Tính ổn định của ánh xạ đa trị chính quy* Milyutin bởi nhiễu Lipschitz

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TÓM TẮT

Bài báo nghiên cứu tính ổn định của ánh xạ đa trị chính quy* Milyutin bị nhiễu bởi một ánh xạ Lipschitz trong ngữ cảnh các khái niệm chính quy Milyutin và chính quy* Milyutin được phỏng lại cho phù hợp với một số tình huống trong thực tiễn.

Từ khóa: *Tính chính quy mêtric, tính chính quy* mêtric, độ dốc mạnh, tính ổn định nhiễu, tính pseudo-Lipschitz*.*

The stability of star Milyutin regularity multifunctions under Lipschitz perturbation

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ABSTRACT

The paper investigates the stability of a star Milyutin regular set-valued mapping perturbed by a Lipschitz mapping in the context of the concepts of Milyutin regularity and star Milyutin regularity that have been adapted to be suitable for some practical situations.

Keywords: *Metric regularity, star metric regularity, strong slope, perturbation stability, star pseudo-Lipschitz*

1. INTRODUCTION

First discovered from classical results: Lyusternik-Graves Theorem, which is formed from two independent results by L. A. Lyusternik (1934) and L. M. Graves (1950), Open Mapping Theorem by Rudin (1973), and Implicit Function Theorem by Cauchy, Dini (1980s),... until now, the local metric regularity for single-valued mappings has been studied and expanded by many mathematicians such as: Borwein, Ioffe, Penot, Frankowska, Aubin,... to set-valued mappings in nonlinear case of high order or in nonlocal forms in works by Arutyunov[1], Gfrerer[2], Frankowska and Quicampoix[3], Mordukhovich and Ouyang[4], Penot[5], Ioffe[6,7], Ngai, Tron, and Théra[8], Ivanov and Zlateva[9], etc. In the most recent paper by Tron, Han, and Ngai[10], models of nonlocal metric regularity of multivalued mappings are considered on an arbitrary subset of product metric space. And then, the infinitesimal characterization for these models as well as the stability of Milyutin regular under perturbation are also established.

Besides, in the process of expansion of Aubin property to the fixed set situation, Ioffe[6] led to a weak version of metric regularity which is called *star metric regularity*. Recall that star metric regularity of a set-valued mapping on fixed subsets of the form $\mathcal{U} \times \mathcal{V}$ is the metric regularity of the mapping whose images are the ones of the original set-valued mapping truncated by $\mathcal{V}$, i.e., a set-valued mapping T between metric spaces is said to be star metric regularity on $\mathcal{U} \times \mathcal{V}$ if there exists $\tau > 0$ such that

$$d(u, T^{-1}(v)) \leq \tau d(v, T(u) \cap \mathcal{V}),$$

for all $(u, v) \in \mathcal{U} \times \mathcal{V}$ and $0 < \tau d(v, T(u) \cap \mathcal{V}) \leq \delta(u)$, where δ is a gauge function that takes positive values on $\mathcal{U}$. In also[6], Ioffe has shown that there exist set-valued mappings that satisfy star metric regularity but are not metric regularity. And so, star metric regularity is claimed to be weaker than metric regularity. Then, for the such mappings, the use of the Milyutin perturbation theorems as mentioned in[10] with the metric regularity assumption of the original set-valued mapping may not useful. Consequently, the purpose of this article is to consider the stability of Milyutin regular when the initial mapping just satisfies star Milyutin regularity.

The paper is organized as follows. In Section 2 we introduce some basic notations and preliminaries. Further we recall the related results by Tron, Han and Ngai[10]. In Section 3 we prove stability theorems of perturbed star Milyutin regularity set-valued mappings.

2. PRELIMINARIES

Throughout the article, we shall mainly be working in the setting of a metric space X, endowded with a metric d. For $u \in X$, we denote by $d(u, A)$ the distance from u to $A \subseteq X$, $d(u, A) := \inf\{d(u, t) \mid t \in A\}$. By $B(C, \rho), \overline{B}(C, \rho)$ we denote respectively an open and a closed neighborhood of C with radius $\rho \in (0, +\infty)$. *A set-valued mapping (or a multifunction) between metric spaces* X, Y denoted by $T : X \rightrightarrows Y$ is a correspondence which associates every u a set $T(u)$, possibly empty. For every set-valued mapping $T : X \rightrightarrows Y$, we associate two sets, the graph of T and the domain of T, are defined by $\operatorname{Graph} T := \{(u, v) \in X \times Y \mid v \in F(u)\}$ and $\operatorname{Dom} T := \{u \in X \mid T(u) \neq \emptyset\}$. The inverse of T is the mapping $T^{-1} : Y \rightrightarrows X$ defined by $T^{-1}(v) = \{u \in X \mid v \in F(u)\}$. Then,

$$(u, v) \in \operatorname{Graph} T \Longleftrightarrow (v, u) \in \operatorname{Graph} T^{-1}.$$

2.1 Some basic notations and notions

In view of variational analysis, stability theory is closely related to the basic notion of metric regularity. The versions of this key property are recalled below, and for more details and further references, readers refer to the works[11,12].

