# Dạng tích phân cho một số mở rộng của bất đẳng thức Aczél


## TÓM TẮT

Bất đẳng thức Aczél xuất hiện lần đầu tiên vào năm 1956. Kể từ đó, nó đã thu hút sự quan tâm của nhiều nhà toán học. Từ đó các kết quả mở rộng và ứng dụng của bất đẳng thức này đã được công bố. Trong bài báo này, chúng tôi trình bày các phiên bản tích phân cho một số mở rộng của bất đẳng thức Aczél. Qua đó, chúng tôi thu được dạng tích phân cho các bất đẳng thức Aczél và Bellman.



**Từ khóa:** *Bất đẳng thức dạng Aczél, bất đẳng thức Popoviciu, bất đẳng thức Bellman.*

# Integral versions of some generalizations of Aczél's inequality


## ABSTRACT

Aczél inequality was first proposed in 1956. Then it has been considered by many mathematicians. Thus its generalizations and applications were published. In this paper we establish integral versions of some generalizations of Aczél's inequality. As a consequence, we obtain integral types of Aczél's inequality and Bellman's inequality.



**Keywords:** *Aczél-type inequality, Bellman's inequality, Popoviciu's inequality.*


## 1. INTRODUCTION

In 1956, a famous result of J. Aczél was published in[1] stated as follows.

**Theorem A** ([1])**.** *If $a_1,\ a_2,\dots,a_n,\ b_1,\ b_2,\dots,b_n$ are positive real numbers such that $a_1^2 > a_2^2 + a_3^2 + \cdots + a_n^2$ and $b_1^2 > b_2^2 + b_3^2 + \cdots + b_n^2$, then*

$$\left. a_1^2 - \sum_{i=2}^{n} a_i^2 \right) \left( b_1^2 - \sum_{i=2}^{n} b_i^2 \right) \leqslant \left( a_1 b_1 - \sum_{i=2}^{n} a_i b_i \right)^2. \tag{1}$$

Inequality (1) was later called 'Aczél's inequality'. In 1959, the first extension of (1) was provided by Popoviciu[3] and later called 'Popoviciu's inequality' stated as follows.

**Theorem B** ([3])**.** *Let $p$, $q$ be positive real numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and let $a_1,\dots,a_n,\ b_1,\dots,b_n$ be positive real numbers such that $a_1^p > a_2^p + \cdots + a_n^p$ and $b_1^q > b_2^q + \cdots + b_n^q$. Then*

$$\left. a_1^p - \sum_{i=2}^{n} a_i^p \right)^{1/p} \left. b_1^q - \sum_{i=2}^{n} b_i^q \right)^{1/q} \leqslant a_1 b_1 - \sum_{i=2}^{n} a_i b_i. \tag{2}$$

The next result is the famous Bellman's inequality. Although this inequality was discovered in 1934 by Hardy et al.[2], it is also considered as a Aczél-type inequality. Let us recall this inequality.

**Theorem C** ([2])**.** *Let $a_1,\dots,a_n,\ b_1,\dots,b_n$ be positive real numbers and $p > 1$. If $a_1^p > a_2^p + \cdots + a_n^p$ and $b_1^p > b_2^p + \cdots + b_n^p$, then*

$$\left. a_1^p - \sum_{i=2}^{n} a_i^p \right)^{1/p} + \left. b_1^p - \sum_{i=2}^{n} b_i^p \right)^{1/p}$$

$$\leqslant \left[(a_1+b_1)^p - \sum_{i=2}^{n}(a_i+b_i)^p\right]^{1/p}. \tag{3}$$

Recently, some generalizations of inequalities (2) and (3) are presented by Chang-Jian Zhao and Wing-Sum Cheung[4]. These results are stated as the following theorems.

