < j \leq n$.

It has been shown[8] that if $\mathring{B}$ is additive consistent then Model (**A2**) will return a unique solution for the considered optimization problem.

Let δ^* be the optimal value and let l_{ij}^* and r_{ij}^* for $1 \leq i < j \leq n$ be optimal deviation values of the optimization model **(M1)**. One can see that if $\delta^* = 0$ then B is an multiplicative consistent intuitionistic fuzzy preference relation. Otherwise, we can improve the multiplicative consistency of B by defining the new intuitionistic fuzzy preference relation $B^* = ((\mu_{ij}^*, \nu_{ij}^*))_{n\times n}$, where

$$\mu_{ij}^* = \begin{cases} \mu_{ij} - l_{ij}^* & \text{if } i<j \\ 0.5 & \text{if } i=j \\ \nu_{ij}^* & \text{if } i>j \end{cases} \qquad \nu_{ij}^* = \begin{cases} \nu_{ij} - r_{ij}^* & \text{if } i<j \\ 0.5 & \text{if } i=j \\ \mu_{ij}^* & \text{if } i>j \end{cases}$$

Based on matrix B^* we can calculate the priority weight vector $\mathbf{w} = (w_1, \ldots, w_n)^T$ by establishing the weight intervals $[w_k^-, w_k^+]$ for each $k = 1, \cdots, n$. In order to do that, we solve the following optimization models:

Model (M2): for each $k = 1, 2, \cdots, n$:

$$(l_k, r_k) = (\min w_k, \max w_k)$$

$$s.t. \quad \begin{cases} \left.\begin{array}{l} \dfrac{w_i}{w_i+w_j} \geq \mu_{ij}^* \\ \dfrac{w_i}{w_i+w_j} \leq 1 - \nu_{ij}^* \end{array}\right\} \quad (*) \\ w_i \geq 0 \quad \text{for } i=1,\cdots,n, \\ \sum_{j=1}^{n} w_j = 1. \end{cases}$$

where (*) hold for all $1 \leq i < j \leq n$..

The second method is based on the modification of the concepts in Section 3.1. In order to transform from probability weight vector into the interval weight vector model, we should modify the normalization condition $\sum_{i=1}^{n} w_i = 1$ for probability vectors, the additive and multiplicative consistency conditions of IFPR and the corresponding optimization problems. In [10] and [11], these conditions were redefined as follows:

- **Normalization:** If an interval weight vector $\mathbf{w} = ([l_1, r_1], \cdots, [l_n, r_n])^T$ satisfies the conditions

$$\sum_{j=1, j\neq i}^{n} l_j + r_i \leq 1 \leq l_i + \sum_{j=1, j\neq i}^{n} r_j \quad (5)$$

  then it is called the *normalized interval weight vector.*

- **Additive consistency:** An IFPR B is *additive consistent* if there exists a normalized interval weight vector $\mathbf{w}$ such that

$$L_B = \mathbf{A}^-(\mathbf{w}) \text{ and } U_B = \mathbf{A}^+(\mathbf{w})$$

  where $\mathbf{A}^-(\mathbf{w}) = (a_{ij}^-)$ and $\mathbf{A}^+(\mathbf{w}) = (a_{ij}^+)$ are the additive consistent FPR defined by

$$(a_{ij}^-, a_{ij}^+) = \begin{cases} (0.5, 0.5) & \text{if } i = j \\ \left(\frac{1+l_i-r_j}{2}, \frac{1+r_i-l_j}{2}\right) & \text{if } i \neq j \end{cases}$$

- **Multiplicative consistency:** An IFPR B is *multiplicative consistent* if there exists a normalized interval weight vector $\mathbf{w}$ such that

$$L_B = \mathbf{P}^-(\mathbf{w}) \text{ and } U_B = \mathbf{P}^+(\mathbf{w})$$

  where $\mathbf{P}^-(\mathbf{w}) = (p_{ij}^-)$ and $\mathbf{P}^+(\mathbf{w}) = (p_{ij}^+)$ are the multiplicative consistent FPR defined by

$$(p_{ij}^-, p_{ij}^+) = \begin{cases} (0.5, 0.5) & \text{if } i = j \\ \left(\frac{l_i}{l_i+r_j}, \frac{r_j}{r_i+l_j}\right) & \text{if } i \neq j \end{cases}$$

  where L_B, U_B are the lower and upper bounds of B.