Let X, Y be metric spaces, $T : X \rightrightarrows Y$ be a multifunction, $(\bar{u}, \bar{v}) \in \operatorname{Graph} T$.

Definition 1. [11,12] *A multifunction T is called metrically regular around $(\bar{u}, \bar{v}) \in \operatorname{Graph} T$ with modulus $\kappa > 0$ if there exists a neighborhood $U \times V$ of $(\bar{u}, \bar{v})$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u)), \text{ for all } (u, v) \in U \times V.$$

We denoted by $\operatorname{reg} T(\bar{u}, \bar{v})$ the infimum of all modulus κ above.

Ioffe[11,6] suggested a nonlocal regularity model of set-valued mapping $T : X \rightrightarrows Y$ associated to a gauge function γ as follows. Let $\mathcal{U} \subset X, \mathcal{V} \subset Y$ and $\gamma : X \to \mathbb{R} \cup \{+\infty\}$ be positive on $\mathcal{U}$.

Definition 2. [6,12] *A multifunction $T : X \rightrightarrows Y$ is called γ-metrically regular on $\mathcal{U} \times \mathcal{V}$ if there is a real number $\kappa > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u)), \qquad (2.1)$$

provided that $u \in \mathcal{U}$, $v \in \mathcal{V}$, and $0 < \kappa d(v, T(u)) < \gamma(u)$. Denote by $\operatorname{reg}_\gamma T(\mathcal{U}|\mathcal{V})$ the lower bound of the κ satisfying (2.1). If no such κ exists, set $\operatorname{reg}_\gamma T(\mathcal{U}|\mathcal{V}) = \infty$.

Furthermore, in the work[10] by Tron, Han and Ngai, a different version of γ-metric regularity which is extended to an arbitrary set $\mathcal{W} \subset X \times Y$ suggested as follows.

Definition 3. [10] *Let $T : X \rightrightarrows Y$ be a multifunction and W be a subset of $X \times Y$. T is called γ-metrically regular on $\mathcal{W}$ with constant κ if there is a real number $r > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u)), \qquad (2.2)$$

for all $(u, v) \in \mathcal{W}$ with $0 < r d(v, T(u)) < \gamma(u)$. The lower bound $\operatorname{reg}_\gamma T(\mathcal{W})$ of κ in (2.3) is the modulus of γ-metric regularity of T on $\mathcal{W}$. If no such κ exists, set $\operatorname{reg}_\gamma T(\mathcal{W}) = \infty$.

The above definition covers the case where the parameters κ and r coincide, which is known as the concept of γ-metric regularity *in the sense of Ioffe*, as shown in the following definition.

Definition 4. [10] *Let X, Y be metric spaces, W be a subset of $X \times Y$ and let $T : X \rightrightarrows Y$ be a set-valued mapping. T is called γ-metrically regular on W if there is $\kappa > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u))$$

for all $(u, v) \in W$ with $0 < \kappa d(v, T(u)) < \gamma(u)$.

Next, we recall a weaker version of metric regularity, star metric regularity, introduced by Ioffe in also[6].

Definition 5. [6] *Set $T_{\mathcal{V}}(u) = T(u) \cap \mathcal{V}$. A multifunction T is said to be γ-regular* (or star γ-regular) on $\mathcal{U} \times \mathcal{V}$ if $T_{\mathcal{V}}$ is γ-regular on $\mathcal{U} \times \mathcal{V}$. Specifically, T is called γ-regular* on $\mathcal{U} \times \mathcal{V}$ if there is a $\kappa > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u) \cap \mathcal{V})$$

for all $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $0 < \kappa d(v, T(u) \cap \mathcal{V}) < \gamma(u)$.

In order to be convenient in some applications, in this paper, we propose an improved version of the above definition in which the parameters "κ" in the regularity inequality and the gauge condition could be distinguished.

Definition 6. *A multifunction $T : X \rightrightarrows Y$ is called γ-metrically regular* on $\mathcal{U} \times \mathcal{V} \subset X \times Y$ with constant κ if there is a real number $r > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, T(u) \cap \mathcal{V}), \qquad (2.3)$$

for all $(u, v) \in \mathcal{U} \times \mathcal{V}$ with $0 < rd(v, T(u) \cap \mathcal{V}) < \gamma(u)$. The lower bound $\operatorname{reg}^*_\gamma T(\mathcal{U}|\mathcal{V})$ of κ in (2.3) is the modulus of γ-metric regularity of T on $\mathcal{U} \times \mathcal{V}$. If no such κ exists, set $\operatorname{reg}^*_\gamma T(\mathcal{U}|\mathcal{V}) = \infty$.*

Remark 7. *In case of $r = \kappa$, Definition 6 leads to the version of γ-metric regularity* on $\mathcal{U} \times \mathcal{V}$ in the sense of Ioffe as in Definition 5.*

γ-openness* and γ-pseudo-Lipschitz* of set-valued mappings are equivalent properties of the regularity* stated as follows.