**Theorem D** ([4])**.** *Let $p$, $q$ be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and let $a_i, b_i, a_{ji}, b_{ji}$ $(i=1,\ldots,n, j=1,\ldots,m)$ be positive real numbers such that*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right) - \left(\sum_{j=1}^{m} a_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} a_{ji}^p\right) > 0,$$

$$\left(b_1^q - \sum_{i=2}^{n} b_i^q\right) - \left(\sum_{j=1}^{m} b_{j1}^q - \sum_{j=1}^{m}\sum_{i=2}^{n} b_{ji}^q\right) > 0,$$

$$\frac{a_{j1}}{b_{j1}} = \cdots = \frac{a_{jn}}{b_{jn}}, \qquad j=1,\ldots,m.$$

*Then*

$$\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right) - \left(\sum_{j=1}^{m} a_{j1}b_{j1} - \sum_{j=1}^{m}\sum_{i=2}^{n} a_{ji}b_{ji}\right)$$
$$\geqslant \left[\left(a_1^p - \sum_{i=2}^{n} a_i^p\right) - \left(\sum_{j=1}^{m} a_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} a_{ji}^p\right)\right]^{1/p}$$
$$\times \left[\left(b_1^q - \sum_{i=2}^{n} b_i^q\right) - \left(\sum_{j=1}^{m} b_{j1}^q - \sum_{j=1}^{m}\sum_{i=2}^{n} b_{ji}^q\right)\right]^{1/q} \tag{4}$$

**Theorem E** ([4])**.** *Let $p \geqslant 1$, $a_i, b_i, a_{ji}, b_{ji}$ $(j=1,\ldots,m, i=1,\ldots,n)$ be positive real numbers such that*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right) - \left(\sum_{j=1}^{m} a_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} a_{ji}^p\right) > 0,$$

$$\left(b_1^p - \sum_{i=2}^{n} b_i^p\right) - \left(\sum_{j=1}^{m} b_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} b_{ji}^p\right) > 0,$$

$$\frac{a_{j1}}{b_{j1}} = \frac{a_{j2}}{b_{j2}} = \cdots = \frac{a_{jn}}{b_{jn}}, \qquad j=1,2,\ldots,m.$$

*Then*

$$\left[\left((a_1+b_1)^p - \sum_{i=2}^{n}(a_i+b_i)^p\right) - \left(\sum_{j=1}^{m}(a_{j1}+b_{j1})^p - \sum_{j=1}^{m}\sum_{i=2}^{n}(a_{ji}+b_{ji})^p\right)\right]^{1/p}$$
$$\geqslant \left[\left(a_1^p - \sum_{i=2}^{n} a_i^p\right) - \left(\sum_{j=1}^{m} a_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} a_{ji}^p\right)\right]^{1/p}$$
$$\times \left[\left(b_1^p - \sum_{i=2}^{n} b_i^p\right) - \left(\sum_{j=1}^{m} b_{j1}^p - \sum_{j=1}^{m}\sum_{i=2}^{n} b_{ji}^p\right)\right]^{1/p}. \tag{5}$$

In the present paper we establish integral versions for inequalities (4) and (5). As a result, respective integral versions of inequalities (1) and (3) are obtained.

## 2. MAIN RESULTS

We first establish an integral version of inequality (4) in Theorem D as follows.

**Theorem 2.1.** *Let $A, B$, $A_j, B_j$ $(j=1,\ldots,m)$ be positive real numbers. Let $f$, $g$, $f_j, g_j$ $(j=1,\ldots,m)$ be positive Riemann integrable functions on $[a,b]$ such that*

$$\left(A^2 - \int_a^b f^2(x)dx\right) - \left(\sum_{j=1}^{m} A_j^2 - \sum_{j=1}^{m}\int_a^b f_j^2(x)dx\right) > 0, \tag{6}$$

$$\left(B^2 - \int_a^b g^2(x)dx\right) - \left(\sum_{j=1}^{m} B_j^2 - \sum_{j=1}^{m}\int_a^b g_j^2(x)dx\right) > 0, \tag{7}$$

$$\frac{f_j(x)}{g_j(x)} = \frac{A_j}{B_j}, \qquad \forall x \in [a,b],\ j=1,2,\ldots,m. \tag{8}$$

