Toán tử hợp có trọng từ không gian kiểu Bloch vào không gian tăng trưởng trên hình cầu đơn vị của không gian Hilbert

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TÓM TẮT

Cho ν, μ là các trọng chuẩn tắc trên hình cầu đơn vị B_X của một không gian Hilbert phức với số chiều tùy ý và ψ là một hàm chỉnh hình trên B_X, φ là một ánh xạ tự chỉnh hình của B_X. Trong bài báo này, chúng tôi khảo sát các đặc trưng cho tính bị chặn và tính compact của toán tử hợp có trọng $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, từ không gian kiểu Bloch (nhỏ) $\mathcal{B}_\nu(B_X)$ đến không gian tăng trưởng $\mathcal{H}_\mu^\infty(B_X)$, $\mathcal{H}_\mu^0(B_X)$ theo tính chất lý thuyết của ψ và ước lượng hàm $\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}$, những hạn chế của các hàm ψ, φ đến các không gian con m chiều với $m \geq 2$. Chúng tôi cũng đạt được một công thức chính xác của chuẩn toán tử $W_{\psi,\varphi}$.

Từ khóa: *Toán tử hợp có trọng, hình cầu đơn vị, không gian Bloch, không gian tăng trưởng, tính bị chặn, tính compact.*

Weighted composition operators from Bloch-type spaces into growth spaces on the unit ball of a Hilbert space

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ABSTRACT

Let ν, μ be normal weights on the unit ball B_X of an Hilbert space X with arbitrary dimension and ψ be a holomorphic function on B_X and φ a holomorphic self-map of B_X. In this work, we characterize the boundedness and the compactness weighted composition operators $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, from the Bloch-type spaces $\mathcal{B}_\nu(B_X)$ to the (little) growth spaces $\mathcal{H}^\infty_\mu(B_X)$, $\mathcal{H}^0_\mu(B_X)$ in terms of function theoretic properties of the symbol ψ and the point evaluation function $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi(z)}$, specifically, of the restrictions of functions ψ, φ to the m-dimensional subspaces for some $m \geq 2$. We obtain also an exact formula of the operator norm of $W_{\psi,\varphi}$.

Keywords: *Weighted composition operator, unit ball, Bloch spaces, growth spaces, boundedness, compactness*

1. INTRODUCTION

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be spaces of holomorphic functions on the unit ball B_X of a Banach space X, ψ be a holomorphic function on B_X and φ a holomorphic self-map of B_X. The weighted composition operator, defined by symbols ψ and φ, maps from $\mathcal{E}_1$ to $\mathcal{E}_2$ and is given by

$$W_{\psi,\varphi}(f) = M_\psi C_\varphi(f) = \psi \cdot (f \circ \varphi)$$

where M_ψ represents the multiplication operator with symbol ψ and C_φ denotes the composition operator with symbol φ.

In recent years, there has been significant interest in studying weighted composition operators. A well-known theorem due to Banach states that for a compact metric space K, the onto linear isometries of $C(K)$ are of the form $Tf = u(f \circ \varphi)$ where $|u(x)| = 1$ for all $x \in K$, and $\varphi : K \to K$ is a homeomorphism. Motivated by this theorem, active research on the description of the isometries of Banach spaces of analytic functions has shown that the weighted composition operators characterize the isometries of many Banach spaces of analytic functions, including the Hardy space H^p (for $1 \leq p \leq \infty$, $p \neq 2$), the weighted Bergman space, and the disk algebra (see [1]).

We refer to a standard reference [2] for various aspects on the theory of (weighted) composition operators acting on several spaces of holomorphic functions. There is a vast literature on the weighted composition operators or integral operators between specific holomorphic function spaces. In order to treat these specific spaces in a unified manner, some frameworks of Banach spaces of holomorphic functions on the unit disk were introduced (see, e.g. [3,4]). For example, in [3], the authors provided some topological and function theoretic conditions for the domain space and then provide boundedness and compactness criteria, as well as estimates of the operator norm and the essential norm of the weighted composition operators mapping to the weighted-type space or the Bloch-type space on the unit disk. In recent years, considerable interest has emerged in the study of the weighted composition operators. Recently, interest has arisen in composition operators and operator-valued multipliers on many vector-valued analytic function spaces as well as in the case X is an infinite dimensional Hilbert space, see, for example [5,6,7,8,9,10].

Our setting in this paper will be to discuss the boundedness, compactness of the weighted composition operators $W_{\psi,\varphi}$ in the case $\mathcal{E}_1$ is a general Banach spaces of holomorphic functions and $\mathcal{E}_2$ is either growth space $\mathcal{H}_\mu^\infty(B_X)$ or the little growth space $\mathcal{H}_\mu^0(B_X)$ determined by the growth of the functions:

$$\mathcal{H}_\mu^\infty(B_X) = \Big\{f \in \mathscr{H}(B_X) : \sup_{z\in B_X} \mu(z)|f(z)| < \infty\Big\},$$

$$\mathcal{H}_\mu^0(B_X) = \Big\{f \in \mathcal{H}_\mu^\infty(B_X) : \lim_{\|z\|\to 1} \mu(z)|f(z)| = 0\Big\},$$

where $\mathscr{H}(B_X)$ is the space of holomorphic functions on B_X and μ is a normal weight on B_X. These growth spaces were first studied by Rubel and Shields [11] in the setting of $X = \mathbb{C}$.