Definition 8. *A multifunction $T : X \rightrightarrows Y$ is γ-open* on $\mathcal{U} \times \mathcal{V}$ with constant κ if there is a real number $\rho > 0$ such that*

$$B(T(u) \cap \mathcal{V}, \rho t) \cap \mathcal{V} \subset T(B(u, \kappa^{-1}\rho t)), \qquad (2.4)$$

whenever $u \in \mathcal{U}$, $0 < t < \gamma(u)$. The upper bound $\operatorname{sur}^*_\gamma T(\mathcal{U}|\mathcal{V})$ of κ in (2.4) is the modulus of γ-surjection of T on $\mathcal{U} \times \mathcal{V}$. If no such κ exists, set $\operatorname{sur}^*_\gamma T(\mathcal{U}|\mathcal{V}) = 0$.*

Definition 9. *A multifunction $T^{-1} : Y \rightrightarrows X$ is γ-pseudo-Lipschitz* on $\mathcal{V} \times \mathcal{U}$ with constant κ if there is a real number $\rho > 0$ such that*

$$d(u, T^{-1}(v)) \leq \kappa d(v, w), \qquad (2.5)$$

provided that $u \in T^{-1}(w) \cap \mathcal{U}$, $v, w \in \mathcal{V}$ and $0 < \rho d(v, w) < \gamma(x)$. The lower bound $\operatorname{lip}^*_\gamma T^{-1}(\mathcal{U}|\mathcal{V})$ of κ in (2.5) is the γ-pseudo-Lipschitz modulus of T^{-1} on $\mathcal{V} \times \mathcal{U}$. If no such κ exists, set $\operatorname{lip}^*_\gamma T^{-1}(\mathcal{U} \times \mathcal{V}) = \infty$.*

The following propositon shows the equivalence of the above three star regular concepts.

Proposition 10. *Let $T : X \rightrightarrows Y$ be set-valued mapping and $\mathcal{U} \subset X$, $\mathcal{V} \subset Y$. The following statements are equivalent:*

(i) T is γ-open on $\mathcal{U} \times \mathcal{V}$ with modulus not smaller than κ^{-1};*

(ii) T is γ-regular on $\mathcal{U} \times \mathcal{V}$ with modulus not greater than κ;*

(iii) T^{-1} is γ-pseudo-Lipschitz on $\mathcal{V} \times \mathcal{U}$ with modulus not greater than κ.*

Proof. To show $(i) \Rightarrow (ii)$, let $(u, v) \in \mathcal{U} \times \mathcal{V}$ be with $0 < \rho d(v, T(u) \cap \mathcal{V}) < \gamma(u)$. Then, for all $\epsilon > 0$, take $\tau = \rho(d(v, T(u) \cap \mathcal{V}) + \epsilon)$ such that $0 < \rho d(v, T(u) \cap \mathcal{V}) < \tau < \gamma(u)$. Then, $u \in \mathcal{U}, 0 < \tau < \gamma(u)$ and $v \in B(T(u) \cap \mathcal{V}, \rho^{-1}\tau) \cap \mathcal{V}$. By (i), $v \in T(B(u, \kappa\rho^{-1}\tau))$. So, there exists $z \in B(u, \kappa\rho^{-1}\tau)$ such that $v \in T(z)$. It follows that $d(u, T^{-1}(v)) \leq d(u, z) \leq \kappa\rho^{-1}\tau = \kappa(d(v, T(u) \cap \mathcal{V}) + \epsilon)$. Let $\epsilon \downarrow 0$, one gets $d(u, T^{-1}(v)) \leq \kappa d(v, T(u) \cap \mathcal{V})$.

The implication $(ii) \Rightarrow (iii)$ is obvious. For $(iii) \Rightarrow (i)$. Let $u \in \mathcal{U}$, $0 < \tau < \gamma(u)$, and let $v \in B(T(u) \cap \mathcal{V}, \rho^{-1}\tau) \cap \mathcal{V}$. Then $u \in U$ and there exists $w \in T(u) \cap \mathcal{V}$ such that $0 < d(v, w) < \rho^{-1}\tau$. It follows $u \in T^{-1}(w) \cap \mathcal{U}$, $v, w \in \mathcal{V}$ and $0 < \rho d(v, w) < \tau < \gamma(u)$. By (iii), $d(u, T^{-1}(v)) \leq \kappa d(v, w) < \kappa\rho^{-1}\tau$. This means that there is $z \in T^{-1}(v)$ such that $d(u, z) < \kappa\rho^{-1}\tau$, that is $v \in T(B(u, \kappa\rho^{-1}\tau))$. So,

$$B(T(u) \cap \mathcal{V}, \rho^{-1}\tau) \cap V \subset T(B(u, \kappa\rho^{-1}\tau)).$$

The proof is complete.

2.2 Auxiliary results

Now, we recall the concept of (strong) slope which is considered as an infinitesimal tool in metric spaces, first introduced in 1980 by De Giorgi, Marino, and Tosques[13].