Similar to the case of probability weight vector, not every intuitionistic fuzzy preference relation is either additive or multiplicative consistent. The relaxation is based on the minimization of the differences $\|L_B - \mathbf{A}^-(\mathbf{w})\|_p$ and $\|U_B - \mathbf{A}^+(\mathbf{w})\|_p$, where

$$\|\mathbf{M}\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |m_{i,j}|^p\right)^{1/p}$$

is the $p-$norm of a $m \times n$ matrix $\mathbf{M}$ for some $p \geq 1$. In case $p = 1$, the optimal interval weight vector problem can be formulated as follows:

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (|\tfrac{l_i-r_j+1}{2} - \lambda_{ij}| + |\tfrac{r_i-l_j+1}{2} - \upsilon_{ij}|)$$
$$\text{s.t.} \begin{cases} \sum\limits_{j=1, j\neq i}^{n} l_j + r_i \leq 1 \leq l_i + \sum\limits_{j=1, j\neq i}^{n} r_j \\ 0 \leq l_i \leq r_i \leq 1, \text{ for } i = 1, ..., n \end{cases}$$

This problem can be transformed into a linear programming problem by the fact:

Lemma 10. *For any $x \in \mathbb{R}$, if $\xi^+ = \frac{|x|+x}{2}$ and $\xi^- = \frac{|x|-x}{2}$ then $\xi^+, \xi^- \geq 0$ and $|x| = \xi^+ + \xi^-$, $x = \xi^+ - \xi^-$.*

Applying the above lemma, the modified additive consistent model for interval weight vector [11] has been redefined as follows:

Model (A3):

$$\min \mathbf{J} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\xi_{ij}^+ + \xi_{ij}^- + \eta_{ij}^+ + \eta_{ij}^-)$$
$$\text{s.t.} \begin{cases} \left.\begin{array}{l} \frac{l_i-r_j+1}{2} - \lambda_{ij} - \xi_{ij}^+ + \xi_{ij}^- = 0 \\ \frac{r_i-l_j+1}{2} - \upsilon_{ij} - \eta_{ij}^+ + \eta_{ij}^- = 0 \\ \xi_{ij}^+, \xi_{ij}^-, \eta_{ij}^+, \eta_{ij}^- \geq 0 \end{array}\right\} (*) \\ \sum\limits_{j=1, j\neq i}^{n} l_j + r_i \leq 1 \leq l_i + \sum\limits_{j=1, j\neq i}^{n} r_j \\ 0 \leq l_i \leq r_i \leq 1, \text{ for } i = 1, ..., n \end{cases}$$

where $(*)$ hold for all $1 \leq i < j \leq n$.

Similarly, the multiplicative consistent model [11] has been modified as follows:

Model (M3):

$$\min \mathbf{K} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\xi_{ij}^+ + \xi_{ij}^- + \eta_{ij}^+ + \eta_{ij}^-)$$
$$\text{s.t.} \begin{cases} \left.\begin{array}{l} l_i - \lambda_{ij}(l_i + r_j) - \xi_{ij}^+ + \xi_{ij}^- = 0 \\ r_i - \upsilon_{ij}(r_i + l_j) - \eta_{ij}^+ + \eta_{ij}^- = 0 \\ \xi_{ij}^+, \xi_{ij}^-, \eta_{ij}^+, \eta_{ij}^- \geq 0 \end{array}\right\} (*) \\ \sum\limits_{j=1, j\neq i}^{n} l_j + r_i \leq 1 \leq l_i + \sum\limits_{j=1, j\neq i}^{n} r_j \\ 0 \leq l_i \leq r_i \leq 1, \text{ for } i = 1, ..., n \end{cases}$$

where $(*)$ hold for all $1 \leq i < j \leq n$.

