Tối ưu hóa các ước lượng trạng thái mũ cho hệ rời rạc dương với trễ thời gian và nhiễu

TÓM TẮT

Trong bài báo này, chúng tôi xem xét đến bài toán ước lượng trạng thái mũ cho hệ dương rời rạc với trễ thời gian và nhiễu. Bằng việc sử dụng một phép đổi trạng thái, chúng tôi chuyển hệ dương rời rạc với trễ thời gian và nhiễu về một hệ tương ứng không còn nhiễu. Bằng cách sử dụng các kỹ thuật tối ưu hóa, chúng tôi đưa ra các ước lượng trạng thái cho hệ nhận được (không có nhiễu), từ đó chúng tôi cũng thu được chặn trạng thái mũ cho hệ ban đầu. Một số ví dụ số đưa ra để minh họa cho các kết quả lý thuyết đạt được.

Từ khóa: *hệ rời rạc dương, trễ thời gian, nhiễu, ước lượng trạng thái mũ, tối ưu hóa.*

Optimization of exponential state estimates for positive discrete-time systems with delays and disturbances

ABSTRACT

In this paper, we consider the problem of exponential state estimate for positive discrete-time system with time delays and disturbances. By using a state transformation, we reformulate a positive discrete-time system with time delays and disturbances to system without disturbances. By using optimization techniques, we derive the optimal exponential state estimate for the obatined systems (no noise), from which we also obtain the exponential state estimate for the original system. Some numerical examples are given to illlustate the obtained theoretical results.

Keywords: *positive discrete-time system, time delays, disturbances, exponential state estimate, optimization.*

1. INTRODUCTION

Positive systems are dynamical systems in which the state vectors are always belong to the nonnegative orthant providing that the initial value functions are nonnegative. This kind of systems has attracted a lot of attention in the mathematics community. Due to the positivity requirement, it is much more complicated and difficult to study on positive systems than on general systems. On one hand, similar to general systems, the time delay appears in the positive system and it can affect the stability of the system. On the other hand, different from general systems with time delays in which the quadratic Lyapunov-Krasovskii functional is used, the copositive Lyapunov-Krasovskii functional is applied to study the stability and the performance of positive time-delay systems. In,[1] for the first time, the stability of linear positive systems with constant delays was considered via using the copositive Lyapunov-Krasovskii functional. This result is extended to linear positive systems with time-varying delay in[2]. The L_1-gain and L_∞-gain are first mentioned by Briat[3] where the input-output gain is represented by linear inequalities. The Lyapunov-Krasovskii functional method is also a useful tool to study the problem of α-exponential stability for positive systems with bounded time-varying delays.[4,5] Another approach to study positive systems is based on the comparison principle. There are many developments of this approach have been introduced in the literature, see, e.g:[6,7]

In practical time-delay systems, disturbance is a factor which appears very often and cannot be avoided. State estimate for time-delay systems with bounded disturbances is one of the key problems in control theory. For positive systems, the main approach for solving this problem is based on the property of Metzler/Hurwitz/Schur matrices.[5,8–11] For discrete-time systems,[4,12] consider the problem of state estimate for positive discrete-time systems with delays without disturbances. The state estimating problem for positive discrete-time systems with delays and bounded disturbances has been studied in.[11,13] However, it should be noted that these state estimates have not been optimized.

Motivated by the above observation, in this paper, we consider the optimization problem of state estimates for positive discrete-time systems with delays and bounded disturbances. This work can be considered as a counterpart of[14] in which the continuous case was studied. Firstly, a state transformation is used to reformulate the problem of finding the optimized exponential state estimate for a positive discrete time-delay system with bounded disturbances into the problem of finding the optimized exponential state estimate for the positive discrete time-delay system without disturbance. Then, we apply an optimization scheme to the method proposed in[11] to obtain a better exponential componentwise estimate for the state vector of the transformed positive time-delay system (without disturbance). Consequently, we receive a more accurate exponential componentwise state estimate for the considered perturbed positive time-delay system.