*Then*

$$\left[\left(AB-\int_a^b f(x)g(x)dx\right)-\right.$$
$$\left.-\left(\sum_{j=1}^m A_jB_j-\sum_{j=1}^m\int_a^b f_j(x)g_j(x)dx\right)\right]^2$$
$$\geqslant\left[\left(A^2-\int_a^b f^2(x)dx\right)-\left(\sum_{j=1}^m A_j^2-\sum_{j=1}^m\int_a^b f_j^2(x)dx\right)\right]$$
$$\times\left[\left(B^2-\int_a^b g^2(x)dx\right)-\left(\sum_{j=1}^m B_j^2-\sum_{j=1}^m\int_a^b g_j^2(x)dx\right)\right]. \tag{9}$$

*Proof.* For any positive integer $n$, we choose an equidistant partition of $[a,b]$ by $n+1$ points

$$x_0<x_1<\cdots<x_n,$$

with $x_0=a$, $x_i=a+i\,\dfrac{b-a}{n}$, $\Delta x=x_i-x_{i-1}=\dfrac{b-a}{n}$, $i=1,2,\ldots,n$. Due to (6) and (7), it follows that there exists a positive integer number $N$ such that for all $n>N$ we have

$$\left(A^2-\sum_{i=1}^n f^2\left(x_{i-1}\right)\Delta x\right)-$$
$$-\left(\sum_{j=1}^m a_j^2-\sum_{j=1}^m\sum_{i=1}^n f_j^2\left(x_{i-1}\right)\Delta x\right)>0,$$

and

$$\left(B^2-\sum_{i=1}^n g^2\left(x_{i-1}\right)\Delta x\right)-$$
$$-\left(\sum_{j=1}^m b_j^2-\sum_{j=1}^m\sum_{i=1}^n g_j^2\left(x_{i-1}\right)\Delta x\right)>0.$$

It follows from (8) that

$$\frac{f_j\left(a+(i-1)\dfrac{b-a}{n}\right)\left(\dfrac{b-a}{n}\right)^{1/2}}{g_j\left(a+(i-1)\dfrac{b-a}{n}\right)\left(\dfrac{b-a}{n}\right)^{1/2}}=\frac{A_j}{B_j},$$

for $j=1,2,\ldots,m$ and $i=2,3\ldots,n$. Applying Theorem D with

$$p=q=\frac{1}{2},\ a_1=A,\ b_1=B,\ a_{j1}=A_j,\ b_{j1}=B_j,$$
$$a_i=f\left(a+(i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)^{1/2},$$
$$b_i=g\left(a+(i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)^{1/2},$$
$$a_{ji}=f_j\left(a+(i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)^{1/2},$$
$$b_{ji}=g_j\left(a+(i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)^{1/2}$$

for $j=1,\ldots,m,\ i=1,\ldots,n$, we obtain

$$\left(AB-\sum_{i=1}^n f\left(x_{i-1}\right)g\left(x_{i-1}\right)(\Delta x)^{\frac12+\frac12}\right)-$$
$$-\left(\sum_{j=1}^m A_jB_j-\sum_{j=1}^m\sum_{i=1}^n f_j\left(x_{i-1}\right)g_j\left(x_{i-1}\right)(\Delta x)^{\frac12+\frac12}\right)$$
$$\geqslant\left[\left(A^2-\sum_{i=1}^n f^2\left(x_{i-1}\right)\Delta x\right)-\right.$$
$$\left.-\left(\sum_{j=1}^m A_j^2-\sum_{j=1}^m\sum_{i=1}^n f_j^2\left(x_{i-1}\right)\Delta x\right)\right]^{1/2}$$
$$\times\left[\left(B^2-\sum_{i=1}^n g^2\left(x_{i-1}\right)\Delta x\right)-\right.$$
$$\left.-\left(\sum_{j=1}^m B_j^2-\sum_{j=1}^m\sum_{i=1}^n g_j^2\left(x_{i-1}\right)\Delta x\right)\right]^{1/2}.$$

