Tính chất Forelli mạnh của các không gian Fréchet và định lý Alexander đối với các chuỗi luỹ thừa hình thức giá trị Fréchet

TÓM TẮT

Chúng tôi đưa ra các điều kiện đủ để một chuỗi luỹ thừa hình thức (tương ứng, một dãy của chuỗi luỹ thừa hình thức) của các đa thức thuần nhất, liên tục, giá trị Fréchet hội tụ trong lân cận của 0 trên không gian Fréchet E (tương ứng, $E = \mathbb{C}^N$) là hội tụ trong lân cận của 0 trên mỗi đường thẳng phức $\ell_a := \mathbb{C}a$ với mỗi $a \in A$ (A là tập không đa cực xạ ảnh trong $\mathbb{C}^N$). Kết quả trong trường hợp $F = \mathbb{C}^N$ là một "tương tự giá trị Fréchet" của định lý Alexander cổ điển nhưng với các giả thiết yếu hơn; Đưa ra lớp các định lý tương tự đinh lý Alexander nhận giá trị Fréchet trong trường hợp $E = \mathbb{C}^N$ nhưng với các giả thiết yếu hơn. Chúng tôi cũng chứng minh rằng mọi không gian Fréchet F có tính chất Forelli mạnh, nghĩa là nếu mọi hàm $f : \Delta_N \to F$ sao cho $f \in C^\infty(0)$ và $f|_{l_a \cap \Delta_N}$ là chỉnh hình với mọi đường thẳng phức $l_a, a \in A$, thì f chỉnh hình trên Δ_N.

Từ khóa: *Hàm đa điều hoà dưới, hàm chỉnh hình, tập đa cực xạ ảnh, chuỗi luỹ thừa hình thức.*

Strong Forelli property of Fréchet spaces and Alexander's theorem for Fréchet-valued formal power series

ABSTRACT

We give sufficient conditions to ensure the convergence on some zero-neighbourhood in a Fréchet space E (resp. $E = \mathbb{C}^N$) of a formal power series (resp. a sequence of formal power series) of Fréchet-valued continuous homogeneous polynomials provided that the convergence holds at a zero-neighbourhood of each complex line $\ell_a := \mathbb{C}a$ for every $a \in A$, a non-projectively-pluripolar set in E. The result in the case $E = \mathbb{C}^N$ is a Fréchet-valued analog of classical Alexander's theorem but under weaker assumptions. It is also shown that every Fréchet space has the strong Forelli property, i.e, for a non-projectively-pluripolar set $A \subset \mathbb{C}^N$, every Fréchet-valued function f on the open unit ball $\Delta_N \subset \mathbb{C}^N$, $f \in C^\infty(0)$, such that its restriction on each complex line ℓ_a, $a \in A$, is holomorphic admits an extension to an entire function.

Keywords: *Plurisubharmonic functions, holomorphic functions, projectively pluripolar sets, formal power series*

1. INTRODUCTION AND PRELIMINARIES

The focus of this paper is to study the Fréchet-valued analogs and the generalizations of the following two classical theorems.

Forelli's Theorem [1]**.** *If f is a function defined in the unit ball $\Delta_N \subset \mathbb{C}^N$, holomorphic on the intersection of Δ_N with every complex line ℓ passing through the origin and if f is of class C^∞ in a neighborhood of this point, then it is holomorphic in Δ_N.*

Alexander's Theorem [2]**.** *Let $\mathscr{F}$ be a family of analytic functions on $\Delta_N \subset \mathbb{C}^N$. If the restriction of $\mathscr{F}$ to each complex line through the origin is normal (resp. at the origin), then $\mathscr{F}$ is normal (resp. at the origin).*

Recall that a family $\mathscr{F}$ of analytic functions on a complex manifold Ω is *normal* if every sequence in $\mathscr{F}$ has a subsequence which converges uniformly on compact subsets of Ω either to an analytic function or to ∞, and that $\mathscr{F}$ is normal at a point $x \in \Omega$ if there exists a neighborhood W of x such that the restriction of $\mathscr{F}$ to W is normal.

Forelli's theorem is a radial analogue of the fundamental theorem of Hartogs. Alexander's theorem allows us to obtain the Hartogs theorem on the convergence of formal power series in several complex variables.

The problems of extensions and generalizations of the above classical theorems for holomorphic maps and vector-valued holomorphic functions have drawn attention of mathematicians.

In this note, we will investigate these results for the Fréchet-valued case in the "strong" sense in which the functions are only required that their restrictions on $\ell \cap \Delta_N$ are holomorphic for every $\ell \in \mathscr{L}$, *a family of sufficiently many complex lines* passing through the origin.

Families of "sufficiently many" complex lines in the paper concern the notions of pluripolar sets and projectively pluripolar sets. These notions require some extra background material for their definition.

Let D be a domain in a locally convex space E. An upper-semicontinuous function $\varphi : D \to [-\infty, +\infty)$ is said to be *plurisubharmonic*, and write $\varphi \in PSH(D)$, if φ is subharmonic on every one dimensional section of D.

A subset $B \subset D$ is said to be *pluripolar* in D if there exists $\varphi \in PSH(D)$ such that $\varphi \not\equiv -\infty$ and $\varphi\big|_B = -\infty$.

A function $\varphi \in PSH(E)$ is called *homogeneous plurisubharmonic* if

$$\varphi(\lambda z) = \log|\lambda| + \varphi(z) \quad \forall \lambda \in \mathbb{C},\ \forall z \in E.$$

We denote by $HPSH(E)$ the set of homogeneous plurisubharmonic functions on E. We say that a subset $A \subset E$ is *projectively pluripolar* if A is contained in the $-\infty$ locus of some element in $HPSH(E)$ which is not identically $-\infty$. It is clear that projective pluripolarity implies pluripolarity. The converse is not true (see [3] [Proposition 3.2 b]).

Some properties, examples and counterexamples of projectively pluripolar sets may be found in [3]. We introduce below a few examples in locally convex spaces.

Example 1.1. Let E be a metrizable locally convex space. Fix $a \in E$. Then, the complex line $\ell_a := \mathbb{C}a = \{\lambda a : \lambda \in \mathbb{C}\}$, hence, every $A \subset \ell_a$, is projectively pluripolar in E.