Definition 11. [13,14] *Let X be a metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. The symbol $[f(x)]_+$ stands for $\max(f(x), 0)$ and $\operatorname{Dom} f := \{x \in X \mid f(x) < +\infty\}$ denotes the domain of f.*

(i) The quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f; otherwise

$$|\nabla f|(x) = \limsup_{u \to x, u \neq x} \frac{f(x) - f(u)}{d(x, u)}.$$

is called the local slope of the function f at $x \in \operatorname{Dom} f$.

(ii) The quantity

$$|\Gamma f|(x) := \sup_{u\neq x} \frac{[f(x)-f(u)]_+}{d(x,u)}$$

is called the nonlocal slope of the function f at $x \in \mathrm{Dom}\, f$.

For $x \notin \mathrm{Dom}\, f$, we set $|\nabla f|(x) = |\Gamma f|(x) = +\infty$. Obviously, $|\nabla f|(x) \le |\Gamma f|(x)$ for all $x \in X$.

In case of X being a normed space and f being Fréchet differentiable function at x then the slope of f coincides with the norm of the derivative ∇f at the point. For a fuller treatment of strong slope, we refer the reader to[13,15,16,17,18,19].

To establish infinitesimal characterizations for regularity, an effective tool that has been used is the lower semicontinuous envelop of the distance function associated to a set-valued mapping $T : X \rightrightarrows Y$ defined by

$$\varphi_y^T(x) := \liminf_{(u,v)\to(x,y)} d(v, T(u)) := \liminf_{u\to x} d(y, T(u)).$$

The following theorem established by Tron, Han, Ngai[10] gives the necessary/ sufficient conditions for the metric regularity via nonlocal slope of the function φ_y^T. Now, let be given a subset W of $X \times Y$, we associate every $v \in Y$ to set $W_v = \{u \in X : (u,v) \in W\}$, and every $u \in X$ to set $W_u = \{v \in Y : (u,v) \in W\}$. Then, denoted by $P_X W := \cup_{v\in Y} W_v$, and $P_Y W := \cup_{u\in X} W_u$. In particular, in the case where the form of W is a box $U \times V$, the sets W_v (with $v \in V$), $P_X W$ are identical to U and the sets W_u (with $u \in U$), $P_Y W$ are identical to V.

Theorem 12. *(Tron-Han-Ngai[10]) Given X is a complete metric space, Y is a metric space and $\mathcal{W} \subset X \times Y$ is a nonempty subset. Let $T : X \rightrightarrows Y$ be a closed set-valued mapping and $\gamma : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a gauge function. Then,*

(i) Suppose that γ is lower semicontinuous. If $\mathcal{W}$ is open and T is γ-metrically regular on $\mathcal{W}$ with constant κ, i.e., there exists a real $r > 0$ such that for every $(x,y) \in \mathcal{W}$, with $0 < rd(y, T(x)) < \gamma(x)$,

$$d(x, T^{-1}(y)) \le \kappa d(y, T(x)),$$

then for each $(x,y) \in \mathcal{W}$, with $0 < r\varphi_y^T(x) < \gamma(x)$, one has

$$|\Gamma \varphi_y^T|(x) \ge \kappa^{-1}$$

(ii) Conversely, assume further that $\gamma : X \to \mathbb{R}_+$ is a Lipschitz continuous function with constant 1. If there are a positive real κ such that

$$\lim_{\delta\downarrow 0} \inf\{|\Gamma\varphi_y^T|(x) : d(x, \mathcal{W}_y) < \delta\gamma(x),\ y \in P_Y\mathcal{W},\ 0 < \varphi_y^T(x) < \delta\gamma(x)\} > \kappa^{-1}.$$

then T is γ-metrically regular on $\mathcal{W}$ with constant κ.

Regarding Definition 4, the theorem below in the work by Tron, Han, Ngai[10] gives a suficient condition for the γ-metric regularity via the nonlocal slope.

Theorem 13. *(Tron-Han-Ngai[10]) Let X be a complete metric space and Y be a metric space, $\mathcal{W} \subset X \times Y$ be a nonempty subset. Let $T : X \rightrightarrows Y$ be a closed set-valued mapping. Suppose that $\gamma : X \to \mathbb{R}_+$ is a Lipschitz function with constant 1. If there exists $\kappa > 0$ such that*

$$|\Gamma\varphi_y^T|(x) \ge \kappa^{-1},$$

$\forall x \in (\mathcal{W}_y)_\gamma, y \in P_y\mathcal{W}, 0 < \kappa\varphi_y^T(x) < \gamma(x)$, where $(\mathcal{W}_y)_\gamma = \cup_{x\in\mathcal{W}_y} B(x, \gamma(x))$, then one has

$$d(x, T^{-1}(y)) \le \kappa d(y, T(x)),$$

for all $(x,y) \in \mathcal{W}$ with $0 < \kappa d(y, T(x)) < \gamma(x)$.

3. PERTURBATION STABILITY OF STAR MILYUTIN REGULARITY MULTIFUNCTIONS

Let X, Y be metric spaces and $\mathcal{W}$ be a nonempty subset of $X \times Y$. Firstly, we recall the defintiton of Milyutin regular on $\mathcal{W}$ given by Tron, Han and Ngai in[10].