Hence

$$\left(AB-\sum_{i=1}^n f\left(x_{i-1}\right)g\left(x_{i-1}\right)\Delta x-\right)$$
$$-\left(\sum_{j=1}^m A_jB_j-\sum_{j=1}^m\sum_{i=1}^n f_j\left(x_{i-1}\right)g_j\left(x_{i-1}\right)\Delta x\right)$$
$$\geqslant\left[\left(A^2-\sum_{i=1}^n f^2\left(x_{i-1}\right)\Delta x\right)-\right.$$
$$\left.-\left(\sum_{j=1}^m A_j^2-\sum_{j=1}^m\sum_{i=1}^n f_j^2\left(x_{i-1}\right)\Delta x\right)\right]^{1/2}$$
$$\times\left[\left(B^2-\sum_{i=1}^n g^2\left(x_{i-1}\right)\Delta x\right)-\right.$$

$$-\left(\sum_{j=1}^{m} B_j^2-\sum_{j=1}^{m}\sum_{i=1}^{n} g_j^2\left(x_{i-1}\right)\Delta x\right)\Bigg]^{1/2}. \tag{9A}$$

Since $f,\ g,\ f_j,\ g_j$ are Riemann integrable on $[a,b]$, so are $f^2$, $g^2$, $fg$, $f_j^2$, $g_j^2$, and $f_jg_j$ $(j=1,\ldots,m)$. Letting $n\to\infty$ in both sides of (9A), we obtain (9). The proof is complete. □

In the case of $m=1$, we get an integral type of Aczél's inequality (1):

**Corollary 2.2.** *Let $A>0$, $B>0$, and let $f,g:[a,b]\to(0,\infty)$ be Riemann integrable functions such that $A^2>\int_a^b f^2(x)dx$ and $B^2>\int_a^b g^2(x)dx$. Then*

$$\left(A^2-\int_a^b f^2(x)dx\right)\left(B^2-\int_a^b g^2(x)dx\right) \leqslant\left(AB-\int_a^b f(x)g(x)dx\right)^2. \tag{10}$$

By using a similar method in the proof of Theorem 2.1, we get the following result, which is an integral version of inequality (5).

**Theorem 2.3.** *Let $p>1$, $A>0$, $B>0$. Let $a_j,b_j,\ (j=1,\ldots,m)$ be positive real numbers. Let $f,\ g,\ f_j,g_j\ (j=1,\ldots,m)$ be positive Riemann integrable functions on $[a,b]$ such that*

$$\left(A^p-\int_a^b f^p(x)dx\right)-\left(\sum_{j=1}^{m} a_j^p-\sum_{j=1}^{m}\int_a^b f_j^p(x)dx\right)>0, \tag{11}$$

$$\left(B^p-\int_a^b g^p(x)dx\right)-\left(\sum_{j=1}^{m} b_j^p-\sum_{j=1}^{m}\int_a^b g_j^p(x)dx\right)>0, \tag{12}$$

$$\frac{f_j(x)}{g_j(x)}=\frac{a_j}{b_j},\qquad x\in[a,b],\ j=1,2,\ldots,m. \tag{13}$$

*Then*

$$\left[\left((A+B)^p-\int_a^b [f(x)+g(x)]^p\,dx\right)-\left(\sum_{j=1}^{m}(a_j+b_j)^p-\sum_{j=1}^{m}\int_a^b [f_j(x)+g_j(x)]^p\,dx\right)\right]^{1/p}$$
$$\geqslant\left[\left(A^p-\int_a^b f^p(x)dx\right)-\left(\sum_{j=1}^{m} a_j^p-\sum_{j=1}^{m}\int_a^b f_j^p(x)dx\right)\right]^{1/p}$$
$$+\left[\left(B^p-\int_a^b g^p(x)dx\right)-\left(\sum_{j=1}^{m} b_j^p-\sum_{j=1}^{m}\int_a^b g_j^p(x)dx\right)\right]^{1/p}. \tag{14}$$