Indeed, let d be the metric defining the topology on E. Consider the function

$$\varphi(z) = -\log d(z, \ell_a) := -\log \inf_{w \in \ell_a} d(z, w).$$

It is easy to check that $\varphi \in HPSH(E)$, $\varphi \not\equiv -\infty$ and $\ell_a \subset \varphi^{-1}(-\infty)$.

Example 1.2. Let E be a Fréchet space which contains a non-pluripolar compact balanced convex subset B. By the same proof as in Example 1.1, the set ∂B is pluripolar. However, ∂B is not projectively pluripolar in E.

Otherwise, we can find a function $\varphi \in HPSH(E)$, $\varphi \not\equiv -\infty$ and $\partial B \subset \varphi^{-1}(-\infty)$. For every $z \in B$ we can write $z = \lambda y$ for some $y \in \partial B$ and $|\lambda| < 1$. Then

$$\varphi(z) = \varphi(\lambda y) = \log|\lambda| + \varphi(y) = -\infty, \quad \forall z \in B.$$

It is impossibe because B is non-pluripolar.

Example 1.3. By Theorem 9 of [4] and Example 1.2, a nuclear Fréchet space having the linear topological invariant $(\widetilde{\Omega})$ which is introduced by Vogt (see [5]) contains a non-projectively-pluripolar set.

We recall that a complex space or a locally convex space X is said to have *Forelli Property* if every map $f : \Delta_N \to X$ such that f is of C^∞- class in a neighborhood of $0 \in \Delta_N$ and $f\big|_{\ell \cap \Delta_N}$ is holomorphic for *all* complex lines ℓ through $0 \in \Delta_N$ then f is holomorphic on Δ_N. In 2005 L. M. Hai and N. V. Khue [6] studied the Forelli property for complex spaces. They also investigated the relation between these spaces with Hartogs spaces and Hartogs holomorphic extension spaces for holomorphically convex Kähler complex spaces.

Definition 1.1. A locally convex space F is said to have the *strong Forelli property* if every function $f : \Delta_N \to F$ satisfying that:

(i) f belongs to C^k-class at $0 \in \mathbb{C}^N$ for $k \geq 0$,

(ii) for some non-projectively-pluripolar subset $A \subset \mathbb{C}^N$, the restriction of f on each complex line ℓ_a, $a \in A$, is holomorphic, then there exists an entire function $\widehat{f}$ on $\mathbb{C}^N$ such that $\widehat{f} = f$ on ℓ_a for all $a \in A$.

Note that, from Proposition 3.1 in [3], in $\mathbb{C}^N$, the following are equivalent:

a) A is projectively pluripolar;

b) $A^\lambda := \{tz : t \in \mathbb{C}, |t| < \lambda, z \in A\}$ is pluripolar for each $\lambda > 0$;

c) $\mu(A^\lambda) = 0$ where μ is the Lebesgue measure;

d) $\nu(\varrho(A^\lambda)) = 0$ where ν is the invariant measure on the projective space $\mathbb{CP}^{N-1}$ and $\varrho : \mathbb{C}^N \setminus \{0\} \to \mathbb{CP}^{N-1}$ is the natural projection.

It follows that the condition (ii) in Definition 1.1 can be replaced by the following condition:

(ii') for some family $\mathscr{L}$ of complex lines through $0 \in \mathbb{C}^N$ such that $\mu\big(\Delta_N \cap \bigcup_{\ell \in \mathscr{L}} \ell\big) > 0$, the restriction of f on each $\ell \in \mathscr{L}$ is holomorphic.

The main theorems of our note are the following.

Theorem 1.1. *Every Fréchet space has the strong Forelli property.*

Theorem 1.2. *Let $A \subset \mathbb{C}^N$ be a non-projectively-pluripolar set and $(f_n)_{n\geq 1}$ be a sequence of formal power series of continuous homogeneous polynomials on $\mathbb{C}^N$ with values in a Fréchet space. Assume that there exists $r_0 \in (0,1)$ such that, for each $a \in A$, the restriction of $(f_n)_{n\geq 1}$ on ℓ_a is a sequence of holomorphic functions which is convergent on the disk $\Delta(r_0)$. Then there exists $r > 0$ such that $(f_n)_{n\geq 1}$ is a sequence of holomorphic functions that converges on $\Delta_N(r)$.*

By the equivalence of a) and d) mentioned above, the hypotheses of Theorem 1.2 may be stated under an alternative form as follows: *Let B be a subset of Δ_N such that $\nu(\varrho(B)) = 0$ where ν is the invariant measure on the projective space $\mathbb{CP}^{N-1}$ and $\varrho : \mathbb{C}^N \setminus \{0\} \to \mathbb{CP}^{N-1}$ is the natural projection. Assume that for some $r_0 \in (0,1)$, the restriction of the sequence $(f_n)_{n\geq 1}$ on each complex line ℓ through $0 \in \Delta_N$ with $\ell \cap B = \{0\}$ is convergent in $\Delta(r_0)$.*

Actually, Theorem 1.2 is not a generalization of Alexander's theorem because our result only refers to uniform convergence, not to the normality of the sequence of formal power series. Therefore, it is still an *open question* that whenever we obtain a truly generalization of Alexander's theorem. In other words, "*Whether or not a version of Theorem 1.2 where the uniform convergence of the sequence $(f_n|_{\ell_a})_{n\geq 1}$ on compact sets of $\Delta(r_0)$ is replaced by* normality *of this sequence on $\Delta(r_0)$ i.e., we allow convergence to ∞ uniformly on compact sets?*"

The proof of the first main theorem will be presented in Section 2. To prepare for the proof, with the help of techniques of pluripotential theory and functional analysis, we investigate the Hartogs Lemma for sequence of plurisubharmonic functions for the infinite dimensional case (Theorem 2.2). This result is also essential to the Section 3 in which we discus a problem closely related to the two classical theorems mentioned above. The main goal of this section (Theorem 3.1) is to study the convergence set of a formal power series of continuous homogeneous polynomials between Fréchet spaces under the hypothesis that it is convergent along a pencil of complex lines through the origin.

Finally, the last section presents the proof of the second main theorem of the paper. Some results concerning to Vitali's theorem for a sequence of Fréchet-valued holomorphic functions (Proposition 4.3) will be shown to help for our proof.