2. NOTATION AND PRELIMINARIES

Notation: $\mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{R}^n_{0,+}$ are respectively the set of nonnegative integers, the n-dimensional vector space and the nonnegative orthant in $\mathbb{R}^n$; $e = [1\ 1\ \cdots\ 1]^\top \in \mathbb{R}^n$; for two vectors $x = [x_1\ x_2\ \cdots\ x_n]^\top$, $y = [y_1\ y_2\ \cdots\ y_n]^\top$ in $\mathbb{R}^n$, two $n\times n$-matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $x \prec y$ ($x \preceq y$) means that $x_i < y_i$ ($x_i \leq y_i$), $\forall i = 1, \cdots, n$ and $A \prec B$ ($A \preceq B$) means that $a_{ij} < b_{ij}$ ($a_{ij} \leq b_{ij}$), $\forall i, j = 1, \cdots, n$; A is a nonnegative matrix if $0 \preceq A$; $x \succeq y$ ($A \succeq B$) means that $y \preceq x$ ($B \preceq A$); $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral radius of A; I_n is the identity matrix of size n. The maximum, minimum of a finite set of vectors (of matrices) are understood componentwise.

Consider the following positive discrete-time system with time-varying delays and bounded disturbances

$$x(t+1) = A_0x(t) + A_1x(t - h_1(t)) + \omega(t),\ t \in \mathbb{N}, \quad (1)$$

$$x(s) = \varphi(s),\ s \in \{-h, -h+1, \ldots, 0\}, \quad (2)$$

where $x(t)$ is the state vector; $h_1(t) \in [0, h]$ is the time-varying delay; h is a known positive scalar; A_0 and A_1 are two known nonnegative matrices; $\omega(t) \in \mathbb{R}^n_{0,+}$ is the vector of disturbance; $\varphi(s) \in \mathbb{R}^n_{0,+},\ s \in \{-h, -h+1, \ldots, 0\}$, is the initial value function. Both $\omega(\cdot)$ and $\varphi(\cdot)$ are unknown but assumed to be bounded by some time-varying functions, i.e.

$$0 \preceq \omega(t) \preceq \bar{\omega}(t),\ s \in \mathbb{N}, \tag{3}$$

$$0 \preceq \varphi(s) \preceq \bar{\varphi}(s),\ s \in \{-h, -h+1, \ldots, 0\}, \tag{4}$$

where $\bar{\omega}(t)$, $\bar{\varphi}(s)$ are two known time-varying functions. Let denote by $x(t, \varphi, \omega)$ the unique solution of (1) with respect to the initial value function $\varphi(s)$ and the vector of disturbance $\omega(t)$.

The aim of this paper is to find the smallest possible exponential estimate with a predefined decay rate $\lambda > 1$ of the solution $x(t, \varphi, \omega)$. More specifically, we tend to find the two smallest possible vectors β_l and β_r such that

$$x(t, \varphi, \omega) \preceq \beta_l + \beta_r \lambda^{-t}, \forall t \in \mathbb{N}. \tag{5}$$

Remark 2.1. Since $\bar{\omega}(t)$ is a known function, with a predefined decay rate λ, it can be found two nonnegative constant vectors $\bar{\omega}_l$ and $\bar{\omega}_r$ such that

$$\bar{\omega}(t) \preceq \bar{\omega}_l + \bar{\omega}_r \lambda^{-t} := \tilde{\omega}(t),\ t \in \mathbb{N}. \tag{6}$$

In this paper, we will assume the existence of $\bar{\omega}_l$ and $\bar{\omega}_r$ satisfying (6).

Remark 2.2. For each positive scalar $\lambda > 0$, let us define the matrix

$$M_\lambda := \lambda A_0 + \lambda^{h+1} A_1. \tag{7}$$

As in,[4,11] to guarantee the existence of exponential state estimate for system (5), it must be assumed that M_λ is a nonnegative and a Schur matrix, i.e. $\rho(M_\lambda) < 1$.

An exponential state estimate under the form (5) for system (1) is obtained via a solution comparison with the following system

$$y(t+1) = A_0 y(t) + A_1 y(t - h_1(t)) + d(t), t \in \mathbb{N}, \tag{8}$$

where $d(t)$ is a vector of disturbance which will be defined later.

The next lemma give us some useful facts related to systems (1) and (8) which will be needed in next parts of the paper.