Definition 14. *(Tron-Han-Ngai[10]) A multifunction $T : X \rightrightarrows Y$ is called Milyutin regular on $\mathcal{W}$ with constant κ if there is a real number $r > 0$ such that*

$$d(T^{-1}(y)) \le \kappa d(y, T(x)),$$

for all $(x,y) \in \mathcal{W}$ with $0 < rd(y, T(x)) < m_{P_X\mathcal{W}}(x)$. The infimum of all above κ denoted by $\mathrm{reg}_m T(\mathcal{W})$.

Next, we consider the definitions of Milyutin regular* associated to the gauge function $\gamma \equiv m_{P_X\mathcal{W}} : X \to \mathbb{R}_+$ defined by $m_{P_X\mathcal{W}}(x) := d(x, X\backslash P_X\mathcal{W})$.

Definition 15. *A multifunction $T : X \rightrightarrows Y$ is called Milyutin regular* on $\mathcal{W}$ with constant κ if there is a real number $r > 0$ such that*

$$d(T^{-1}(y)) \leq \kappa d(y, T(x) \cap P_X\mathcal{W}),$$

for all $(x, y) \in \mathcal{W}$ with $0 < rd(y, T(x) \cap P_X\mathcal{W}) < m_{P_X\mathcal{W}}(x)$. The infimum of all above κ denoted by $\operatorname{reg}^*_m T(\mathcal{W})$ is the modulus of Milyutin regular of T on $\mathcal{W}$. If the above constant κ does not exists, set $\operatorname{reg}^*_m T(\mathcal{W}) = \infty$.*

Remark 16. *In the above definition, taking $r = \kappa$ one obtains the definition of Milyutin regular* on $\mathcal{W}$ in the sense of Ioffe.*

It is easily seen that $m_{P_X\mathcal{W}}(x)$ is positive on $P_X\mathcal{W}$ if and only if $P_X\mathcal{W}$ is an open set, which follows from $\mathcal{W}$ is open. And then, the results of Theorem 12 and Theorem 13 are also applied to the function $m_{P_X\mathcal{W}}$ due to Lipschitz property with constant 1 of this one.

In this part, we shall investigate the stability of Milyutin regular under perturbation by single-valued mappings and the original set-valued mapping is assumed to be Milyutin regular*.

Theorem 17. *Let X be a complete metric space and Y be a Banach space. Let $\mathcal{U} \subset X$, $\mathcal{V} \subset Y$ be open sets. Let a closed set-valued mapping $T : X \rightrightarrows Y$ and a single-valued mapping $h : X \to Y$ be Lipschitz on $\mathcal{U}$ with constant $\lambda \in (0, \kappa^{-1})$. If T is Milyutin regular* on $\mathcal{U} \times \mathcal{V}$ with constant κ, i.e, there exists $r > 0$ such that for all $(x, y) \in \mathcal{U} \times \mathcal{V}$ with $0 < rd(y, T(x) \cap \mathcal{V}) < m_{\mathcal{U}}(x)$,*

$$d(x, T^{-1}(y)) \leq \kappa d(y, T(x) \cap \mathcal{V}).$$

Then, for every $\eta > 0$, $T + h$ is Milyutin regular on $\mathcal{W}^{\lambda\eta}$ with $\operatorname{reg}_m(T + h)(\mathcal{W}^{\lambda\eta}) \leq (\kappa^{-1} - \lambda)^{-1}$, where

$$\mathcal{W}^{\lambda\eta} = \{(x, y) \in X \times Y \mid x \in \mathcal{U}, B(y - h(x), \lambda\eta m_{\mathcal{U}(x)}) \subset \mathcal{V}\}.$$

Proof. Let $\eta > 0$ be given. According to Theorem 12, we only need to prove that

$$\lim_{\delta\downarrow 0} \inf\{|\Gamma\varphi_y^{T+h}|(x) : d(x, \mathcal{W}_y^{\lambda\eta}) < \delta m_{P_X\mathcal{W}^{\lambda\eta}}(x), y \in P_Y\mathcal{W}^{\lambda\eta}, 0 < \varphi_y^{T+h}(x) < \delta m_{P_X\mathcal{W}^{\lambda\eta}}(x)\} > \kappa^{-1} - \lambda. \tag{3.1}$$

Indeed, choose δ such that $\frac{\delta}{1-\delta} < \min\{1, \eta\}$, $0 < r\delta < 1$, $\frac{(\lambda+1)\delta}{1-\delta} < \lambda\eta$.