Growth spaces are an interesting and important class of Banach spaces of holomorphic functions. They have been explored in many different contexts and there are many general and more specific references such as, for example [12,13]. Some well known properties of these spaces, for B_X is the unit disk $\mathbb{B} \subset \mathbb{C}$, are that:

- For a normal weight μ, $\mathcal{H}_\mu^\infty(\mathbb{B})$ is strictly bigger than $\mathcal{H}^\infty$ (the space of bounded holomorphic functions on $\mathbb{B}$) if and only if $\lim_{|z|\to 1} \mu(z) = 0$. If on the other hand, $\limsup_{|z|\to 1} \mu(z) > 0$, then $\mathcal{H}_\mu^0 = \{0\}$;

- The topologies on $\mathcal{H}_\mu^\infty(\mathbb{B})$ is stronger than the compact open topology τ_{co};

- The bidual $[\mathcal{H}_\mu^0(\mathbb{B})]''$ isometrically isomorphic to $\mathcal{H}_\mu^\infty(\mathbb{B})$;

- The point evaluation functionals on $\mathcal{H}_\mu^0(\mathbb{B})$, denoted by $\delta_z^{\mathcal{H}}$, is bounded and is uniquely extended to point evaluation functional on $\mathcal{H}_\mu^\infty(\mathbb{B})$ with equal norms;

- The differentiation operator sending $f \mapsto f''$ is an isometric isomorphism between $\mathcal{B}_\mu^0(\mathbb{B})$, the subspace of the Bloch space $\mathcal{B}_\mu(\mathbb{B})$ of functions with $f(0) = 0$, and $\mathcal{H}_\mu^\infty(\mathbb{B})$. Note that the Bloch space contains functions determined by the growth of the functions derivatives, so it is closely related to growth spaces;

- The differentiation operator sending $f \mapsto f''$ is an isometric isomorphism between $\mathcal{H}_\alpha^0$, the subspace of the $\mathcal{H}_\alpha^\infty$ of functions with $f(0) = 0$, and $\mathcal{H}_{\alpha+1}^\infty$, where $\mathcal{H}_\alpha^\infty$ is the growth space $\mathcal{H}_\mu^\infty(\mathbb{B})$ with the weight $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$;

- The maps $z \mapsto \delta_z^{\mathcal{H}}$ is continuous, and $\|\delta_z^{\mathcal{H}}\|$ goes to infinity as $|z| \to 1$.

These are just a few of the reasons motivating our research.

In Section 2, we recall crucial conditions for spaces of holomorphic functions that shall be used to establish the boundedness, the compactness as well as for providing essential norm estimates of these operators in our settings.

To characterize the boundedness and compactness, basing on the idea in [5,9] with minor modifications, in Section 3, we give the connection between functions in the growth space $\mathcal{H}_\mu^\infty(B_X)$ and their restrictions to finite-dimensional ones, which leads to the fact that if the restrictions of the function to the m-dimensional subspaces, $m \geq 2$, have their growth-norms uniformly bounded, then the function is in the growth spaces $\mathcal{H}_\mu^\infty(\mathbb{B}_m)$ and conversely.

In Section 4, we characterize the boundedness and the compactness of $W_{\psi,\varphi}$ from $\mathcal{B}_\nu(B_X)$ into $\mathcal{H}_\mu^\infty(B_X)$ and into $\mathcal{H}_\mu^0(B_X)$ as well as calculate the operator norms. We will show that these characterizations are completely determined by their behaviour on $\psi^{[m]}$ and on the point evaluation functions $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi^{[m]}(z)}$ and $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z)}$, where $\psi^{[m]}$ and $\varphi^{[m]}$ are the restrictions of ψ and φ, respectively, on the m-dimensional subspaces and $\varphi_{(m)} = (\varphi_1, \dots, \varphi_m)$, $m \geq 2$.

Throughout this paper, we use the notions $a \lesssim b$ and $a \asymp b$ for non negative quantities a and b to mean $a \leq Cb$ and, respectively, $C^{-1}b \leq a \leq Cb$ for some inessential constant $C > 0$.

2. PRELIMINARIES AND AUXILIARY RESULTS

Let X be a complex Hilbert space of arbitrary dimension, Y a Banach space. Denote by B_X the closed unit ball of X, and we write $\mathbb{B}_n$ instead of $B_{\mathbb{C}^n}$. Let $(e_k)_{k\in\Gamma}$ be an orthonormal basis of X that we fix at once. Then every $z \in X$ can be written as

$$z = \sum_{k\in\Gamma} z_k e_k, \quad \overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k.$$

2.1. Möbius transformations

The analogues of Möbius transformations on a Hilbert space X are the mappings $\Phi_a : B_X \to B_X$, $a \in B_X$, defined as follows:

$$\Phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathcal{B}_X \tag{2.1}$$

where $s_a = \sqrt{1 - \|a\|^2}$, P_a is the orthogonal projection from X onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from X onto $X \ominus [a]$. It is clear that

$$P_a(z) = \frac{\langle z, a\rangle}{\|a\|^2} a, \ (z \in X) \quad \text{and}$$

$$Q_a(z) = z - \frac{\langle z, a\rangle}{\|a\|^2} a, \ (z \in B_X).$$

When $a = 0$, we define $\Phi_a(z) = -z$.

Denote by $Aut(B_X)$ the group of automorphisms of the unit ball B_X.

For details concerning Möbius transformations we refer to the book of K. Zhu [14].

2.2. Banach spaces of holomorphic functions

By $\mathscr{H}(B_X, Y)$ we denote the vector space of Y-valued holomorphic functions on B_X. A holomorphic function $f \in \mathscr{H}(B_X, Y)$ is called locally bounded holomorphic on B_X if for every $z \in B_X$ there exists a neighbourhood U_z of $0 \in X$ such that $f(U_z)$ is bounded. Put
$\mathscr{H}_{LB}(B_X, Y) =$
$\{f \in \mathscr{H}(B_X, Y) : f \text{ is locally bounded on } B_X\}$.
Given $f \in \mathscr{H}(B_X)$ and $z \in B_X$. We will denote, as usual, by $\nabla f(z)$ the gradient of f at z; that is, the unique element in E representing the linear operator $f'(z) \in X'$. We can write

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_k}(z)\Big)_{k\in\Gamma}$$

and hence

$$f'(z)(x) = \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(x_k e_k) = \langle x, \overline{\nabla f(z)}\rangle, \quad x \in X.$$

We define the radial derivative of f at $z \in B_X$ as follows:

$$Rf(z) := \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(z_k e_k) = \langle z, \overline{\nabla f(z)}\rangle.$$

It is obvious that

$$|Rf(z)| \le \|\nabla f(z)\|\|z\|, \quad z \in B_X.$$

Now, let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space.

For each $z \in B_X$, denote $\delta_z^{\mathcal{E}}$ the point-evaluation functional at z defined by $\delta_z^{\mathcal{E}}(f) := f(z)$ for all $f \in \mathcal{E}$. Thus, for any function $f \in \mathcal{E}$ and $z \in B_X$,

$$|f(z)| \le \|f\|\|\delta_z^{\mathcal{E}}\|, \tag{2.2}$$

where $\|\delta_z^{\mathcal{E}}\| = \sup\{|f(z)| : f \in \mathcal{E}, \|f\| \le 1\}$.