The standard notation of the theory of locally convex spaces used in this note is presented as in the book of Jarchow [7]. A locally convex space is always a complex vector space with a locally convex Hausdorff topology. For a locally convex space E we use E'_{bor} to denote E' equipped with the bornological topology associated with the strong topology β.

The locally convex structure of a Fréchet space is always assumed to be generated by an increasing system $(\|\cdot\|_k)_{k\geq 1}$ of seminorms. For an absolutely convex subset B of E, by E_B we denote the linear hull of B which becomes a normed space in a canonical way if B is bounded (with the norm $\|\cdot\|_B$ is the gauge functional of B).

We say that a Fréchet space E has the property (LB_∞), and write $E \in (LB_\infty)$ for short, if $\forall \varrho_N \uparrow \infty, \exists p \,\forall q \,\exists k(q) \geq q$, $C(q) > 0, \forall x \in E, \exists m$ with $q \leq m \leq k(q)$:

$$\|x\|_q^{1+\varrho_m} \leq C(q)\|x\|_m \|x\|_p^{\varrho_m}.$$

This property is a linear toplogical invariant which plays a very important role in modern theory of Fréchet spaces. Khue, Hai, Hoan [8] [Theorem 4.1] proved that if $E \in (LB_\infty)$ then $(E'_{\text{bor}})'_\beta \in (LB_\infty)$.

For further terminology from complex analysis we refer to [9].

We use throughout this paper the following notations: $\Delta_N(r) = \{z \in \mathbb{C}^N : \ \|z\| < r\}$; $\Delta_N := \Delta_N(1); \Delta(r) = \Delta_1(r)$; $\Delta := \Delta_1$; and ℓ_a is the complex line $\mathbb{C}a$.

2. THE STRONG FORELLI PROPERTY OF FRÉCHET SPACES

This section is devoted to the proof of Theorem 1.1. First we investigate the Hartogs Lemma for a sequence of plurisubharmonic functions in the infinite dimensional case. This is essential to our proofs.

Lemma 2.1. *Let $(P_n)_{n\geq 1}$ be a sequence of continuous homogeneous polynomials on a Baire locally convex space E of degree $\leq n$. Assume that*

$$\limsup_{n\to\infty} \frac{1}{n} \log |P_n(z)| \leq 0$$

for each $z \in E$. Then for every $\varepsilon > 0$ and every compact set K in E there exists n_0 such that

$$\frac{1}{n} \log |P_n(z)| < \varepsilon \quad \forall n > n_0, \ \forall z \in K.$$

Proof. Since

$$\limsup_{n\to\infty} |P_n(z)|^{\frac{1}{n}} \leq 1 \quad \forall z \in E$$

the formula

$$f(z)(\lambda) = \sum_{n\geq 1} P_n(z)\lambda^n$$

defines a function $f : E \to H(\Delta)$, the Fréchet space of holomorphic functions on the open unit disc $\Delta \subset \mathbb{C}$.

Let us check f is holomorphic on E. Given $z \in E \setminus \{0\}$ and consider $f(\cdot\, z) : \mathbb{C} \to H(\Delta)$ with

$$f(\xi z)(\lambda) = \sum_{n\geq 1} P_n(z)\lambda^n \xi^{k_n}$$

where $k_n = \deg P_n \leq n$. Then $f(\cdot\, z)$ is holomorphic because for $0 < r < 1$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{|\lambda|\leq r} (|P_n(z)||\lambda|^n)^{\frac{1}{k_n}} \\
&= \limsup_{n\to 0} (|P_n(z)|^{\frac{1}{n}} r)^{\frac{n}{k_n}} \\
&\leq \limsup_{n\to 0} |P_n(z)|^{\frac{1}{n}} r \leq r < 1.
\end{aligned}$$

This means that f is Gâteaux holomorphic on E.

Now for each $k \geq 1$ we put

$$A_k := \{z \in E : \ |P_n(z)| \leq k^{k_n} \quad \forall n \geq 1\}.$$

By the continuity of P_n, the sets A_k are closed in E. Moreover, $E = \bigcup_{k\geq 1} A_k$. Since E is a Baire space, there exists $k_0 \geq 1$ such that $\text{Int} A_{k_0} \neq \varnothing$. Then f is holomorphic on $\frac{1}{k_0}\text{Int} A_{k_0}$ because

$$\sum_{n\geq 1} |P_n(z)||\lambda|^n \leq \sum_{n\geq 1} \frac{k_0^{k_n}}{k_0^{k_n}} r^n = \sum_{n\geq 1} r^n < \infty$$

for $0 < r < 1$. Hence, by Zorn's theorem [10] [Theorem 1.3.1], f is holomorphic on E.

Now given $K \subset E$ a compact set and $\varepsilon > 0$. Take $0 < r < 1$ and denote

$$C := \sup\{|f(z)(\lambda)| : \ z \in K, |\lambda| \le r\} < \infty.$$

Then we have

$$|P_n(z)| = \left| \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(z)(\xi)d\xi}{\xi^{n+1}} \right| \le \frac{C}{r^n} \, \forall z \in K,$$

i.e,

$$|P_n(z)|^{\frac{1}{n}} \le \frac{C^{\frac{1}{n}}}{r}.$$

Choose n_0 sufficiently large we obtain

$$|P_n(z)|^{\frac{1}{n}} \le \frac{C^{\frac{1}{n}}}{r} < e^{\varepsilon} \quad \forall n > n_0.$$

The lemma is proved. □

The Proposition 5.2.1 in [11] says that a non-empty family $(u_\alpha)_{\alpha \in I}$ of plurisubharmonic functions from the Lelong class such that the set $\{z \in \mathbb{C}^N : \sup_{\alpha \in I} u_\alpha(z) < \infty\}$ is not $\mathcal{L}$-polar is locally uniformly bounded from above.

The next is similar to the above result in the infinite dimensional case.

Theorem 2.2. *Let B be a balanced convex compact subset of a Fréchet space E and $(P_n)_{n \ge 1}$ be a sequence of continuous homogeneous polynomials on E of degree $\le n$. Assume that the set*

$$\left\{ z \in E_B : \sup_{n \ge 1} \frac{1}{n} \log |P_n(z)| < \infty \right\}$$

is not projectively pluripolar in E_B. Then the family $(\frac{1}{n} \log |P_n|)_{n \ge 1}$ is locally uniformly bounded from above on E_B.