Lemma 2.3. (i) Systems (1) and (8) are nonnegative;

(ii) With two initial value functions $0 \preceq \varphi_l(s) \preceq \varphi_r(s),\ s \in \{-h, \ldots, 0\}$, and two vectors of disturbance $0 \preceq \omega_l(t) \preceq \omega_r(t),\ t \in \mathbb{N}$, we then have

$$x(t, \varphi_l, \omega_l) \preceq x(t, \varphi_r, \omega_l),$$
$$x(t, \varphi_r, \omega_l) \preceq x(t, \varphi_r, \omega_r),$$
$$x(t, \varphi_r, \omega_r) \preceq y(t, \varphi_r, \omega_r),$$
$$y(t, \varphi_l, \omega_l) \preceq y(t, \varphi_l, \omega_r).$$

Proof. The proof of this lemma can be conducted similarly as in.[5] □

3. EXPONENTIAL STATE ESTIMATE FOR POSITIVE DISCRETE-TIME SYSTEMS WITH DELAYS (WITHOUT DISTURBANCE)

Let us consider the following positive discrete-time system (without disturbance)

$$z(t+1) = A_0 z(t) + A_1 z(t - h_1(t)),\ t \in \mathbb{N}, \tag{9}$$

$$z(s) = \phi(s), s \in \{-h, -h+1, \ldots, 0\}, \tag{10}$$

where the initial value function $\phi(\cdot)$ is unknown but assumed to be upper bounded by a known time-varying function $\bar{\phi}(\cdot)$, i.e.,

$$0 \preceq \phi(s) \preceq \bar{\phi}(s),\ s \in \{-h, -h+1, \ldots, 0\}. \tag{11}$$

In this section, under the assumption that $\rho(M_\lambda) < 1$, we present a method to obtain an λ-exponential state estimate for the solution $z(t, \phi)$ of the system (1). By Lemma 2.3, one has

$$z(t, \phi) \preceq z(t, \bar{\phi}), t \in \mathbb{N}. \tag{12}$$

As in,[11] if there exist a vector $p \succ 0$, a number $\delta \in (0,1)$ such that

$$(A_0 + A_1)p \prec \delta p \quad (13)$$

and a nonnegative scalar γ such that

$$\bar{\phi}(s) \preceq \gamma p \lambda^{-s}, s \in \{-h, -h+1, \ldots, 0\},\ \lambda = \delta^{\frac{-1}{h+1}}, \quad (14)$$

we then have

$$z(t, \bar{\phi}) \preceq \gamma p \lambda^{-t}, t \in \mathbb{N}. \quad (15)$$

It should be noted that condition (13) is equivalent to $\rho(M_\lambda) < 1$ where M_λ is defined in (7). Combining two inequalities (12) and (15), we get

$$z(t, \phi) \preceq \gamma p \lambda^{-t}, t \in \mathbb{N}. \quad (16)$$

A vector $(p, \gamma) \in \mathbb{R}^n_+ \times \mathbb{R}_{0,+}$ satisfying the condition $(M_\lambda - I)\, p \prec 0$ and (14) can be found as below

$$p = \left(I - \lambda A_0^\top - \lambda^{h+1} A_1^\top\right)^{-1} e, \quad (17)$$

$$\gamma = \max\left\{ \max_{s=-h,\ldots,0} \frac{\|\bar{\phi}(s)\|_\infty}{p_1}, \ldots, \max_{s=-h,\ldots,0} \frac{\|\bar{\phi}(s)\|_\infty}{p_n} \right\}. \quad (18)$$

It can be seen that the factor vector (p, γ) of the exponential state estimate (16) has not been optimized. From this inequality, for each $i \in \{1, \ldots, n\}$, the exponential estimate of the i-th element of the state vector $z(t, \phi)$ can be obtained as below

$$z_i(t, \phi) \preceq \gamma p_i \lambda^{-t}, t \in \mathbb{N}. \quad (19)$$

For each $i \in \{1, \ldots, n\}$, our aim is to find a vector $(p, \gamma) \in \mathbb{R}^n_+ \times \mathbb{R}_{0,+}$ such that the coefficient γp_i in (19) is minimized.