For all $\Phi = (\Phi_j)_{j\in\Gamma} \in Aut(B_X)$, for every $j \ge 1$, $m \ge 2$ and all $f \in \mathcal{E}$, we write

$$\Phi_{(m)} = (\Phi_1, \ldots, \Phi_m),$$

$$f \cdot \Phi_{(m)} = (f \cdot \Phi_1, \ldots, f \cdot \Phi_m).$$

We state below a comprehensive list of conditions some of which will be needed to characterize boundedness, compactness, or determine the essential norm of the operators under consideration in this work.

- **(e1)** $\mathcal{E}$ contains the constant functions.
- **(e2)** The closed unit ball $B_{\mathcal{E}}$ is compact in the compact open topology τ_{co}.
- **(e3)** There are $m \ge 2$ and constant $C > 0$ such that for all $\Phi \in Aut(B_X)$, for all $f \in \mathcal{E}$, $\Phi_j \cdot f \in \mathcal{E}$,

$$\|\Phi_j \cdot f\| \le C\|f\|, \quad j \in \{1, \ldots, m\}.$$

Remark 2.1. It follows from (e1) that $\inf_{z\in B_X} \|\delta_z^{\mathcal{E}}\| > 0$, and in particular, the following equivalent conditions are satisfied:

- **(e1a)** $\|\delta_z^{\mathcal{E}}\|$ is bounded from below by a positive constant on compact sets;
- **(e1b)** For each point $z \in B_X$ the functions in $\mathcal{E}$ do not all vanish at z.

Indeed, since the function $1 \in \mathcal{E}$, for every $z \in B_X$ we have $\|\delta_z^{\mathcal{E}}\| \ge \frac{1}{\|1\|}$. It is obvious that (e1a) $\Rightarrow$ (e1b). Now, assume that (e1b) holds but (e1a) fails. Then there exist a compact subset K of B_X, a sequence $\{z_n\}_{n\ge1} \in K$ and $z_0 \in K$ such that $z_n \to z_0$ and $\|\delta_{z_n}^{\mathcal{E}}\| \to 0$. This clearly implies that $f(z_0) = 0$ for all $f \in \mathcal{E}$, which contradicts (e1b).

By the uniform boundedness principle, we can easily prove the following:

Proposition 2.1. Let $\mathcal{E}$ be a Banach space of holomorphic functions on B_X. Then the mapping $\delta^{\mathcal{E}} : B_X \to \mathbb{C}$, $z \mapsto \|\delta_z^{\mathcal{E}}\|$, is bounded on compact subsets of B_X.

3. GROWTH SPACES AND BLOCH-TYPE SPACES

For a normal weight ν on B_X, we denote

$$I_\nu^1(z) := \int_0^{\|z\|} \frac{dt}{\nu(t)}.$$

Remark 3.1. Since ν is positive, continuous, $m_{\nu,\delta} := \min_{t\in[0,\delta]} \nu(t) > 0$. Moreover, it follows from (W_1) that ν is strictly decreasing on $[\delta, 1)$, hence, $\max_{t\in[0,1)} \nu(t) =: M_\nu < \infty$. Then, it is easy to check that

$$\nu(z) I_\nu^1(z) < R_\nu := \delta \frac{M_\nu}{m_{\nu,\delta}} + 1 - \delta < \infty. \tag{3.1}$$

for every $z \in B_X$.

We define bounded holomorphic spaces $\mathcal{H}^\infty(B_X)$, growth holomorphic spaces $\mathcal{H}_\mu^\infty(B_X)$, little growth holomorphic spaces $\mathcal{H}_\mu^0(B_X)$, Bloch-type spaces $\mathcal{B}_\nu(B_X)$, and little Bloch-type spaces $\mathcal{B}_{\nu,0}(B_X)$ on the unit ball B_X as follows:

$$\mathcal{H}^\infty(B_X) = \Big\{ f \in \mathscr{H}(B_X) : \sup_{z\in B_X} |f(z)| < \infty \Big\},$$

$$\mathcal{H}_\mu^\infty(B_X) = \Big\{ f \in \mathscr{H}(B_X) : \sup_{z\in B_X} \mu(z)|f(z)| < \infty \Big\},$$

$$\mathcal{H}_\mu^0(B_X) = \Big\{ f \in \mathcal{H}_\mu^\infty(B_X) : \lim_{\|z\|\to1} \mu(z)|f(z)| = 0 \Big\},$$

$$\mathcal{B}_\nu(B_X) := \Big\{ f \in \mathscr{H}(B_X) :$$
$$\|f\|_{s\mathcal{B}_\nu(B_X)} := \sup_{z\in B_X} \nu(z)|Rf(z)| < \infty \Big\}.$$

It is easy to check that $\mathcal{H}^\infty(B_X)$, $\mathcal{H}^\infty_\mu(B_X)$ and $\mathcal{B}_\nu(B_X)$ are Banach under following norms

$$\|f\|_\infty := \sup_{z\in B_X} |f(z)|,$$
$$\|f\|_{\mathcal{H}^\infty_\mu} := \sup_{z\in B_X} \mu(z)|f(z)|,$$
$$\|f\|_{\mathcal{B}_\nu(B_X)} := |f(0)| + \|f\|_{s\mathcal{B}_\nu(B_X)},$$

respectively.