Let $(x, y) \in X \times Y$ such that $d(x, \mathcal{W}_y^{\lambda\eta}) < \delta m_{P_X\mathcal{W}^{\lambda\eta}}(x)$, $y \in P_X\mathcal{W}^{\lambda\eta}$ and $0 < \varphi_y^{T+h}(x) < \delta m_{P_X\mathcal{W}^{\lambda\eta}}(x)$. Then there exists $u \in \mathcal{W}^{\lambda\eta}$ such that

$$d(x, u) < \delta m_{P_X\mathcal{W}^{\lambda\eta}}(x) \leq \delta m_{\mathcal{U}}(x).$$

So, $u \in \mathcal{U}$, $B(y - h(u), \lambda\eta m_{\mathcal{U}}(x)) \subset \mathcal{V}$, and since $m_{\mathcal{U}}$ is Lipschitz with constant 1, it follows that

$$d(x, u) < \delta m_{\mathcal{U}}(u) + \delta d(x, u).$$

By the choice of δ, one has

$$d(x, u) < \frac{\delta}{1-\delta} m_{\mathcal{U}}(u) < m_{\mathcal{U}}(u) \tag{3.2}$$

which gives $x \in \mathcal{U}$.

Let now $\{u_n\} \subset X$ be such that $u_n \to x$ and

$$d(y, (T + h)(u_n)) \to \varphi_y^{T+h}(x) \text{ as } n \to \infty.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$0 < d(y, (T + h)(u_n)) < \delta m_{\mathcal{U}}(u_n) \tag{3.3}$$

and, as $u_n \to x \in \mathcal{U}$, we have $u_n \in \mathcal{U}$ due to the openness of $\mathcal{U}$. And then, by the choice of δ when n is sufficiently large, we have

$$0 < d(y, (T + h)(u_n)) < r^{-1} m_{\mathcal{U}}(u_n). \tag{3.4}$$

Furthermore, for n large enough, we find that $d(y_n, T(u_n)) = d(y_n, T(u_n) \cap \mathcal{V})$. Indeed, fixing $n \in \mathbb{N}^*$, we take a sequence $\{a_k\} \subset T(u_n)$ such that

$$d(y - h(u_n), a_k) \to d(y - h(u_n), T(u_n)), \; k \to \infty.$$

By (3.2), (3.3) and the continuity of distance function, we conclude that

$$\begin{aligned} d(y - h(u_n), a_k) &< \delta m_{\mathcal{U}}(u_n) \\ &\leq \delta m_{\mathcal{U}}(u) + \delta d(u_n, u) \\ &\leq \delta m_{\mathcal{U}}(u) + \frac{\delta^2}{1-\delta} m_{\mathcal{U}}(u) \\ &= \frac{\delta}{1-\delta} m_{\mathcal{U}}(u). \end{aligned} \tag{3.5}$$

From (3.2), (3.5) and the choice of δ, it follows that for $n \geq n_0$,

$$\begin{aligned} d(a_k, y - h(u)) &\leq d(a_k, y - h(u_n)) \\ &\quad + d(y - h(u_n), y - h(u)) \\ &\leq \frac{\delta}{1-\delta} m_{\mathcal{U}}(u) + \lambda d(u_n, u) \\ &\leq \frac{\delta}{1-\delta} m_{\mathcal{U}}(u) + \lambda \frac{\delta}{1-\delta} m_{\mathcal{U}}(u) \\ &= \frac{(\lambda+1)\delta}{1-\delta} m_{\mathcal{U}}(u) \\ &\leq \lambda \eta m_{\mathcal{U}}(u) \end{aligned}$$

which gives $a_k \in B(y - h(u), \lambda \eta m_{\mathcal{U}}(u)) \subset \mathcal{V}$, and thus $a_k \in T(u_n) \cap \mathcal{V}$. Consequently, $d(y - h(u_n), a_k) \geq d(y - h(u_n), T(u_n) \cap \mathcal{V})$, So, $d(y - h(u_n), T(u_n)) \leq d(y - h(u_n), T(u_n) \cap \mathcal{V})$. And then, $d(y - h(u_n), T(u_n)) = d(y - h(u_n), T(u_n) \cap \mathcal{V})$ when n is sufficiently large.

Then from (3.4), we see that

$$\begin{aligned} 0 < d(y - h(u), T(u_n) \cap \mathcal{V}) &= d(y - h(u), T(u_n)) \\ &< r^{-1} m_{\mathcal{U}}(u_n). \end{aligned}$$

Moreover, by (3.2), for n is large enough, we conclude from the continuity of distance function that

$$\begin{aligned} d(y - h(u_n), y - h(u)) &< \lambda d(u_n, u) \\ &\leq \lambda d(x, u) \\ &\leq \lambda \frac{\delta}{1-\delta} m_{\mathcal{U}}(u) \\ &\leq \lambda \eta m_{\mathcal{U}}(u), \end{aligned}$$

where the last inequality is followed from the choice of δ. Consequently,

$$y - h(u_n) \in B(y - h(u), \lambda \eta m_{\mathcal{U}}(u)) \subset \mathcal{V}.$$

Then from the fact that T is Milyutin regular* on $\mathcal{U} \times \mathcal{V}$ with constant κ, we obtain

$$\begin{aligned} d(u_n, T^{-1}(y - h(u_n))) &\leq \kappa d(y - h(u_n), T(u_n) \cap \mathcal{V}) \\ &= d(y - h(u_n), T(u_n)), \ \forall n \geq n_0. \end{aligned}$$