For simplicity, let us consider the case $i = 1$. Since the function $\bar{\phi}(s)$ is given, for each $i \in \{1, \ldots, n\}$, we can define the number

$$a_i := \max_{s \in \{-h, -h+1, \ldots, 0\}} \frac{\|\bar{\phi}_i(s)\|_\infty}{\lambda^{-s}}. \quad (20)$$

Let $a = [a_1, a_2, \ldots, a_n]$. Then, condition (14) is equivalent to $\gamma p \succeq a$.
Let

$$\Omega := \left\{(p, \gamma) \in \mathbb{R}^n_+ \times \mathbb{R}_{0,+} |\, (M_\lambda - I)\, p \prec 0, \gamma p \succeq a\right\}$$

and $f(p, \gamma) := \gamma p_1$.
The smallest factor γp_1 of the exponential estimate of the first element in (19) is the optimal value of the following optimization problem:

$$\min f(p, \gamma) = \gamma p_1 \text{ such that } (p, \gamma) \in \Omega. \quad (\mathbf{OP}_1)$$

It should be noted that ($\mathbf{OP}_1$) is a nonconvex optimization problem. Hence, this problem is quite difficult to be solved. However, ($\mathbf{OP}_1$) can be reformulated under the form of the following linear programming:

$$\min g(u) = u_1 \text{ such that } u \in \Lambda, \quad (\mathbf{LP}_1)$$

where

$$\Lambda := \left\{u \in \mathbb{R}^n_+ |\, (M_\lambda - I)\, u \prec 0, u \succeq a\right\}. \quad (21)$$

It can be seen that the two problems ($\mathbf{OP}_1$) and ($\mathbf{LP}_1$) have the same optimal value.

Similarly, for each $i \in \{2, 3, \ldots, n\}$, by solving linear programming problems

$$\min g(u) = u_i \text{ such that } u \in \Lambda, \quad (\mathbf{LP}_i)$$

where Λ is defined in (21), we find the smallest factor u_i^*, of the i-th element of the exponential state estimate $z_i(t, \phi)$ under the form (19). Combine the above procedures, we receive the minimized vector $u_r = [u_1^*, u_2^*, \ldots, u_n^*]^\top$ of the following exponential state estimate of the system (9)

$$z(t, \phi) \preceq u_r \lambda^{-t}, t \in \mathbb{N}. \quad (22)$$

From the above development, the main result of this section is summarized in the following theorem.

Theorem 3.1. Assume that $\rho(M_\lambda) < 1$ and $0 \preceq \phi(s) \preceq \bar{\phi}(s)$. The solution $z(t,\phi)$ of the system (9) has an exponential state estimate under the form

$$z(t,\phi) \preceq u_r \lambda^{-t},\ t \in \mathbb{N} \tag{23}$$

where $u_r = [u_1^*, u_2^*, \ldots, u_n^*]^\top$ is the optimal factor vector and $u_i^*, i = 1, \ldots, n$, is the optimal value of the problem ($\mathbf{LP}_i$).

4. EXPONENTIAL STATE ESTIMATE FOR POSITIVE DISCRETE-TIME SYSTEMS WITH DELAYS AND BOUNDED DISTURBANCES

In this section, we will establish an exponential state estimate for the positive discrete-time system with time-varying delays and bounded disturbances system (1). Choose $\lambda > 1$ such that $\rho(M_\lambda) < 1$. Let us define two nonnegative vectors

$$q_l := (I - M_1)^{-1}\overline{\omega}_l, \tag{24}$$

$$q_r := (I - M_\lambda)^{-1}\overline{\omega}_r. \tag{25}$$

Let

$$\psi(s) := \max\left\{\bar{\varphi}(s),\ q_l + q_r\lambda^{-s+1}\right\}, \tag{26}$$

$$\phi(s) := \psi(s) - q_l - q_r\lambda^{-s+1}, \tag{27}$$

$$d(t) := \bar{\omega}_l + \bar{\omega}_r\lambda^{-t} + \lambda^{-t}(\lambda^{h+1}A_1 - \lambda^{h_1(t)+1}A_1)q_r. \tag{28}$$

Then, by comparing with conditions (3), (4), (6), and noting that $h_1(t) \le h, A_1 \succeq 0$, it can be verified that $\varphi(s) \preceq \bar{\varphi}(s) \preceq \psi(s)$, $0 \preceq \phi(s)$ và $\omega(t) \preceq \tilde{\omega}(t) \preceq d(t)$. From Lemma 2.3, one has

$$x(t,\varphi,\omega) \preceq x(t,\psi,\omega) \preceq y(t,\psi,\tilde{\omega}) \preceq y(t,\psi,d). \tag{29}$$

From the above inequalites, we just need to find an exponential state estimate for the solution $y(t,\psi,d)$ of the system (8).