Now we consider the holomorphic function

$$g(z) := 1 + \sum_{k>k_0} 2^k z^{n_k}, \quad z \in \mathbb{B}_1, \qquad (3.2)$$

where $k_0 = \big[\log_2 \frac{1}{\nu(\delta)}\big]$, $n_k = \big[\frac{1}{1-r_k}\big]$ with $r_k = \nu^{-1}(1/2^k)$ for every $k \geq 1$. Here the symbol $[x]$ means the greatest integer not bigger than x. By [15], Theorem 2.3, $g(t)$ is increasing on $[0,1)$ and

$$|g(z)| \leq g(\|z\|) \in \mathbb{R}, \quad z \in \mathbb{B}_1,$$
$$0 < C_1 := \inf_{t\in[0,1)} \nu(t)g(t) \leq \sup_{t\in[0,1)} \nu(t)g(t)$$
$$\leq \sup_{z\in\mathbb{B}_1} \nu(z)|g(z)| =: C_2 < \infty. \qquad (3.3)$$

Lemma 3.1. Let ν be a normal weight on B_X. Then there exists $C > 0$ such that for every $z \in B_X$ we have

$$|f(z)| \leq \mu(z)^{-1}\|f\|_{\mathcal{H}^\infty_\mu(B_X)} \quad \forall f \in \mathcal{H}^\infty_\mu(B_X), \qquad (3.4)$$
$$|f(z)| \leq C(1+I^1_\nu(z))\|f\|_{\mathcal{B}_\nu(B_X)} \quad \forall f \in \mathcal{B}_\nu(B_X). \qquad (3.5)$$

Proof. The inequality (3.4) is obvious. The inequality (3.5) was proved in [9] (Proof of Theorem 3.2). □

Lemma 3.2. Let ν be a normal weight on B_X. Then,

(1) $\|\delta_z^{\mathcal{H}^\infty_\nu(B_X)}\| = 1/\nu(z)$;

(2) $\|\delta_z^{\mathcal{B}_\nu(B_X)}\| \asymp 1 + I^1_\nu(z)$.

Proof. (1) It is obvious.

(2) It follows easily from the definition of $\delta_z^{\mathcal{B}_\nu(B_X)}$ and (3.5) that

$$\|\delta_z^{\mathcal{B}_\nu(B_X)}\| \lesssim 1 + I^1_\nu(z).$$

Now we consider the function f_0 given by

$$f_0(z) = (1+C_2)^{-1}\Big(1+\int_0^{\|z\|} g(t)dt\Big), \quad z \in B_X,$$

where g defined by (3.2). It is clear that $f_0 \in \mathcal{B}_\nu(B_X)$ and by (3.3), it is easy to see that $\|f_0\|_{\mathcal{B}_\nu(B_X)} \leq 1$. Then, in view of (3.3) again, this yields that
$\|\delta_z^{\mathcal{B}_\nu(B_X)}\| \geq |f_0(z)|$

$$\geq \max\Big\{\frac{1}{1+C_2}, \frac{C_1}{1+C_2}\Big\}(1 + I^1_\nu(z)).$$

□

It is easy to prove the following:

Corollary 3.3. $\mathcal{H}^\infty_\nu(B_X)$, $\mathcal{B}_\nu(B_X)$ satisfy the properties (e1), (e2), (e3).

We will show below that the study of growth spaces on the unit ball can be reduced to studying functions defined on finite dimensional subspaces. Note that, the similar results for Bloch-type spaces have just been studied in [9].

For each finite subset $F \subset \Gamma$, in symbol $|F| = m < \infty$, we denote by $\mathbb{B}_{[F]}$ the unit ball of $\mathrm{span}\{e_k,\ k \in F\}$. Without loss of generality we may assume that $F = \{1, \ldots, m\}$, and hence $\mathbb{B}_{[F]} = \mathbb{B}_m$. For each $m \in \mathbb{N}$, we denote

$$\mu^{[m]} = \mu\big|_{\mathrm{span}\{e_1,\ldots,e_m\}},$$
$$z_{[m]} := (z_1, \ldots, z_m) \in \mathbb{B}_m.$$

For $m \geq 2$, by

$$OS_m := \{x = (x_1, \ldots, x_m),$$
$$x_k \in X, \langle x_k, x_j\rangle = \delta_{kj}\}.$$

we denote the family of orthonormal systems of order m. It is clear that OS_1 is the unit sphere of X.

For every $x \in OS_m$, $f \in \mathscr{H}(B_X)$, we define

$$f_x(z_{[m]}) = f\Big(\sum_{k=1}^m z_k x_k\Big).$$

Then

$$\Big\|\nabla f_x(z_{[m]})\Big\| = \Big\|\nabla f\Big(\sum_{k=1}^m z_k x_k\Big)\Big\|. \quad (3.6)$$

Definition 3.1. Let $\mathbb{B}_1$ be the open unit ball in $\mathbb{C}$ and $f \in \mathscr{H}(B_X)$. We define an affine norm as follows

$$\|f\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)} := \sup_{\|x\|=1} \|f(\cdot\, x)\|_{\mathcal{H}^\infty_\mu(\mathbb{B}_1)},$$

where $f(\cdot\, x) : \mathbb{B}_1 \to \mathbb{C}$ given by $f(\cdot\, x)(\lambda) = f(\lambda x)$ for every $\lambda \in \mathbb{B}_1$, and

$$\|f(\cdot\, x)\|_{\mathcal{H}^\infty_\mu(\mathbb{B}_1)} = \sup_{\lambda\in\mathbb{B}_1} \mu(\lambda x)|f(\lambda x)|.$$

It is easy to see that the space $\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)$

$$:= \{f \in \mathscr{H}(B_X) : \ \|f\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)} < \infty\}$$

is Banach under the norm $\|\cdot\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)}$.

Proposition 3.4. Let $f \in \mathscr{H}(B_X)$. The following are equivalent:

(1) $f \in \mathcal{H}^\infty_\mu(B_X)$;

(2) $\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty$ for every $m \geq 2$;

(3) $\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty$ for some $m \geq 2$.

Moreover, for each $m \geq 2$

$$\|f\|_{\mathcal{H}^\infty_\mu(B_X)} = \sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)}. \quad (3.7)$$

Proof. (1) $\Rightarrow$ (2): Let $m \geq 2$ and $z_{[m]} := (z_1, \ldots, z_m) \in \mathbb{B}_m$. Since $\big\|\sum_{j=1}^m z_j e_j\big\| = \big\|z_{[m]}\big\|$, we get

$$\begin{aligned}\|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(B_X)} &= \sup_{z_{[m]}\in\mathbb{B}_m} \mu^{[m]}(z_{[m]})|f_x(z_{[m]})| \\ &\leq \sup_{z\in B_X} \mu(z)\Big|f\Big(\sum_{j\in F} z_j e_j\Big)\Big| \\ &\leq \|f\|_{\mathcal{H}^\infty_\mu(B_X)} < \infty. \end{aligned} \quad (3.8)$$

In particular, we obtain (2).