Proof. Suppose that the family $(\frac{1}{n} \log |P_n|)_{n \ge 1}$ is not locally uniformly bounded from above on B. Then there exists a sequence $(u_j)_{j \ge 1} = (\frac{1}{n_j} \log |P_{n_j}|)_{j \ge 1}$ such that

$$M_j := \sup_{z \in B} u_j(z) \ge j \quad \forall j \ge 1.$$

Take $w \in E_B \setminus B$ and for each $j \ge 1$ consider the function

$$v_j(\zeta) := u_j(\zeta^{-1} w) - M_j - \log^+(|\zeta|^{-1} \|w\|_B),$$

for $\zeta \in \Delta(\|w\|_B) \setminus \{0\}$. Obviously, v_j is subharmonic and, it is easy to see that $v_j(\zeta) \le O(1)$ as $\zeta \to 0$. Hence, in view of Theorem 2.7.1 in [11], v_j extends to a subharmonic function, say $\widetilde{v}_j$, on $\Delta(\|w\|_B)$. Now, by the maximum principle, $\widetilde{v}_j \le 0$ on $\Delta(\|w\|_B)$. In particular,

$$v_j(1) = \widetilde{v}_j(1) = u_j(w) - M_j - \log^+ \|w\|_B \le 0.$$

Hence

$$u_j(z) - M_j \le \log^+ \|z\|_B \quad \text{for } z \in E_B, \forall j \ge 1. \tag{2.1}$$

Then there exists $z_0 \in E_B$ such that

$$\limsup_{j \to \infty} \exp\left(u_j(z_0) - M_j\right) =: \delta > 0. \tag{2.2}$$

For otherwise we would have

$$\limsup_{j \to \infty} \exp(u_j(z) - M_j) \le 0$$

at each point $z \in E_B$. By Lemma 2.1, the sequence $(\exp(u_j(z) - M_j))_{j \ge 1}$ is bounded from above on any compact set in E_B. This would imply from [10][Lemma 1.1.12] that $\exp\left(u_j(z) - M_j\right) < \frac{1}{2}$ for all $z \in B$ and all sufficiently large j. But then the last estimate would contradict the definition of the constants M_j.

Now we choose a subsequence $(u_{j_k})_{k \ge 1} \subset (u_j)_{j \ge 1}$ such that

$$\lim_{k \to \infty} \exp(u_{j_k}(z_0) - M_{j_k}) = \delta \quad \text{and} \quad M_{j_k} \ge 2^k$$

for all $k \ge 1$. Consider the function

$$w(z) := \sum_{k \ge 1} 2^{-k}(u_{j_k} - M_{j_k}), \quad z \in E_B.$$

In view of (2.1) we have the extimate

$$w_k(z) := 2^{-k}(u_{j_k} - M_{j_k}) - 2^{-k} \log R \le 0$$

for $z \in E_B$, $\|z\|_B \le R$ and $R \ge 1$. Thus w_k is plurisubharmonic on $\{z \in E_B : \ \|z\|_B < R\}$ and $w_k \le 0$. Hence, the function $\sum_{k \ge 1} w_k = w - \log R$, $R > 1$, is either plurisubharmonic on $\{z \in E_B : \ \|z\|_B < R\}$ or identically $-\infty$.

Consequently, as R can be chosen arbitrarily large, w is either plurisubharmonic or identically $-\infty$. Therefore, since $w(z_0) > -\infty$, $w \in PSH(E_B)$. It is easy to see that $w \in HPSH(E_B)$.

If $z \in E_B$, $\sup_{n\geq 1} \frac{1}{n}\log|P_n(z)| < \infty$ then $\sum_{k\geq 1} 2^{-k}u_{j_k}(z) < \infty$ and, hence

$$w(z) \leq \sum_{k\geq 1} 2^{-k}u_{j_k}(z) - \sum_{k\geq 1} 1 = -\infty$$

which proves that the set

$$\Big\{z \in E_B : \sup_{n\geq 1} \frac{1}{n}\log|P_n(z)| < \infty\Big\}$$

is projectively pluripolar in E_B. This contradicts the hypothesis. □

Corollary 2.3. *Let B, E and $(P_n)_{n\geq 1}$ be as in Theorem 2.2; in addition assume that B contains a non-projectively-pluripolar subset. Then the family $(\frac{1}{n}\log|P_n|)_{n\geq 1}$ is locally uniformly bounded from above on E.*

Proof. It suffices to prove that E_B is dense in E. Indeed, if the closure of the subspace E_B is not equal to E then, by the Hahn-Banach theorem, there exists $\varphi \in E'$, $\varphi \neq 0$, such that $\varphi(E_B) = 0$. Then it is easy to see that $v := \log|\varphi| \in HPSH(E)$, $v \not\equiv 0$, $B \subset E_B \subset \{z : \ v(z) = -\infty\}$. This contradicts the fact that B contains a non-projectively-pluripolar subset. □

It is known that a subset with non-empty interior in a Fréchet space is not pluripolar, hence it is not projectively pluripolar. Then by Corollary 2.3 we have the following.

Corollary 2.4. *Let B be a balanced convex compact subset of a Fréchet space E which contains a non-projectively-pluripolar subset and $(P_n)_{n\geq 1}$ be a sequence of continuous homogeneous polynomials on E of degree $\leq n$. If the set*

$$\Big\{z \in E_B : \sup_{n\geq 1} \frac{1}{n}\log|P_n(z)| < \infty\Big\}$$

has the non-empty interior in E then the family $(\frac{1}{n}\log|P_n|)_{n\geq 1}$ is locally uniformly bounded from above on E.

We are now in a position to prove the first main theorem.