Set

$$z(t) := y(t) - q_l - q_r\lambda^{-t+1},\ t \ge -h. \tag{30}$$

From (8), we then have

$$\begin{aligned}
z(t+1) &= y(t+1) - q_l - q_r\lambda^{-t} \\
&= A_0 y(t) + A_1 y\left(t - h_1(t)\right) + d(t) \\
&\quad - q_l - q_r\lambda^{-t} \\
&= A_0\left(z(t) + q_l + q_r\lambda^{-t+1}\right) \\
&\quad + A_1\left(z\left(t - h_1(t)\right) + q_l + q_r\lambda^{-t+h_1(t)+1}\right) \\
&\quad + d(t) - q_l - q_r\lambda^{-t} \\
&= A_0 z(t) + A_1 z\left(t - h_1(t)\right) + (A_0 + A_1 - I)q_l \\
&\quad + \lambda^{-t}(\lambda A_0 + \lambda^{h+1}A_1 - I)q_r \\
&\quad - \lambda^{-t}\left(\lambda^{h+1}A_1 - \lambda^{h_1(t)+1}A_1\right)q_r + d(t) \\
&= A_0 z(t) + A_1 z\left(t - h_1(t)\right) \\
&\quad + (M_1 - I)(I - M_1)^{-1}\bar{\omega}_l \\
&\quad + \lambda^{-t}(M_\lambda - I)(I - M_\lambda)^{-1}\bar{\omega}_r \\
&\quad - \lambda^{-t}\left(\lambda^{h+1}A_1 - \lambda^{h_1(t)+1}A_1\right)q_r + d(t) \\
&= A_0 z(t) + A_1 z\left(t - h_1(t)\right) - \bar{\omega}_l - \bar{\omega}_r\lambda^{-t} \\
&\quad - \lambda^{-t}\left(\lambda^{h+1}A_1 - \lambda^{h_1(t)+1}A_1\right)q_r + d(t) \\
&= A_0 z(t) + A_1 z\left(t - h_1(t)\right).
\end{aligned}$$

This means that we obtain the positive discrete-time system with delay and without disturbance. We then deduce from (30) that

$$y(t,\psi,d) = z(t,\phi) + q_l + q_r\lambda^{-t+1}, \tag{31}$$

where, $\phi(\cdot)$ is defined in (27). Combine inequalites (29) and (31), one gets

$$x(t,\varphi,\omega) \preceq z(t,\phi) + q_l + q_r\lambda^{-t+1}. \tag{32}$$

From (22) and (32), we have

$$\begin{aligned}
x(t,\varphi,\omega) &\preceq u_r\lambda^{-t} + q_l + q_r\lambda^{-t+1} \\
&= q_l + (u_r + \lambda q_r)\lambda^{-t}.
\end{aligned}$$

From this, we obtain an exponential state estimate (5) for system (1) with factor vectors defined by

$$\beta_l := q_l,\ \beta_r := u_r + \lambda q_r, \tag{33}$$

where the vector u_r is found as in Theorem 3.1. The main result of this section is summarized in the following theorem.

Theorem 4.1. *Assume that* $\rho(M_\lambda) < 1$, (3) *and* (4) *hold. The solution* $x(t,\varphi,\omega)$ *is estimated via the formula* (5) *where the vectors* β_l, β_r *are found by* (33).

5. NUMERICAL EXAMPLES

In this section, we present two numerical examples to illustrate the results obtained in Theorems 3.1 and 4.1.