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): Assume that there exists $m \geq 2$ such that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty.$$

We fix $z \in B_X$, $z \neq 0$. Consider $x = (\frac{z}{\|z\|}, x_2, \ldots, x_m) \in OS_m$ and put $z_{[m]} := (\|z\|, 0, \ldots, 0) \in \mathbb{C}^m$. Then $\|z_{[m]}\| = \|z\|$ and

$$\big|f_x(z_{[m]})\big| = \Big|f\Big(\sum_{k=1}^m z_k x_k\Big)\Big| = |f(z)|.$$

This implies that

$$\begin{aligned}\|f\|_{\mathcal{H}^\infty_\mu(B_X)} &= \sup_{z\in B_X} \mu(z)|f(z)| \\ &\leq \sup_{z\in B_X} \mu^{[m]}(z_{[m]})|f_x(z_{[m]})| \\ &\leq \sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty. \end{aligned} \quad (3.9)$$

Thus $f \in \mathcal{H}^\infty_\mu(B_X)$.

On the other hand, it is obvious that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} \leq \|f\|_{\mathcal{H}^\infty_\mu(B_X)}$$

for every $m \geq 2$. (3.10)

Hence, we obtain (3.7) from (3.8), (3.9) and (3.10). $\square$

Remark 3.2. The proposition is not true for the case $m = 1$.

Indeed, let X be a Hilbert space with the orthonormal basis $\{e_n\}_{n\geq1}$. Consider $\mu(z) := 1 - \|z\|^2$, and $f : B_X \to \mathbb{C}$ given by

$$f(z) := \sum_{n=1}^\infty \Big\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n\Big\rangle, \quad z \in B_X.$$

Then $f \in \mathscr{H}(B_X)$ because $\sum_{n=1}^\infty \Big|\Big\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n\Big\rangle\Big|^2$

$$\leq \sum_{n=1}^\infty \frac{1}{n^2} + \|z\|^2 + \sum_{n=1}^\infty \frac{2}{n^{3/2}} < \infty.$$

For each $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \in OS_1$ and for every $z_{[1]} := z_k \in \mathbb{B}_1$ for some $k \geq 1$, we have

$$f_x(z_{[1]}) = f(z_k x_k) = \frac{1}{k} - \frac{z_k x_k}{\sqrt{k}},$$

and thus, since $|f_x(z_{[1]})| \leq 2$, we get $\sup_{x \in OS_1} \|f_x\|_{\mathcal{H}_\mu^\infty(\mathbb{B}_1)}$

$$= \sup_{x \in OS_1} (1 - \|z_{[1]}\|^2) |f_x(z_{[1]})| \leq 2.$$

However, $f \notin \mathcal{H}_\mu^\infty(B_X)$ because for every $z \in B_X$, we have
$(1 - \|z\|^2)|f(z)|$

$$= (1 - \|z\|^2) \left| \sum_{n=1}^{\infty} \left\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n \right\rangle \right| \to \sum_{n=1}^{\infty} \frac{1}{n}$$

as $z \to 0$.

Using a similar argument to that in the proof of Proposition 2.3 in [9], we easily obtain the following result, for which the proof will be omitted.

Proposition 3.5. The spaces $\mathcal{H}_\mu^\infty(B_X)$ and $\mathcal{H}_{\mu,\mathrm{aff}}^\infty(B_X)$ coincide. Moreover,

$$\|f\|_{\mathcal{H}_\mu^\infty(B_X)} \leq \|f\|_{\mathcal{H}_{\mu,\mathrm{aff}}^\infty(B_X)} \lesssim \|f\|_{\mathcal{H}_\mu^\infty(B_X)} \quad \forall f \in \mathcal{H}_\mu^\infty(B_X).$$

4. THE BOUNDEDNESS AND THE COMPACTNESS OF $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$

In this section we consider the weighted composition operator $W_{\psi,\varphi}$ from $\mathcal{B}_\nu(B_X)$ into $\mathcal{H}_\mu^\infty(B_X)$ and into $\mathcal{H}_\mu^0(B_X)$ defined by

$$(W_{\psi,\varphi} f)(z) := \psi(z) \cdot (f \circ \varphi)(z), \quad z \in B_X.$$

The component operators are the multiplication operator $M_\psi f = \psi \cdot f$ and the composition operator $C_\varphi f = f \circ \varphi$, which correspond to the cases when the composition symbol φ is the identity function on $\mathbb{B}$ and the multiplication symbol ψ is the constant function 1, respectively.

Theorem 4.1. The following are equivalent:

(1) $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded;

(2) $M_{\psi,\varphi,\mu}^{[m]} := \sup_{z \in \mathbb{B}_m} \mu^{[m]}(z) |\psi^{[m]}(z)| \|\delta_{\varphi^{[m]}(z)}^{\mathcal{B}_\nu(B_X)}\| < \infty$ for some $m \geq 2$;

(3) $M_{\psi,\varphi,\mu} := \sup_{z \in B_X} \mu(z) |\psi(z)| \|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| < \infty.$

Moreover, in this case

$$\|W_{\psi,\varphi}\| = M_{\psi,\varphi,\mu}. \tag{4.1}$$

Proof. (3) $\Rightarrow$ (2): It is clear.

(1) $\Rightarrow$ (3): Suppose $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded. Fix $z \in B_X$. For each $f \in \mathcal{B}_\nu(B_X)$ with $\|f\|_{\mathcal{B}_\nu(B_X)} \leq 1$, we have

$$\mu(z)|\psi(z)||f(\varphi(z))| \leq \|W_{\psi,\varphi} f\|_{\mathcal{H}_\mu^\infty(B_X)} \leq \|W_{\psi,\varphi}\| \|f\|_{\mathcal{B}_\nu(B_X)} \leq \|W_{\psi,\varphi}\|.$$

By definition of $\delta^{\mathcal{B}_\nu(B_X)}$ (see Proposition 2.1), taking the supremum over all f in the closed unit ball of $\mathcal{B}_\nu(B_X)$, we obtain

$$\mu(z)|\psi(z)| \|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| \leq \|W_{\psi,\varphi}\|.$$