From the model **(A3)** or **(M3)**, we find out the interval weight vectors $w_i = (l_i, r_i)$ which is assigned to the i^{th} alternatives, respectively.

3.3. Ranking methods for interval weight vectors

Many useful methods have been developed to compare two interval weights as well as to arrange a set of interval weights in a linear order. Given two intervals $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$, where $0 < a_1 \leq a_2$ and $0 < b_1 \leq b_2$, the following methods can be applied in order to compare $\mathbf{a}$ and $\mathbf{b}$:

Score function: The first idea is to evaluate each interval $[l, r]$, where $0 \le l \le r \le 1$ by a real value $S([l, r])$ called *score function* or briefly *score*. Most of score functions were originally defined for IFV, and for the use of this paper, they will be reformulated for interval weights. In real life applications, the following score functions can be applied:

- The simplest score function is defined by:

$$S_1([l, r]) = \frac{l + r}{2} \in [0, 1]$$

- In [12], a parameterized score function was defined by:

$$S_\lambda([l, r]) = (l + r - 1)(l + 1 - r) + \lambda \cdot (r - l)^2$$

 where $\lambda \in [-1, 1]$ is the risk parameter given by the DM's in consensus, reflecting a DM's attitude towards risk.

- Another double score function $S([l, r]) = (H(l, r), L(l, r))$ has been proposed in [13]:

$$H([l, r]) = l + 1 - r; \quad L([l, r]) = \frac{r}{1 + r - l}$$

 where H and L are called the *Accuracy* and the *Similarity functions*. Using those functions, two intervals $\alpha = [l_1, r_1]$ and $\beta = [l_2, r_2]$ can be compared as follows:

$$\begin{aligned}&\textbf{if } L(\alpha) > L(\beta) \textbf{ then } \alpha > \beta\\ &\textbf{if } L(\alpha) = L(\beta) \textbf{ and}\\ &\quad \textbf{if } H(\alpha) > H(\beta) \textbf{ then } \alpha > \beta\\ &\quad \textbf{if } H(\alpha) = H(\beta) \textbf{ then } \alpha = \beta\end{aligned}$$

Likelihood function: Wang, Zhang and Xu (2005)[14] introduced comparisons and rankings of interval weights based on calculating the degree of preference of $\mathbf{a} = [l_a, r_a]$ over $\mathbf{b} = [l_b, r_b]$, where $0 \le l_a \le r_a \le 1$ and $0 \le l_b \le r_b \le 1$.

$$p(\mathbf{a} \ge \mathbf{b}) = \frac{\max(0, r_a - l_b) - \max(0, l_a - r_b)}{r_a - l_a + r_b - l_b}$$

The value $p(\mathbf{a} \ge \mathbf{b}) \in [0, 1]$ can be interpreted as a likelihood of the event that $x \in \mathbf{a}$, $y \in \mathbf{b}$ and $x > y$. This function satisfies the condition:

$$p(\mathbf{a} \ge \mathbf{b}) + p(\mathbf{b} \ge \mathbf{a}) = 1$$

therefore if $p(\mathbf{a} \ge \mathbf{b}) > 0.5$, $\mathbf{a}$ is said to be superior to $\mathbf{b}$ to the degree of $p(\mathbf{a} \ge \mathbf{b})$, and it is denoted by

$$\mathbf{a} \overset{p(\mathbf{a} \ge \mathbf{b})}{\ge} \mathbf{b}.$$

4. Example and Applications

In [1], we proposed a new method for feature ranking called RAFAR (Rough-fuzzy Algorithm For Attribute Ranking). This is a hybrid method that combines discernibility relation of the rough set theory and the ranking method described in the previous section. The RAFAR method consists of two main steps: (1) construction of Intuitionistic Fuzzy Preference Relation (IFPR) for the set of features and (2) searching for the optimal features ranking that is consistent with this IFPR. The general framework of our proposition is presented in the Fig. 1:

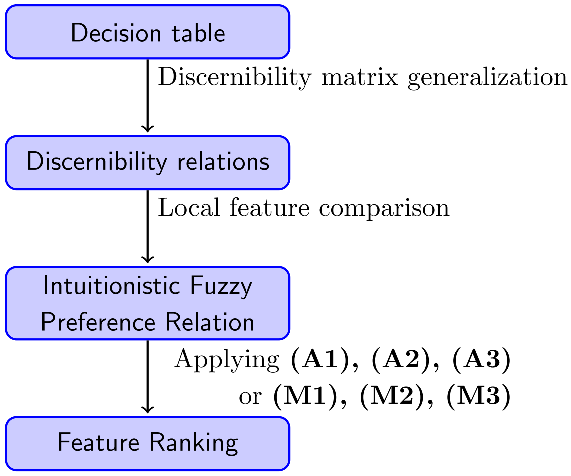

Fig. 1: *The general framework RAFAR.*

In [1], only model **(A1)** and **(M1)** were applied to generate the probability weight vector for features of different data sets. In this paper, we will show that the interval weight vector approach can be more efficient.

Firstly, let's consider the Example given in [1]. The following 5×5 matrix B is the IFPR representing the preference relation between 5 features named by a_1, a_2, a_3, a_4, a_5. Recall that this matrix has been generated by our RAFAR method.

$$B = \begin{bmatrix} (0.50, 0.50) & (0.55, 0.22) & (0.36, 0.24) & (0.51, 0.25) & (0.65, 0.06) \\ (0.22, 0.55) & (0.50, 0.50) & (0.16, 0.41) & (0.44, 0.43) & (0.64, 0.28) \\ (0.24, 0.36) & (0.16, 0.41) & (0.50, 0.50) & (0.47, 0.29) & (0.66, 0.20) \\ (0.25, 0.51) & (0.43, 0.44) & (0.29, 0.47) & (0.50, 0.50) & (0.56, 0.25) \\ (0.06, 0.65) & (0.28, 0.64) & (0.20, 0.66) & (0.25, 0.56) & (0.50, 0.50) \end{bmatrix}$$

Based on the RAFAR method, the optimal coefficients for this matrix are

$$(w_1, w_2, w_3, w_4, w_5) = (0.327, 0.227, 0.322, 0.124, 0.000)$$

and the order of the features is

$$a_1 > a_3 > a_2 > a_4 > a_5.$$

Applying the property of additive consistent IFPR and the interval IFPR method, based on the **Model (A3)** and **Model (M3)**, the optimal intuitionistic fuzzy weight vectors $w_i = (l_i, r_i)$ assigned to alternative a_i are defined as in the following table:

	Model (A3)		**Model (M3)**	
	l_i	r_i	l_i	r_i
a_1	0.2600	0.6400	0.1976	0.4304
a_2	0.0800	0.3400	0.1214	0.1902
a_3	0.1600	0.5400	0.1359	0.3513
a_4	0.1200	0.2400	0.1435	0.1545
a_5	0.0000	0.0000	0.0515	0.1064

Tab. 1: *Results from model* **(A3)** *and* **(M3)**

Let's choose two simple ranking methods for interval weight vectors presented in 3.3 and get the results below:

	$S_1([l,r])$	$S([l,r]) = (H([l,r]), L([l,r]))$
w_1	0.45	(0.62,0.46)
w_2	0.21	(0.74,0.27)
w_3	0.35	(0.62,0.39)
w_4	0.18	(0.88,0.21)
w_5	0.00	(1.00,0.00)

Tab. 2: *Score values from Model* **A3**

	$S_1([l,r])$	$S([l,r]) = (H([l,r]), L([l,r]))$
w_1	0.3140	(0.7672,0.3491)
w_2	0.1558	(0.9312,0.1779)
w_3	0.2436	(0.7846,0.2890)
w_4	0.1490	(0.9890,0.1528)
w_5	0.0789	(0.9451,0.1009)

Tab. 3: *Score values from Model* **M3**

The good thing is that both methods applied in model (**A3**) and (**M3**) give the same order of alternatives similar to the RAFAR method: $a_1 > a_3 > a_2 > a_4 > a_5$.