*Proof.* For any positive integer $n$, we choose an equidistant partition of $[a,b]$ by $n+1$ points

$$x_0<x_1<\cdots<x_n,$$

with $x_0=a$, $x_i=a+i\,\dfrac{b-a}{n}$, $\Delta x=x_i-x_{i-1}=\dfrac{b-a}{n}$, $i=1,2,\ldots,n$. Owing to (11) and (12), there exists a positive integer number $N$ such that for all $n>N$

$$\left(A^p-\sum_{i=1}^{n} f^p\left(x_{i-1}\right)\Delta x\right)- -\left(\sum_{j=1}^{m} a_j^p-\sum_{j=1}^{m}\sum_{i=1}^{n} f_j^p\left(x_{i-1}\right)\Delta x\right)>0,$$

and

$$\left(B^p-\sum_{i=1}^{n} g^p\left(x_{i-1}\right)\Delta x\right)- -\left(\sum_{j=1}^{m} b_j^p-\sum_{j=1}^{m}\sum_{i=1}^{n} g_j^p\left(x_{i-1}\right)\Delta x\right)>0.$$

Since (13) , it follows that

$$\frac{f_j(x_i)\left(\Delta x\right)^{1/p}}{g_j(x_i)\left(\Delta x\right)^{1/p}}=\frac{A_j}{B_j},\ j=1,2,\ldots,m.$$

Applying Theorem E with $a_1 = A$, $b_1 = B$, and for $j = 1, \ldots, m,\ i = 1, \ldots, n$

$$a_{j1} = A_j, \qquad b_{j1} = B_j,$$
$$a_i = f(x_{i-1})(\Delta x)^{1/p},$$
$$b_i = g(x_{i-1})(\Delta x)^{1/p},$$
$$a_{ji} = f_j(x_{i-1})(\Delta x)^{1/p},$$
$$b_{ji} = g_j(x_{i-1})(\Delta x)^{1/p}$$

we get

$$\Bigg\{\left((A+B)^p - \sum_{i=1}^{n}[f(x_{i-1}) + g(x_{i-1})]^p \Delta x\right) - \left(\sum_{j=1}^{m}(A_j + B_j)^p - \sum_{j=1}^{m}\sum_{i=1}^{n}[f_j(x_{i-1}) + g_j(x_{i-1})]^p \Delta x\right)\Bigg\}^{1/p}$$
$$\geqslant \left[\left(A^p - \sum_{i=1}^{n} f^p(x_{i-1})\Delta x\right) - \left(\sum_{j=1}^{m} A_j^p - \sum_{j=1}^{m}\sum_{i=1}^{n} f_j^p(x_{i-1})\Delta x\right)\right]^{1/p}$$
$$+ \left[\left(B^p - \sum_{i=1}^{n} g^p(x_{i-1})\Delta x\right) - \left(\sum_{j=1}^{m} B_j^p - \sum_{j=1}^{m}\sum_{i=1}^{n} g_j^p(x_{i-1})\Delta x\right)\right]^{1/p}. \tag{14A}$$

Since $f,\ g, f_j, g_j$ are Riemann integrable on $[a,b]$, it follows that $f^p$, $g^q$, $(f+g)^p$, $f_j^p$, $g_j^q$, $(f_j + g_j)^p$, $j = 1, \ldots, m$ are also Riemann integrable on $[a,b]$. Letting $n \to \infty$ in both sides of (14A), we obtain (14). The proof of Theorem 2.3 is complete. □

By getting $m = 1$ in (14), we obtain an integral version of the famous Bellman's integral as follows.

**Corollary 2.4.** *Let $p > 1$, $A > 0$, $B > 0$. Let $f$ and $g$ be positive Riemann integrable functions on $[a,b]$ such that $A^p > \int_a^b f^p(x)dx$ and $B^p > \int_a^b g^p(x)dx$. Then*

$$\left(A^p - \int_a^b f^p(x)dx\right)^{1/p} + \left(B^p - \int_a^b g^p(x)dx\right)^{1/p} \leqslant \left((A+B)^p - \int_a^b [f(x)+g(x)]^p\, dx\right)^{1/p}. \tag{15}$$

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