Example 5.1. Consider the following time-delay system

$$\begin{aligned} x(t+1) &= A_0x(t) + A_1x(t-h_1(t)) \quad t \in \mathbb{N}, \\ x(s) &= \phi(s), \quad s = -2,-1,0, \end{aligned} \tag{34}$$

where, $x(t) \in \mathbb{R}^3, h_1(t) \in \{0,1,2\}$, A_0 and A_1 are two nonnegative matrices with coefficients

$$A_0 = \begin{bmatrix} 0.21 & 0.21 & 0.12 \\ 0.04 & 0.12 & 0.14 \\ 0.12 & 0.05 & 0.26 \end{bmatrix}, A_1 = \begin{bmatrix} 0.32 & 0.01 & 0.18 \\ 0.03 & 0.12 & 0.02 \\ 0.04 & 0.01 & 0.21 \end{bmatrix},$$

the initial value function $\phi(\cdot)$ is unknown and satisfies $|\phi(s)| \preceq \bar{\phi}(s)$ where

$$\begin{aligned} \bar{\phi}(-2) &= \begin{bmatrix} 2 & 3 & 0.5 \end{bmatrix}^\top, \\ \bar{\phi}(-1) &= \begin{bmatrix} 3.1 & 2.4 & 3.2 \end{bmatrix}^\top, \\ \bar{\phi}(0) &= \begin{bmatrix} 0.3 & 0.5 & 0.1 \end{bmatrix}^\top. \end{aligned}$$

By using Remark 4 in,[4] we deduce that the range of the decay rate λ is $[1, 1.1375]$.

Let us consider the case $\lambda = 1.1$. By applying Theorem 3.1, we receive the following table.

Methods	$\lambda = 1.10$		
	u_1	u_2	u_3
Our new method	4.9163	2.4793	2.9091
Method in[11]	9.6567	3.2000	5.6305

The above table shows that with the decay rate $\lambda = 1.10$, the elements of the factor vectors obtained by our new method is smaller than the ones obtained by .[11]

Example 5.2. Consider the following positive discrete-time system with delays and disturbances

$$\begin{aligned} x(t+1) &= A_0x(t) + A_1x(t-h_1(t)) + \omega(t) \quad t \in \mathrm{N}, \\ x(s) &= \varphi(s), \quad s = -2,-1,0, \end{aligned} \tag{35}$$

where, $x(t) \in \mathbb{R}^3, h_1(t) \in \{0,1,2\}$, A_0 are A_1 nonnegative matrices with

$$A_0 = \begin{bmatrix} 0.21 & 0.21 & 0.12 \\ 0.04 & 0.12 & 0.14 \\ 0.12 & 0.05 & 0.26 \end{bmatrix}, A_1 = \begin{bmatrix} 0.32 & 0.01 & 0.18 \\ 0.03 & 0.12 & 0.02 \\ 0.04 & 0.01 & 0.21 \end{bmatrix},$$

$\omega(t) \in \mathbb{R}^n_{0,+}$ is the vector of disturbance satisfying $0 \preceq \omega(t) \preceq \bar{\omega(t)}, t \in \mathbb{N}$; the initial value function $\varphi(s)$, $s = -2,-1,0$, satisfying condition $|\varphi(s)| \preceq \bar{\varphi}(s)$ with

$$\bar{\varphi}(s) = \begin{bmatrix} 13 \\ 8 \\ 10.2 \end{bmatrix}, \bar{\omega}(t) = \begin{bmatrix} 0.6 \\ 1 \\ 0.7 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} \lambda^{-t}.$$

By using Remark 4 in,[4] we find that the range of decay rate is $[1, 1.1375]$.

Let us consider the case $\lambda = 1.1$. By applying the development in Section 4, we find that

$$\begin{aligned} \beta_l &= \begin{bmatrix} 4.1627 & 2.2965 & 2.8374 \end{bmatrix}^\top, \\ \beta_r &= \begin{bmatrix} 13.0887 & 5.7035 & 7.7428 \end{bmatrix}^\top. \end{aligned}$$

This gives us the exponential state estimate for the positive discrete-time system with delays and disturbances (35).

6. Conclusions

In this paper, we have considered the problem of exponential state estimate for positive discrete-time systems with delays and disturbances. A state transformation is used to transform positive discrete-time systems with delays and disturbances to systems without disturbance. By applying an optimization techique, we have found the smallest possible exponential state estimate for the transfromed systems from which the estimates for the considered systems are derived. The approach in this paper can be used to study some other classes of positive systems comprising disturbances.

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