Proof of Theorem 1.1. Let F be a Fréchet space, $f : \Delta_n \to F$ be a function which belongs to C^k-class at $0 \in \mathbb{C}^n$ for $k \geq 0$ and $A \subset \mathbb{C}^n$ be a non-projectively-pluripolar set. If the restriction of f on each complex line ℓ_a, $a \in A$, is holomorphic. By the hypothesis, for each $k \geq 0$ there exists $r_k \in (0,1)$ such that f is a C^k-function on $\Delta_N(r_k)$. We may assume that $r_k \searrow 0$. Put

$$P_k(z) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(\lambda z)d\lambda}{\lambda^{k+1}}, \quad z \in \Delta_N(r_k).$$

Then, for each $k \geq 0$ and $p \geq k$, P_m is a bounded C^p-function on $\Delta_N(r_p)$. Since $\lambda \mapsto f(\lambda a)$ is holomorphic for all $a \in A$ we deduce that

$$P_k(\lambda a) = \lambda^k P_k(a) \quad \text{for } a \in A, \lambda \in \mathbb{C}. \quad (2.3)$$

By the boundedness of P_k on $\Delta_N(r_k)$ we have

$$P_k(w) = O(|w|^k) \quad \text{as } w \to 0.$$

On the other hand, since $P_k \in C^{k+1}(\Delta_N(r_{k+1}))$, the Taylor expansion of P_k at $0 \in \Delta_N(r_{k+1})$ has the form

$$P_k(z) = \sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z) + |z|^k\varrho(z) \quad (2.4)$$

where $P_{k,\alpha,\beta}$ is a polynomial of degree α in z and degree β in $\overline{z}$ and $\varrho(z) \to 0$ as $z \to 0$.

In (2.4), replacing z by λz, $|\lambda| < 1$, from (2.3) we obtain

$$\sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z)\lambda^\alpha\overline{\lambda}^\beta + |\lambda|^k|z|^k\varrho(\lambda z)$$

$$= \sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z)\lambda^k + \lambda^k|z|^k\varrho(z) \quad (2.5)$$

for $z \in r_{k+1}A$.

This yields that $\varrho(\lambda z) = \varrho(z)$ for $\lambda \in [0,1)$, and hence, $\varrho(z) = \varrho(0) = 0$ for $z \in r_{k+1}A$. Thus

$$P_{k,\alpha,\beta}(z) = 0 \quad \text{for } \beta > 0 \text{ and } z \in r_{k+1}A.$$

Note that $r_{k+1}A$ is also not projectively pluripolar. It is easy to check that

$$P_{k,\alpha,\beta} = 0 \quad \text{for } \beta > 0.$$

Indeed, for every $\varphi \in F'$, the function

$$u(w) = \frac{1}{\deg P_{k,\alpha,\beta}} \log |(\varphi \circ P_{k,\alpha,\beta})(w)|$$

is homogeneous plurisubharmonic on $\mathbb{C}^N$, $u \equiv -\infty$ on $r_{k+1}A$. Since $r_{k+1}A$ is not projectively pluripolar, it implies that $u \equiv -\infty$ and hence $\varphi \circ P_{k,\alpha,\beta} \equiv 0$ on $\mathbb{C}^N$ for every $\varphi \in F'$. It implies that $P_{k,\alpha,\beta} \equiv 0$ on $\mathbb{C}^N$ for $\beta > 0$.

Thus, from (2.4) we have

$$P_k(z) = P_{k,k,0}(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$$

for $z \in \Delta_N(r_{k+1})$ and P_k is a homogeneous holomorphic polynomial of degree k.

Now, let $(\|\cdot\|_m)_{m\geq 1}$ be an increasing fundamental system of continuous semi-norms defining the topology of F. By the hypothesis, for every $m \geq 1$

$$\limsup_{k\to\infty} \frac{1}{k} \log \|P_k(z)\|_m = -\infty \quad \text{for } z \in A.$$

Then, by Corollary 2.3, the sequence $(\frac{1}{k} \log \|P_k(z)\|_m)_{k\geq 1}$ is locally uniformly bounded from above on $\mathbb{C}^n$ for all $m \geq 1$. Thus we can define

$$u_m(z) = \limsup_{k\to\infty} \frac{1}{k} \log \|P_k(z)\|_m, \quad z \in \mathbb{C}^N.$$

By [12] the upper semicontinuous regularization u_m^* of u_m belongs to the Lelong class $\mathcal{L}(\mathbb{C}^N)$ of plurisubharmonic functions with logarithmic growth on $\mathbb{C}^N$. Moreover, by Bedford-Taylor's theorem [13]

$$S_m := \{z \in \mathbb{C}^N : \ u_m^*(z) \neq u_m(z)\}$$

is pluripolar for all $m \geq 1$.

On the other hand, by [3], $A^* := \{ta : \ t \in \mathbb{C}, a \in A\}$ is not pluripolar. This yields that $u_m^* \equiv -\infty$ for all $m \geq 1$ because $u_m^* = u_m = -\infty$ on $A^* \setminus S_m$ and $A^* \setminus S_m$ is non-pluripolar. Since $u_m^* \geq u_m$ we have $u_m \equiv -\infty$ for $m \geq 1$. Hence the series $\sum_{k\geq 0} P_k(z)$ is convergent for $z \in \mathbb{C}^N$ and it defines a holomorphic extension $\widehat{f}$ of $f\big|_{\ell_a}$ for every $a \in A$. $\square$

3. THE CONVERGENCE OF A FORMAL POWER SERIES BETWEEN FRÉCHET SPACES

In mathematics, a formal power series is a generalization of polynomials as a formal object, where the number of terms is allowed to be infinite.

The theory of formal power series has drawn attention of mathematicians working in different branches because of their various applications. One can find applications of formal power series in classical mathematical analysis and in the theory of Riordan algebras. Specially, this theory lays the foundation for substantial parts of combinatorics and real and complex analysis.

A formal power series $f(z_1, \dots, z_N) = \sum c_{\alpha_1,\dots,\alpha_N} z_1^{\alpha_1} \dots z_N^{\alpha_N}$ in $\mathbb{C}^N$, $N \geq 2$, with coefficients in $\mathbb{C}$ is said to be convergent if it converges absolutely in a zero-neighborhood in $\mathbb{C}^N$. A classical result of Hartogs [14] states that a series f converges if and only if $f_z(t) = f(tz_1, \dots, tz_N)$ converges, as a series in t, for all $z = (z_1, \dots, z_N) \in \mathbb{C}^N$. In other words, a formal power series in several complex variables is convergent if it converges on *all* lines through the origin. This can be interpreted as a formal analog of Hartogs' theorem on separate analyticity. Because a divergent power series still may converge in certain directions, it is natural and desirable to consider the set of all $z \in \mathbb{C}^N$ for which f_z converges. Since $f_z(t)$ converges if and only if $f_w(t)$ converges for all $w \in \mathbb{C}^N$ on the affine line through z, ignoring the trivial case $z = 0$, the set of directions along which f converges can be identified with a subset of the projective space $\mathbb{CP}^{N-1}$. The convergence set $\mathrm{Conv}(f)$ of a divergent power series f is defined to be the set of all directions $\xi \in \mathbb{CP}^{N-1}$ such that $f_z(t)$ is convergent for some $z \in \varrho^{-1}(\xi)$ where