Now we choose some $z_n \in T^{-1}(y - h(u_n))$ (i.e., $y - h(u_n) \in T(z_n)$) such that

$$d(u_n, z_n) \leq (\kappa + n^{-1}) d(y - h(u_n), T(u_n)). \quad (3.6)$$

From (3.3) and the choice of δ, for all $n \geq n_0$, one has

$$d(u_{n, z_n}) < (\kappa + n^{-1}) \delta m_{\mathcal{U}}(u_n) < m_{\mathcal{U}}(u_n).$$

This yields $z_n \in \mathcal{U}$, and thus from the Lipschitz property of h on $\mathcal{U}$, we have

$$d(h(u_n), h(z_n)) \leq \lambda d(u_n, z_n). \quad (3.7)$$

Since $\varphi_y^{T+h}(x) > 0$, the closeness of T, and $\lim_{n\to\infty} u_n = x$, we see that $\liminf_{n\to\infty} d(u_n, z_n) > 0$. Note that $d(y - h(u_n), T(z_n)) = 0$ since $y - h(u_n) \in T(z_n)$, and from (3.6), (3.7), we conclude that

$$\begin{aligned} |\Gamma \varphi_y^{T+h}|(x) &\geq \limsup_{n\to\infty} \frac{\varphi_y^{T+h}(x) - \varphi_y^{T+h}(z_n)}{d(x, z_n)} \\ &\geq \limsup_{n\to\infty} \frac{d(y, (T+h)(u_n)) - d(y, (T+h)(z_n))}{d(u_n, z_n)} \\ &= \limsup_{n\to\infty} \frac{d(y - h(u_n), T(u_n)) - d(y - h(z_n), T(z_n))}{d(u_n, z_n)} \\ &\geq \limsup_{n\to\infty} \frac{d(y - h(u_n), T(u_n))}{d(u_n, z_n)} - \lambda \\ &\geq \limsup_{n\to\infty} \frac{1}{\kappa + n^{-1}} - \lambda = \kappa^{-1} - \lambda. \end{aligned}$$

This finishes the proof.

The next theorem is a version of the above one in which the definition of Milyutin regular* is replaced by the definition of Milyutin regular* in the sense of Ioffe.

Theorem 18. *Given X is a complete metric space, Y is a Banach space and $\mathcal{U} \subset X$, $\mathcal{V} \subset Y$ are open sets. Let a closed set-valued mapping $T : X \rightrightarrows Y$ and a single-valued mapping $h : X \to Y$ be Lipschitz on $\mathcal{U}$ with constant $\lambda \in (0, \kappa^{-1})$. If T is Milyutin regular* on $\mathcal{U} \times \mathcal{V}$ with constant κ, i.e, for all $(x, y) \in \mathcal{U} \times \mathcal{V}$ with $0 < \kappa d(y, T(x) \cap \mathcal{V}) < m_{\mathcal{U}}(x)$,*

$$d(x, T^{-1}(y)) \leq \kappa d(y, T(x) \cap \mathcal{V}).$$

Then, $T+h$ is Milyutin regular on $\mathcal{W}$ with $\operatorname{reg}_m(T+h)(\mathcal{W}) \leq (\kappa^{-1} - \lambda)^{-1}$*, where*

$$\begin{aligned} \mathcal{W} = \{(x, y) \in X \times Y \mid \ & x \in \mathcal{U}, \\ & B(y - h(x), (2\kappa^{-1} - \lambda) m_{\mathcal{U}}(x)) \subset \mathcal{V}\}. \end{aligned}$$

Proof. Set $(\mathcal{W}_y)_m := \cup_{u \in \mathcal{W}_y} B(u, m_{P_X \mathcal{W}}(u))$. According to Theorem 13, now we shall show that for any $x \in (\mathcal{W}_y)_m$, $y \in P_Y \mathcal{W}$ with $0 < (\kappa^{-1} - \lambda)^{-1} \varphi_y^{T+h}(x) < m_{P_X \mathcal{W}}(x)$,

$$|\ \Gamma \varphi_y^{T+h}\ |\ (x) \geq \kappa^{-1} - \lambda.$$

Indeed, take $(x, y) \in X \times Y$ such that $x \in (\mathcal{W}_y)_m$, $y \in P_Y\mathcal{W}$ with $0 < (\kappa^{-1} - \lambda)^{-1}\varphi_y^{T+h}(x) < m_{P_X\mathcal{U}}(x)$. Then, there is $u \in \mathcal{W}_y$ such that

$$d(x, u) < m_{P_X\mathcal{W}}(u) \le m_{\mathcal{U}}(u). \tag{3.8}$$

So, $u \in U, B(y - h(u), \lambda m_{\mathcal{U}}(u)) \subset \mathcal{V}$, and $x \in \mathcal{U}$.