Applying likelihood function, we calculate the degree of preference of w_i over w_j as follows:

$p(a_1 \geq a_3)$	0.5217
$p(a_3 \geq a_2)$	0.7187
$p(a_2 \geq a_4)$	0.8461
$p(a_4 \geq a_5)$	1.0000

The values of likelihood function in the above table also give the alternatives ranking order which is suitable to the order if we apply mentioned methods. We can present this fact as follows:

$$a_1 \overset{0.5217}{\geq} a_3 \overset{0.7187}{\geq} a_2 \overset{0.8461}{\geq} a_4 \overset{1}{\geq} a_5$$

In the next example, we present the result of application of models **(A1)** and **(A3)** on the the WDBC data set [15]. The WDBC dataset contains features extracted from digitized image of a fine needle aspirate of a breast mass which describes the characteristics of the cell nuclei in the image. This dataset consists of 569 instances with 30 attributes and two decision classes. The features are encoded by $V1, V2, \cdots, V30$. The result of model **(A1)** is presented in column w_i in Table4. We can notice that it returns non-zero value to 6 features and 0 value to the other 24 features. This fact means these 24 features are not comparable by model **A1**.

i	w_i	l_i	r_i	S_1	L	p
V11	0	0	0	0	0	
V12	0	0	0	0	0	0.5
V13	0	0	0	0	0	0.5
V14	0	0	0	0	0	0.5
V15	0	0	0	0	0	0.5
V16	0	0	0	0	0	0.5
V17	0	0	0	0	0	0.5
V18	0	0	0	0	0	0.5
V19	0	0	0	0	0	0.5
V20	0	0	0	0	0	0.5
V29	0	0	0	0	0	0.5
V30	0	0	0	0	0	0.5
V10	0	0	0.028	0.014	0.029	1.000
V9	0	0	0.052	0.026	0.055	0.648
V5	0	0	0.056	0.028	0.059	0.516
V2	0	0	0.282	0.141	0.393	0.835
V24	0	0	0.321	0.161	0.473	0.532
V25	0	0	0.328	0.164	0.488	0.505
V4	0	0	0.350	0.175	0.538	0.516
V26	0	0	0.355	0.178	0.551	0.504
V22	0	0	0.410	0.205	0.696	0.536
V6	0	0	0.450	0.225	0.817	0.523
V1	0.142	0	0.503	0.251	1.012	0.528
V3	0.006	0	0.528	0.264	1.118	0.512
V27	0	0	0.554	0.277	1.241	0.512
V23	0.146	0.004	0.558	0.281	1.250	0.504
V21	0.193	0.011	0.565	0.288	1.266	0.506
V7	0	0.028	0.581	0.305	1.303	0.515
V8	0.216	0.141	0.694	0.418	1.556	0.602
V28	0.296	0.262	0.816	0.539	1.829	0.610

Tab. 4: *Example of RAFAR using 2 additive consistence ranking methods for WDBC data set*

The result of model **(A3)** is shown in columns l_i and r_i. One can notice that in this case, only 12 features are not comparable. The next 2 columns in Table 4 present the values of two score: S_1 and L

and the features $V1, \cdots, V30$ are ranked with respect to score S_1. The column p presents the values of the likelihood function indicating the probability that the given feature is better than the feature in previous line. One can notice the fact that in this example, all three score functions are consistent. Moreover, features V7 and V27 are quite highly ranked by model **(A3)**, while they are treated as not important by the model **(A1)**.

5. Conclusion

In this paper, interval weight vector method is proposed to features ranking problem. The paper shows how an IFPR can be represented by a pair of FPRs and how to apply other methods of additive or multiplicative consistency to find out interval weight vectors. In addition, the paper provides an example illustrating the proposed method and indicates the advantage over original RAFAR method presented in[1]. The advantage is more visible especially when the amount of alternatives is large and the interval weight values are similar.

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