$\varrho : \mathbb{C}^N \setminus \{0\} \to \mathbb{CP}^{N-1}$ is the natural projection. In the two-variables case, Lelong [15] proved that Conv(f) is an F_σ-polar set (i.e. a countable union of closed sets of vanishing logarithmic capacity) in $\mathbb{CP}^1$, and moreover, every F_σ-polar subset of $\mathbb{CP}^1$ is contained in the Conv(g) of some formal power series g. The optimal result was later obtained by Sathaye [16] who showed that the class of convergence sets of divergent power series in two-variables is precisely the class of F_σ-polar sets in $\mathbb{CP}^1$. Levenberg and Molzon, in [17], showed that if the restriction of f on sufficiently many (non-pluripolar) sets of complex line passing through the origin is convergent on small neighborhood of $0 \in \mathbb{C}$ then f is actually represent a holomorphic function near $0 \in \mathbb{C}^N$. By using delicate estimates on volume of complex varieties in projective spaces, Alexander's theorem mentioned above was proved. This follows readily that if the restriction of a formal power series f on every complex line passing through the origin in $\mathbb{C}^N$ is convergent then f is convergent [2] [Theorem 6.3].

The main result of this section is following.

Theorem 3.1. *Let A be a non-projectively-pluripolar set which is contained in a balanced convex compact subset of a Fréchet space E and $f = \sum_{n\geq 1} P_n$ be a formal power series where P_n are continuous homogeneous polynomials of degree n on E with values in a Fréchet space F. Assume that for each $a \in A$, the restriction of f on the complex line ℓ_a is convergent. Then f is convergent in a neighbourhood of $0 \in E$ if one of the following holds:*

(a) E is Schwartz.

(b) $F \in (LB_\infty)$.

Proof. We divide the proof into three steps:

(i) *Step 1: We consider the case where* $F = \mathbb{C}$. It follows from the hypothesis that

$$\limsup_{n\to\infty} |P_n(z)|^{\frac{1}{n}} < \infty \quad \forall z \in A.$$

Then, by Corollary 2.3 there exists a zero-neighbourhood U in E such that

$$\sup\{|P_n(z)|^{\frac{1}{n}} : \ z \in U, n \geq 1\} =: M < \infty.$$

This implies that f is uniformly convergent on $(2M)^{-1}U$.

(ii) *Step 2: We consider the case where F is Fréchet.* By the step 1 we can define the linear map

$$T : F'_{\rm bor} \to H(0_E)$$

by letting

$$T(u) = \sum_{n\geq 1} u(P_n)$$

where $H(0_E)$ denotes the space of germs of scalar holomorphic functions at $0 \in E$. Suppose that $u_\alpha \to u$ in $F'_{\rm bor}$ and $T(u_\alpha) \to v$ in $H(0_E)$ as $\alpha \to \infty$. This implies, in particular, that $[T(u_\alpha)](z) \to v(z)$ for all z in some zero-neighbourhood U in E. However, for $z \in U$ we have

$$\begin{aligned}
[T(u_\alpha - u)]\,(z) &= \sum_{n\geq 1}(u_\alpha - u)(P_n(z)) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(u_\alpha - u)(P_n(z)) \\
&= (u_\alpha - u)\Big(\lim_{n\to\infty}\sum_{k=1}^{n} P_n(z)\Big) \\
&= (u_\alpha - u)\Big(\sum_{n\geq 1} P_n(z)\Big).
\end{aligned}$$

Then $[T(u_\alpha)](z) \to [T(u)](z)$ for all $z \in U$. This implies that $v = T(u)$. Hence T has a closed graph.

Meanwhile, since F is Fréchet, by [7] [Theorem 13.4.2] we have $\beta(F', F)_{\rm bor} = \eta(F', F)$ on F', where $\eta(F', F)$ is the corresponding locally convex inductive limit topology on $F' = \bigcup_{U\in\mathscr{U}} F'_{U^\circ}$ with $\mathscr{U}$ consists of closed and absolutely convex sets in F. This implies that F'_{bor} is ultrabornological. On the other hand, because E is metrizable, we have

$$H(0_E) = \varinjlim_{n\to\infty} (H^\infty(V_n), \|\cdot\|_n)$$

where $(V_n)_{n\geq}$ is a countable fundamental neighbourhood system at $0 \in E$, and $\|\cdot\|_n$ is the norm on the Banach space $H^\infty(V_n)$ given by $\|f\|_n = \sup_{z\in V_n} |f(z)|$. Hence, by the closed graph theorem of Grothendieck [18] [Introduction, Theorem B], T is continuous.

Next, we shall show that there exists a neighbourhood V of $0 \in E$ such that $T : F'_{\text{bor}} \to H^\infty(V)$ is continuous linear.

We consider two cases: (a) E is Schwartz; (b') E is Banach and $F \in (LB_\infty)$.

The case (a): Since E is Schwartz, by [19] [Theorem 2 and Corollary 9], $H(0_E)$ has a continuous norm. Using Proposition 1.4 in [20] we deduce that there exists a neighbourhood V of $0 \in E$ such that $T : F'_{\text{bor}} \to H^\infty(V)$ is continuous linear.

The case (b'): Since E is Banach, it follows from [21] [Theorem 1] that $H(0_E) \in (\Omega)$. Then, because $(F'_{\text{bor}})'_\beta \in (LB_\infty)$, using Theorem 3.2 in [5] we deduce that there exists a neighbourhood V of $0 \in E$ such that $T : F'_{\text{bor}} \to H^\infty(V)$ is continuous linear.