Now, we take $\{u_n\} \subset X$ such that $u_n \to x$ and $d(y, (T + h)(u_n)) \to \varphi_y^{T+h}(x)$ as $n \to \infty$. Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\begin{aligned} 0 < d(y, (T + h)(u_n)) &\le (\kappa^{-1} - \lambda)m_{P_X\mathcal{W}}(x) \\ &\le (\kappa^{-1} - \lambda)m_{\mathcal{U}}(x) \\ &\le (\kappa^{-1} - \lambda)m_{\mathcal{U}}(u_n) \quad (3.9) \\ &< \kappa^{-1}m_{\mathcal{U}}(u_n), \quad (3.10) \end{aligned}$$

and that $u_n \in \mathcal{U}$ follows from the openness of $\mathcal{U}$ and $u_n \to x \in \mathcal{U}$.

Furthermore, $d(y - h(u_n), T(u_n)) = d(y - h(u_n), T(u_n) \cap \mathcal{V})$ for n large enough. Indeed, fixing $n \in \mathbb{N}^*$, we choose a sequence $\{a_k\} \subset T(u_n)$ such that $d(y - h(u_n), a_k) \to d(y - h(u_n), T(u_n)),\ k \to \infty$. By (3.8), (3.9), and the continuity of the distance function, we conclude that

$$\begin{aligned} d(y - h(u_n), a_k) &< (\kappa^{-1} - \lambda)m_{\mathcal{U}}(u_n) \\ &\le (\kappa^{-1} - \lambda)m_{\mathcal{U}}(u) + (\kappa^{-1} - \lambda)d(u_n, u)) \\ &\le (2\kappa^{-1} - \lambda)m_{\mathcal{U}}(u), \end{aligned} \tag{3.11}$$

which yields $a_k \in B(y - h(u_n), (2\kappa^{-1} - \lambda)m_{\mathcal{U}}(u)) \subset \mathcal{V}$, and thus $a_k \in T(u_n) \cap \mathcal{V}$. Consequently, $d(y - h(u_n), a_k) \ge d(y - h(u_n), T(u_n) \cap \mathcal{V})$. So, $d(y - h(u_n), T(u_n)) \ge d(y - h(u_n), T(u_n) \cap \mathcal{V})$. This gives $d(y - h(u_n), T(u_n)) = d(y - h(u_n), T(u_n) \cap \mathcal{V})$ when n is sufficiently large.

Then from (3.10), we see that

$$\begin{aligned} 0 < d(y - h(u_n), T(u_n) \cap \mathcal{V}) &= d(y - h(u_n), T(u_n)) \\ &< \kappa^{-1}m_{\mathcal{U}}(u_n). \end{aligned}$$

Otherwise, by (3.8) and for n large enough, one also have

$$\begin{aligned} d(y - h(u_n), y - h(u)) &\le \lambda d(u_n, u) \\ &\le \lambda d(u_n, x) + \lambda d(x, u) \\ &\le \lambda m_{\mathcal{U}}(u) \\ &\le (2\kappa^{-1} - \lambda)m_{\mathcal{U}}(u) \end{aligned}$$

which leads to $y - h(u_n) \in B(y - h(u), \lambda m_{\mathcal{U}}(u)) \subset \mathcal{V}$.

So, due to the Milyutin regularity* of T on $\mathcal{U} \times \mathcal{V}$ with constant κ, one obtains

$$d(u_n, T^{-1}(y - h(u_n))) \le \kappa d(y - h(u_n), T(u_n) \cap \mathcal{V})$$

We now choose $z_n \in T^{-1}(y - h(u_n))$ (i.e., $y - h(u_n) \in T(z_n)$) such that

$$\begin{aligned} d(u_n, z_n) &\le (\kappa + n^{-1})d(y - h(u_n), T(u_n) \cap \mathcal{V}) \\ &= (\kappa + n^{-1})d(y - h(u_n), T(u_n)) \quad (3.12) \\ &\le (\kappa + n^{-1})\kappa^{-1}m_{\mathcal{U}}(u_n) \\ &< m_{\mathcal{U}}(u_n), \end{aligned}$$

where the last inequality is obtained when n is large enough. It follows that $z_n \in \mathcal{U}$, and thus from the Lipschitz property of h on $\mathcal{U}$, we have

$$d(y - h(u_n), y - h(z_n)) \le \lambda d(u_n, z_n). \tag{3.13}$$

Since $\varphi_y^{T+h}(x) > 0$, the closeness of T, and $\lim_{n\to\infty} u_n = x$, we have $\liminf_{n\to\infty} d(u_n, z_n) > 0$. From (3.12), (3.13), and note that $y - h(u_n) \in T(z_n)$, similar as in the proof of Theorem 17, one concludes that

$$\begin{aligned} | \Gamma\varphi_y^{T+h} | (x) &\ge \limsup_{n\to\infty} \frac{1}{\kappa + n^{-1}} - \lambda \\ &= \kappa^{-1} - \lambda. \end{aligned}$$

The proof is completed.

4. CONCLUSIONS

This artical suggests the models of star regularity on an any subset of product metric spaces as well as established the equivalence of star regular concepts: star openness, star metrically regular and star pseudo-Lipschitz in the literature. Regarding the star Milyutin regularity, we have proved that the stability of Milyutin regularity under small Lipschitz perturbation also attains when the assumption of star Milyutin regularity is imposed on the original set-valued mapping.

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