Taking the supremum over all $z \in B_X$ yields

$$M_{\psi,\varphi,\mu} \leq \|W_{\psi,\varphi}\| < \infty. \tag{4.2}$$

(2) $\Rightarrow$ (1): Assume $M_{\psi,\varphi,\mu}^{[m]} < \infty$ for some $m \geq 2$. Let $f \in \mathcal{B}_\nu(B_X)$ with $\|f\|_{\mathcal{B}_\nu(B_X)} \leq 1$. For each $x \in OS_m$, we write $z_x := \sum_{k=1}^m z_k x_k$. Note that $\|z_x\| = \|z_{[m]}\|$ and hence $\mu^{[m]}(z_{[m]}) = \mu^{[m]}(z_x)$. Then $\|(W_{\psi,\varphi}(f))_x\|_{\mathcal{H}_{\mu^{[m]}}^\infty(\mathbb{B}_m)}$

$$= \sup_{z_x \in \mathbb{B}_m} \mu^{[m]}(z_x) |\psi^{[m]}(z_x) (f \circ \varphi)_x(z_{[m]})| \leq M_{\psi,\varphi,\mu}^{[m]} < \infty$$

for every $x \in OS_m$. By (3.7), $W_{\psi,\varphi}$ is bounded because $\|W_{\psi,\varphi}(f)\|_{\mathcal{H}_\mu^\infty(B_X)}$

$$= \sup_{x \in OS_m} \|(W_{\psi,\varphi}(f))_x\|_{\mathcal{H}_{\mu^{[m]}}^\infty(\mathbb{B}_m)} \leq M_{\psi,\varphi,\mu}^{[m]} < \infty.$$

(4) $\Rightarrow$ (2): For $z \in B_X$, we have

$$\mu(z)|\psi(z)||f(\varphi(z))| \leq \mu(z)|\psi(z)|\|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| \leq M_{\psi,\varphi,\mu} < \infty.$$

Therefore, taking the supremum over all $z \in B_X$, we obtain

$$\|W_{\psi,\varphi} f\|_{\mathcal{H}_\mu^\infty(B_X)} \leq M_{\psi,\varphi,\mu} < \infty. \quad (4.3)$$

Finally, from (4.2), (4.3) we deduce (4.1). □

We next characterize the compactness of the operators $W_{\psi.\varphi}$. As in [10] we can prove the following:

Lemma 4.2 ([10], Lemma 2.10). Let $\mathcal{E}, \mathcal{F}$ be two Banach spaces of holomorphic functions on B_X. Suppose that

(1) The point evaluation functionals on $\mathcal{E}$ are continuous;

(2) The closed unit ball of $\mathcal{E}$ is a compact subset of $\mathcal{E}$ in the topology of uniform convergence on compact sets;

(3) $T : \mathcal{E} \to \mathcal{F}$ is continuous when $\mathcal{E}$ and $\mathcal{F}$ are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{E}$ such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $\mathcal{F}$.

Theorem 4.3. Assume that $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded. Then, the following are equivalent:

(1) There exists $m \geq 2$ such that

$$\lim_{r\to 1} \sup_{\|\varphi_{(m)}(z)\|>r} \mu(z)|\psi(z)|\|\delta_{\varphi_{(m)}(z)}^{\mathcal{B}_\nu(B_X)}\| = 0, \quad (4.4)$$

where $\varphi_{(m)} := (\varphi_1, \dots, \varphi_m)$.

(2) $W_{\psi,\varphi}$ is compact.

Proof. First, we show that $\psi \in \mathcal{H}_\mu^\infty(B_X)$. Indeed, since $W_{\psi,\varphi}$ is bounded, by Theorem 4.1, $M_{\psi,\varphi,\mu} < \infty$. Then, by Remark 2.1, $\inf_{z\in B_X} \|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| =: \alpha > 0$. Consequently,

$$\alpha\mu(z)|\psi(z)| < M_{\psi,\varphi,\mu}, \quad z \in B_X.$$

This means $\psi \in \mathcal{H}_\mu^\infty(B_X)$.

(2) $\Rightarrow$ (1): Suppose $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is compact. Fix $m \geq 2$. It is obvious that (4.4) holds if $\varphi_{(m)}(B_X)$ is relatively compact in B_X. So assume $\overline{\varphi_{(m)}(B_X)} \cap \partial B_X \neq \varnothing$.

Let $\{z^n\}_{n\geq 1}$ be a sequence in B_X such that $\|\varphi_{(m)}(z^n)\| \to 1$. By the definition of $\delta_{\varphi_{(m)}(z^n)}^{\mathcal{B}_\nu(B_X)}$, with $\varepsilon > 0$ is given we can find a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{B}_\nu(B_X)$ with $\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq 1$ for every $n \geq 1$ satisfying

$$|f_n(\varphi_{(m)}(z^n))| > \|\delta_{\varphi_{(m)}(z^n)}^{\mathcal{B}_\nu(B_X)}\| - \varepsilon. \quad (4.5)$$

By the condition (e2), without loss of generality, we may assume that $f_n \to 0 \in \mathcal{B}_\nu(B_X)$ uniformly on compact subsets of B_X and $\{f_n\}_{n\geq 1}$ is uniformly bounded on compact sets.

For each $n \geq 1$, denote $a^n := \varphi(z^n)$ and consider the automorphism $\Phi_{a^n} \in Aut(B_X)$ defined by (2.1). For each $j \in \{1, \dots, m\}$, put

$$G_{a^n,j} := (a^n_{(m)})_j \cdot f_n - ((\Phi_{a^n})_{(m)})_j \cdot f_n.$$

By (e3), $G_{a^n,j} \in \mathcal{B}_\nu(B_X)$. It is an easy calculation that for every $w \in B_X$,

$$|G_{a^n,j}(w)| = |(a^n_{(m)})_j \cdot f_n(w) - ((\Phi_{a^n})_{(m)}(w))_j| \leq \frac{3\sqrt{1 - \|a^n_{(m)}\|^2}}{1 - \|w\|}|f_n(w)|.$$

Then, by (2.2),

$$|G_{a^n,j}(w)| \leq \frac{3\sqrt{1 - \|a^n_{(m)}\|^2}}{1 - \|w\|}\|\delta_w\|,$$

consequently, by Proposition 2.1, and since $\|a^n_{(m)}\| = \|\varphi_{(m)}(z^n)\| \to 1$ as $n \to \infty$, the sequence $\{G_{a^n,j}\}_{n\geq 1}$ is a sequence of holomorphic functions converging to 0 uniformly on compact subsets of B_X for each $j \in \{1, \dots, m\}$.