Now we define the map $\widehat{f} : V \to F''_{\text{bor}}$ by the formula

$$[\widehat{f}(z)](u) = [T(u)](z), \quad z \in V,\ u \in F'_{\text{bor}}.$$

Since T is continuous and point evaluations on $H(V)_{\text{bor}}$ are continuous (see [9] [Proposition 3.19]) it follows that $\widehat{f}(z) \in F''_{\text{bor}}$ for all $z \in V$. Moreover, for each fixed $u \in F'_{\text{bor}}$ the mapping

$$z \in V \mapsto [T(u)](z)$$

is holomorphic, that is

$$\widehat{f} : V \to (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$$

is a continuous mapping. For all $a \in V, b \in E$ and all $u \in F'_{\text{bor}}$ the mapping

$$\{t \in \mathbb{C} :\ a + tb \in V\} \ni \lambda \mapsto u \circ \widehat{f}(a + \lambda b)$$

is a Gâteaux holomorphic mapping and hence

$$\widehat{f} : V \to (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$$

is holomorphic.

By [7] [8.13.2 and 8.13.3], F'_{bor} is a complete locally convex space. Hence by [22] [Theorem 4, p.210] applied to the complete space F'_{bor} we see that $(F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$ and $(F'_{\text{bor}})'_\beta$ have the same bounded sets. An application of [23] [Proposition 13] shows that

$$\widehat{f} : V \to (F'_{\text{bor}})'_\beta$$

is holomorphic.

Let j denote the canonical injection from F into F''. If $z \in B := V \cap \{ta :\ t \in \mathbb{C}, a \in A\}$ and $\widehat{f}(z) \neq j(f(z))$ then there exists $u \in F'$ such that

$$\widehat{f}(z)(u) \neq j(f(z))(u) = u(f(z)).$$

This, however, contradicts the fact that for all $z \in B$ we have

$$\widehat{f}(z)(u) = [T(u)](z) = \sum_{n\geq 1} u(P_n)(z) = u(f(z)).$$

We now fix a non-zero $z \in B$. Then there exists a unique sequence in F'', $(a_{n,z})_{n=1}^\infty$, such that for all $\lambda \in \mathbb{C}$

$$\widehat{f}(\lambda z) = \sum_{n=0}^\infty a_{n,z}\lambda^n.$$

Since $\widehat{f}(0) = f(0) = a_{0,z}$ it follows that $a_{0,z} \in F$. Now suppose that $(a_{j,z})_{j=0}^n \subset F$. When $|\lambda| \leq 1$, $\widehat{f}(\lambda z) = f(\lambda z) \in F$. Hence, if $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, then

$$\frac{\widehat{f}(\lambda z) - \sum_{j=0}^n a_{j,z}\lambda^j}{\lambda^{n+1}} = \sum_{j=n+1}^\infty a_{j,z}\lambda^{j-n-1} \in F.$$

Since F is complete we see, on letting λ tend to 0, that $a_{n+1,z} \in F$. By induction $a_{n,z} \in F$ for all n and hence $\widehat{f}(\lambda z) \in F$ for all $\lambda \in \mathbb{C}$ and all $z \in B$. Since $\widehat{f}$ is continuous and F is a closed subspace of $(F'_{\text{bor}})'_\beta$ (see [24] [Lemma 2.1]) we have shown that $\widehat{f} : V \to F$ is holomorphic.

Hence, the series $\sum_{n\geq 1} P_n$ is convergent on V to f.

The proof of the case (a) is complete here. We continue the last step for the proof of the case (b) as follows:

(iii) *Step 3:* Let $\{U_n\}_{n\geq}$ be a decreasing basis of neighbourhoods of $0 \in E$. By $\mathcal{K}(E)$ we denote the family of all balanced convex compact subsets of E. By the case (b') in Step 2, for each $K \in \mathcal{K}(E)$ there exists $\varepsilon_K > 0$ such that f is uniformly convergent on $\varepsilon_K K$. Put

$$W = \bigcup_{K\in\mathcal{K}(E)} \varepsilon_K K.$$

Obviously, f is convergent on W. It remains to check that W is a neighbourhood of $0 \in E$. Assume the contrary, that W is not a neighbourhood of $0 \in E$. Then for each $n \geq 1$ there exists $x_n \in U_n \setminus W$. Put

$$K_0 := \overline{\text{conv}}\{0, x_1, x_2, \ldots\}.$$

By [25] [Corollary 6.5.4] we can find $K_1 \in \mathcal{K}(E)$, $K_0 \subset K_1$, such that K_0 is relatively compact in E_{K_1}. It implies that $x_n \to 0$ in E_{K_1}. Thus there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have $x_n \in \varepsilon_{K_1} K_1 \subset W$. This is incompatible with x_n being disjoint from W. □

4. ALEXANDER'S THEOREM FOR FRÉCHET-VALUED FORMAL POWER SERIES

We will present the proof of Theorem 1.2 in this section. Our work requires some extra results concerning to Vitali's theorem for a sequence of Fréchet-valued holomorphic functions.

Remark 4.1. In exactly the same way, Theorem 2.1 in [26] is true for the Fréchet-valued case.

Lemma 4.1. *Let E, F be Fréchet spaces, $D \subset E$ be an open set. Let $f : D \to F$ be a locally bounded function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W \subset F'$, where W is separating. Then f is holomorphic.*

The proof of Lemma runs as in the proof of Theorem 3.1 in [26], but here we use Vitali's theorem in [27] [Proposition 6.2] which states for a sequence of holomorphic functions on an open connected subset of a locally convex space.

Lemma 4.2. *Let D be a domain in a Fréchet space E and $f : D \to F$ be holomorphic, where F is a barrelled locally convex space. Assume that $D_0 = \{z \in D : f(z) \in G\}$ is not nowhere dense in D, where G is a closed subspace of F. Then $f(z) \in G$ for all $z \in D$.*

Proof. (i) We first consider the case $G = \{0\}$. On the contrary, suppose that $f(z^*) \neq 0$ for some $z^* \in D \setminus D_0$. By the Hahn-Banach theorem, we can find $\varphi \in F'$ such that $(\varphi \circ f)(z^*) \neq 0$. Let $z_0 \in (\text{int}\overline{D_0}) \cap D$ and let W be a balanced convex neighbourhood of $0 \in E$ such that $z_0 + W \subset \overline{D_0}$. Then by the continuity of f we deduce that $f = 0$ on $z_0 + W$. Hence, it follows from the identity theorem (see [27] [Proposition 6.6]) that $f = 0$ on D. This contradicts above our claim $(\varphi \circ f)(z^*) \neq 0$.

(ii) For the general case, consider the quotient space F/G and the holomorphic function $\omega \circ f : D \to F/G$ where $\omega : F \to F/G$ is the canonical map. Then $\omega \circ f \equiv 0$ on D_0. By the case (i), $\omega \circ f \equiv 0$ on D. This means that $f(z) \in G$ for all $z \in D$. □

Proposition 4.3. *Let E, F be Fréchet spaces and $D \subset E$ a domain. Assume that $(f_n)_{n\geq 1}$ is a locally bounded sequence of holomorphic functions on D with values in F. Then the following assertions are equivalent:*

(i) *The sequence $(f_n)_{n\geq 1}$ converges uniformly on all compact subsets of D to a holomorphic function $f : D \to F$;*

(ii) *The set* $D_0 = \{z \in D : \lim_n f_n(z)$ *exists*$\}$ *is not nowhere dense in* D.

Proof. It suffices to prove the implication (ii) $\Rightarrow$ (i) because the case (i) $\Rightarrow$ (ii) is trivial. Define $\widetilde{f} : D \to \ell^\infty(\mathbb{N}, F)$ by $\widetilde{f}(z) = (f_n(z))_{n\geq 1}$, where $\ell^\infty(\mathbb{N}, F)$ is the Fréchet space with the topology induced by the system of semi-norms

$$\|x\|_k = \|(x_i)_{i\geq 1}\|_k = \sup_i \|x_i\|_k, \ \forall k,$$

$\forall x = (x_i)_{i\geq 1} \in \ell^\infty(\mathbb{N}, F)$.

For each $k \in \mathbb{N}$ we denote $pr_k : \ell^\infty(\mathbb{N}, F) \to F$ is the k-th projection with $pr_k((w_i)_{i\in\mathbb{N}}) = w_k$. Obviously

$$W = \{\varphi \circ pr_k;\ \varphi \in F',\ k \in \mathbb{N}\} \subset \ell^\infty(\mathbb{N}, F)'$$

is separating and

$$\varphi \circ pr_k \circ \widetilde{f} = \varphi \circ pr_k \circ (f_n)_{n\geq 1} = \varphi \circ f_k$$

is holomorphic for every $k \geq 1$. Then by Lemma 4.1, $\widetilde{f}$ is holomorphic.

Since the space

$$G = \{(w_i)_{i\geq 1} \in \ell^\infty(\mathbb{N}, F) : \lim_{i\to\infty} w_i \text{ exists}\}$$

is closed, by the hypothesis, $\widetilde{f}(z) \in G$ for all $z \in D_0$. It follows from Lemma 4.2 that $\widetilde{f}(z) \in G$ for all $z \in D$. Thus $f(z) = \lim_{i\to\infty} f_i(z)$ exists for all $z \in D$. Note that $\Phi : G \to F$ given by $\Phi((y_i)_{i\in\mathbb{N}}) = \lim_{i\to\infty} y_i$ defines a bounded operator. Therefore $f = \Phi \circ \widetilde{f}$ is holomorphic.

Finally, in order to prove that $(f_i)_{i\geq 1}$ converges uniformly on compact sets in D to f, it suffices to show that $(f_i)_{i\geq 1}$ is locally uniformly convergent in D to f. Since $(f_i)_{i\geq 1}$ is locally bounded, by [27] [Proposition 6.1], $(f_i)_{i\geq 1}$ is equicontinuous at every $a \in D$. Let a be fixed point of D. Then for every balanced convex neighbourhood V of 0 in F there exists a neighbourhood U_a^1 of a in D such that

$$f_i(z) - f_i(a) \in 3^{-1}V,\ \forall z \in U_a^1,\ \forall i \geq 1. \quad (4.1)$$

Since $\lim_{i\to\infty} f_i = f$ in D, we can find $i_0 \geq 1$ such that

$$f_i(a) - f(a) \in 3^{-1}V, \quad \forall i \geq i_0. \quad (4.2)$$

By the continuity of f, there exists a neighbourhood U_a^2 of a in D such that

$$f(a) - f(z) \in 3^{-1}V, \quad \forall z \in U_a^2. \quad (4.3)$$

From (4.1), (4.2) and (4.3), for all $z \in U_a = U_a^1 \cap U_a^2$ for all $i \geq i_0$ we have

$$f_i(z) - f(z) \in V. \quad (4.4)$$

The proof of the proposition is complete. $\square$

We now can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. As in the proof of Theorem 3.1, for each $n \geq 1$, define the continuous linear map $T_n : F'_{\text{bor}} \to H(0_{\mathbb{C}^N})$ given by

$$T_n(u) = u \circ f_n, \quad u \in F'_{\text{bor}}.$$

By Theorem 3.5 in [3], the sequence $(T_n(u))_{n\geq 1}$ converges in $H(0_{\mathbb{C}^N})$ for every $u \in F'_{\text{bor}}$. Since F'_{bor} is barrelled (see [7] [13.4.2]) it follows that the sequence $(T_n)_{n\geq 1}$ is equicontinuous in $L(F'_{\text{bor}}, H(0_{\mathbb{C}^N}))$ equipped with the strong topology. As in the proof of Theorem 3.1, by Theorem 3.2 in [5] we deduce that there exists a neighbourhood U of $0 \in F'_{\text{bor}}$ such that

$$\bigcup_{n\geq 1} T_n(U)$$

is bounded in $H(0_{\mathbb{C}^N})$. By the regularity of $H(0_{\mathbb{C}^N})$, we can find $r \in (0, r_0)$ such that $\bigcup_{n\geq 1} T_n(U)$ is contained and bounded in $H^\infty(\Delta_N(r))$. This yields that $(f_n)_{n\geq 1}$ is contained and bounded in $H^\infty(\Delta_N(r), F)$. Since for each $z \in \Delta_N(r)$ the sequence $(f_n|_{\ell_z})_{n\geq 1}$ is convergent in $\Delta_1(r_0) \subset \ell_z$, by Remark 4.1, the sequence $(f_n(z))_{n\geq 1}$ is convergent for every $z \in \Delta_N(r)$. On the other hand, because $(f_n)_{n\geq 1}$ is bounded in $H^\infty(\Delta_N(r), F)$, by Proposition 4.3 it follows that the sequence $(f_n)_{n\geq 1}$ is convergent in $H(\Delta_N(r), F)$. $\square$

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