Now by the condition (e3), there exists $C > 0$ such that for all $j \in \{1, \dots, m\}$, we

have
$\|G_{a^n,j}\|_{\mathcal{B}_\nu(B_X)}$

$$\leq \|a^n_{(m)}\|\|f_n\|_{\mathcal{B}_\nu(B_X)} + \|((\Phi_{a^n})_{(m)})_j \cdot f_n\|_{\mathcal{B}_\nu(B_X)}$$
$$\leq (C+1)\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq C+1 \quad \forall n \geq 1.$$

By (2.2), any bounded sequence in $\mathcal{B}_\nu(B_X)$ is uniformly bounded on compact sets and any sequence in $\mathcal{B}_\nu(B_X)$ that converges to 0 in norm, also converges uniformly on compact sets. Therefore, since $W_{\psi,\varphi}$ is compact, by Lemma 4.2, $\|\psi \cdot ((G_{a^n})_j \circ \varphi)\|_{\mathcal{H}^\infty_\mu(B_X)} \to 0$ as $n \to \infty$ for every $j \in \{1, \dots, m\}$. Note that $\Phi_{a^n}(a^n) = 0$. Therefore, by (4.5), we have

$$\mu(z^n)|\psi(z^n)|\|\varphi_{(m)}(z^n)\|(\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\| - \varepsilon)$$
$$\leq \mu(z^n)|\psi(z^n)|\|\varphi_{(m)}(z^n)\||f_n(\varphi_{(m)}(z^n))|$$
$$= \mu(z^n)|\psi(z^n)|\sqrt{\sum_{j=1}^m |(G_{a^n,j})(\varphi_{(m)}(z^n))|^2}$$
$$= \sqrt{\sum_{j=1}^m \|\psi \cdot ((G_{a^n,j}) \circ \varphi)\|^2_{H^\infty_\mu(B_X)}}$$
$$\to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\lim_{n\to\infty} \mu(z^n)|\psi(z^n)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\|$$
$$< \varepsilon \lim_{n\to\infty} \mu(z^n)|\psi(z^n)| \leq \varepsilon\|\psi\|_{H^\infty_\mu(B_X)}.$$

Since ε is arbitrary, it follows that (4.4) holds.

(1) $\Rightarrow$ (2): Assume that there exists $m \geq 2$ such that (4.4) holds. By Lemma 4.2, it suffices to show that if $\{f_n\}_{n\geq 1}$ is a sequence in $\mathcal{B}_\nu(B_X)$ converging to 0 uniformly on compact subsets of B_X such that $\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq 1$ for all $n \geq 1$ then $\|W_{\psi,\varphi} f_n\|_{H^\infty_\mu(B_X)} \to 0$ as $n \to \infty$.

Let $\{f_n\}_{n\geq 1}$ be such a sequence, fix $\varepsilon > 0$ and choose a number $r \in (0,1)$ such that $\mu(z)|\psi(z)|\|\delta_{\varphi_{(m)}(z)}\| < \varepsilon$ whenever $\|\varphi_{(m)}(z)\| > r$. Since for all $w \in B_X$, $|f_n(w)| \leq \|\delta^{\mathcal{B}_\nu(B_X)}_w\|$, if $\|\varphi_{(m)}(z)\| > r$, then $\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon$. Thus $\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon$ if $\|\varphi(z)\| > r$, because $\|\varphi(z)\| \geq \|\varphi_{(m)}(z)\| > r$ for every $z \in B_X$.

Now, we consider the case $\|\varphi(z)\| \leq r$. Then $\|\varphi_{(m)}(z)\| \leq r$. Note that
$B[\varphi_{(m)}, r]$

$$:= \{\varphi_{(m)}(y) : \|\varphi_{(m)}(y)\| < r, y \in B_X\}$$
$$\subset \mathbb{B}_m \subset \mathbb{C}^m$$

is relatively compact for every $0 \leq r < 1$, by the hypothesis, $f_n \to 0$ uniformly on $\overline{B[\varphi_{(m)}, r]}$. Then, there exists a natural number N such that $|f_n(w)| < \varepsilon/\|\psi\|_{H^\infty_\mu(B_X)}$ for all $n \geq N$ whenever $w \in \overline{B[\varphi_{(m)}, r]}$. Thus,

$$\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon \quad \text{if } \|\varphi(z)\| \leq r.$$

□

We now discuss the boundedness and the compactness of the weighted composition operator mapping into $\mathcal{H}^0_\mu(B_X)$.

Theorem 4.4. The following are equivalent:

(1) $\psi \in \mathcal{H}^0_\mu(B_X)$, and there exists $m \geq 2$,

$$\varphi_{(m)}(rB_X) \text{ is relatively compact for every } 0 \leq r < 1, \tag{4.6}$$
$$\lim_{\|z\|\to 1} \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\| = 0, k \geq 1; \tag{4.7}$$

(2) $W^0_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}^0_\mu(B_X)$ is compact.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. Fix $f \in \mathcal{B}_\nu(B_X)$. We show that $W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi) \in \mathcal{H}^\infty_\mu(B_X)$. Since $\mu(z)|\psi(z)|f(\varphi_{(k)}(z)) \to \mu(z)|\psi(z)|f(\varphi(z))$ as $k \to \infty$ for each $z \in B_X$, and $\mathcal{H}^0_\mu(B_X)$ is closed in $\mathcal{H}^\infty_\mu(B_X)$, it suffices to check that $\psi \cdot (f \circ \varphi_{(k)}) \in \mathcal{H}^0_\mu(B_X)$ for every $k \geq 1$. Given $k \geq 1$. By the hypothesis (1), for given $\varepsilon > 0$ there exists $r \in (0,1)$ such that
$\mu(z)|\psi(z)||f(\varphi_{(k)}(z))|$

$$\begin{aligned}&\leq \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\|\|f\|_{\mathcal{B}_\nu(B_X)} \\ &\leq \varepsilon\|f\|_{\mathcal{B}_\nu(B_X)} \quad \text{for } \|z\| > r.\end{aligned} \tag{4.8}$$

On the other hand, it follows from assumption (1) that

$\sup_{\|z\|\leq r} \mu(z)|\psi(z)||f(\varphi_{(k)}(z))|$

$$\leq \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\|\|f\|_{\mathcal{B}_\nu(B_X)} < \infty. \tag{4.9}$$

Consequently, $\psi\cdot(f\circ\varphi_{(k)}) \in \mathcal{H}^\infty_\mu(B_X)$. Moreover, by (4.8), $\psi \cdot (f\circ\varphi_{(k)}) \in \mathcal{H}^0_\mu(B_X)$.

We also obtain from (4.8) and (4.9) that $W^0_{\psi,\varphi}$ is bounded.

The compactness of the operator $W^0_{\psi,\varphi}$ now follows by arguing as in the proof of Theorem 4.3 and the condition (4.6).

(2) $\Rightarrow$ (1): First, since $W^0_{\psi,\varphi}$ is bounded and $1 \in \mathcal{B}_\nu(B_X)$ it is easy to check that $\psi \in H^0_\mu(B_X)$.

In order to prove (4.6), first we have to show the following claim:

$$\frac{1}{2}\|z-w\| \leq \|\delta^{\mathcal{H}^\infty_\mu(B_X)}_z - \delta^{\mathcal{H}^\infty_\mu(B_X)}_w\|, \quad z,w\in B_X. \tag{4.10}$$

Indeed, it is easy to check by direct calculation that

$$\frac{1}{2}\|z-w\| \leq \sqrt{1-\frac{(1-\|z\|^2)(1-\|w\|^2)}{|1-\langle z,w\rangle|^2}} = \varrho_X(z,w),$$

where ϱ_X is the pseudohyperbolic metric in B_X (see [16] p.99). On the other hand, $\varrho_X(z,w) = \sup\{\varrho(f(z),f(w)) :$

$$f\in\mathcal{H}^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}$$

(see (3.4) in [5]), where $\varrho(x,y) = |\frac{x-y}{1-\bar{x}y}|$, $x,y\in\mathbb{B}_1$, is the pseudohyperbolic metric in $\mathbb{B}_1$. Note that, since the function $\eta\mapsto\frac{\eta}{1-\overline{f(z)}f(w)}$ is holomorphic from $\mathbb{B}_1$ into $\mathbb{B}_1$ and $f(z)-f(w)\mapsto 0$, it follows from Schwarz's lemma that $\varrho(f(z),f(w)) \leq |f(z)-f(w)|$ for every $z,w\in B_X$. Consequently,

$$\begin{aligned}
&\varrho_X(z,w)\\
&\leq \sup\{|f(z)-f(w)| : \\
&\qquad \text{for } f\in\mathcal{H}^\infty(B_X) \text{ with } \|f\|_\infty\leq 1\}\\
&\leq \sup\{|\delta^{\mathcal{H}^\infty_\mu(B_X)}_z(f) - \delta^{\mathcal{H}^\infty_\mu(B_X)}_w(f)| : \\
&\qquad \text{for } f\in\mathcal{H}^\infty(B_X) \text{ with } \|f\|_\infty\leq 1\}\\
&= \|\delta^{\mathcal{H}^\infty_\mu(B_X)}_z - \delta^{\mathcal{H}^\infty_\mu(B_X)}_w\|.
\end{aligned}$$

Hence, (4.10) is proved.

Next, we prove (4.6). For $0<r<1$, the set $V_r := \{\delta^{\mathcal{H}^\infty_\mu(B_X)}_z : \|z\|\leq r\} \subset (\mathcal{H}^\infty_\mu(B_X))'$ is bounded. Then, by the compactness of $W_{\psi,\varphi}$, the set

$$(W_{\psi,\varphi})^*(V_z) = \left\{\psi(z)\delta^{\mathcal{B}_\nu(B_X)}_{\varphi(z)} : \|z\|\leq r\right\}$$

is relatively compact in $[\mathcal{B}_\nu(B_X)]'$.

It should be noted that, for every subset K of the dual of a Banach space $\mathcal{B}_\nu(B_X)$ and every bounded subset $D\subset\mathbb{C}$, if the set $\{t\eta : t\in D, \eta\in A\}$ is relatively compact in $\mathcal{B}_\nu(B_X)$ then $A\subset[\mathcal{B}_\nu(B_X)]'$ is relatively compact. With this fact in hand, since the set $\{\psi(z) : \|z\|\leq r\}$ is bounded, the set $\{\delta^{\mathcal{B}_\nu(B_X)}_z, \|z\|\leq r\}$ is relatively compact. Then, it follows from the inequality (4.10) that $\varphi(rB_X)$ is relatively compact, so is $\varphi_{(m)}(rB_X)$ for $m\geq 2$.

Finally, we prove (4.7). Assume that there exist $m\geq 1$, $\varrho>0$ and a sequence $\{z^n\}_{n\geq 1}\subset B_X$ such that $\|z^n\|\to 1$ and $\mu(z^n)|\psi(z^n)|\|\delta_{\varphi_{(m)}(z^n)}\| > \varrho$ for all $n\geq 1$. Then, we may choose $\{f_n\}_{n\geq 1}\subset\mathcal{B}_\nu(B_X)$ such that $\|f_n\|_{\mathcal{B}_\nu(B_X)}\leq 1$ and $|f_n(\varphi_{(m)}(z^n))| > \|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\| - \varrho/2$ for every $n\geq 1$. Thus

$$\mu(z^n)|\psi(z^n)||f(\varphi_{(m)}(z^n))| > \varrho - \varrho/2\mu(z^n)|\psi(z^n)|.$$

Therefore, since $\psi\in\mathcal{H}^0_\mu(B_X)$, $W^0_{\psi,\varphi}f_n \notin \mathcal{H}^0_\mu(B_X)$. This contradicts the boundedness of $W^0_{\psi,\varphi}$. □

Remark 4.1. In the case of $\dim X<\infty$, and following the proof of Theorem 4.4, the following statements are equivalent:

(1) $\lim_{\|z\|\to 1} \mu(z)|\psi(z)|\|\delta_{\varphi_{(k)}(z)}^{\mathcal{B}_\nu(B_X)}\| = 0$ for every $k \geq 1$ and $\psi \in \mathcal{H}_\mu^\infty(B_X)$;

(2) $W_{\psi,\varphi}^0 : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^0(B_X)$ is compact;

(3) $W_{\psi,\varphi}^0 : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^0(B_X)$ is bounded.

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