Một điều kiện tương đương để một họ ma trận chuẩn tắc là chéo hóa tương đẳng đồng thời được

TÓM TẮT

Trong bài báo này, chúng tôi đưa ra một điều kiện cần và đủ để một họ hữu hạn các ma trận chuẩn tắc là chéo hóa đồng thời được qua phép $*$-tương đẳng (gọi tắt là *SDC*). Bên cạnh đó, chúng tôi cũng xây dựng một gói lệnh MATLAB tương ứng để kiểm tra một họ hữu hạn các ma trận chuẩn tắc có là SDC hay không, và xác định ma trận tương đẳng làm chéo hóa đồng thời các ma trận ban đầu nếu các ma trận là SDC. Một số ví dụ minh họa và kiểm tra gói lệnh này cũng được trình bày trong bài báo.

Từ khóa: **-tương đẳng, chéo hóa tương đẳng đồng thời, chéo hóa tương đương đồng thời, ma trận chuẩn tắc.*

An alternative equivalent condition for a finite family of normal matrices to be simultaneously diagonalizable via congruence

ABSTRACT

This paper gives an alternative equivalent condition for a finite family of normal matrices to be simultaneously diagonalizable via $*$-congruence. The matrices do not need pairwise commute. A corresponding MATLAB package is developed. Some numerical tests for this package are also presented.

Keywords: **-congruence, simultaneous diagonalization via congruence, simultaneous diagonalization via similarity, normal matrices.*

1. INTRODUCTION

The problems of simultaneously diagonalizing a family of matrices (via congruence or similarity) are known to be long-standing due to their applications, for examples, signal processing, data analysis and multi-linear algebra,[1] quadratic equations and optimization,[2, 3] ...

There is a relationship between two concepts of simultaneous diagonalizations via similarity (SDS) and via congruence (SDC); see, e.g., in the book of Horn and Johnson.[4] One should, hence, need to distinguish these existing concepts as follows.

Notations and definitions. Let $\mathbb{F}$ denote the field of real numbers $\mathbb{R}$ or complex ones $\mathbb{C}$, and $\mathbb{F}^{n\times n}$ be the set of all square matrices of order n with entries in $\mathbb{F}$. Let $\mathcal{S}^n$, $\mathcal{H}^n$, and $\mathcal{N}^n$ denote the sets of real symmetric, Hermitian, and normal matrices in $\mathbb{F}^{n\times n}$, respectively. By $.^*$, $.^T$, we denote the conjugate transpose and transpose of a matrix, respectively. For $A \in \mathcal{H}^n$, we write $A \succeq 0$ (resp., $A \succ 0$) for the meaning that A is positive semidefinite (resp., positive definite). As usual, $I_{n\times d}$ denotes the $n \times d$ identity matrix, and we shortly write I_n if $n = d$.

Matrices $C_1, \ldots, C_m \in \mathbb{F}^{n\times n}$ are said to be

(i) *simultaneously diagonalizable via similarity* on $\mathbb{F}$, shortly $\mathbb{F}$-*SDS*, if there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that $P^{-1}C_iP$'s are all diagonal matrices in $\mathbb{F}^{n\times n}$. When $m = 1$, we will say "C_1 is $\mathbb{F}$-*DS*", or $\mathbb{F}$-diagonalizable as usual;

(ii) *simultaneously diagonalizable via* $*$*-congruence* on $\mathbb{F}$, abbreviated $*$-*SDC*, if there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that P^*C_iP is diagonal for every $i = 1, \ldots, m$. When $m = 1$, we will say "C_1 is $*$-*DC*";

In case C_i's are all Hermitian, it is worth mentioning that the diagonal matrices P^*C_iP's are always real due to the Hermitianian of C_i's. Moreover, P can be chosen to be real if C_i's are all real.[5]

(iii) *simultaneously diagonalizable via* T*-congruence* on $\mathbb{F}$, abbreviated T-*SDC*, if there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that P^TC_iP is diagonal for every $i = 1, \ldots, m$. When $m = 1$, we will say "C_1 is T-*DC*".

Unlike the $*$-SDC case, the diagonal matrices P^*C_iP's do not need to be real even C_i's are real symmetric. The readers are referred to the work by Bustamante et. al. [6] for the T-SDC properties;

(iv) *commuting* if they pairwise commute: $C_iC_j = C_jC_i$ for every $i, j = 1, \dots, m$.

In the rest of this paper, the term "SDC" will mean either "*simultaneous diagonalization via congruence*" or "*simultaneously diagonalizing via congruence*", or "*simultaneously diagonalizable via congruence*", and depending upon the situation, we will recognize the $*$- or $T-$congruence. It is analogous to the term "*SDS*".

An overview of the SDC problem. The SDC problem is known that first appeared in 1868 by Weierstrass,[7] in the 1930s by Albert,[8] Finsler,[9] Hertenes,[10] and later studies developed some conditions ensuring that two quadratic forms are SDC (see, e.g., in., works by More [11] and Pong [12] and references therein). However, these works provide only sufficient conditions, except for a few ones.[4, 13]

From the practical point of view, Bunse-Gerstne et al. [14] proposed a Jacobi-like algorithm for SDC two commuting normal matrices, and this is numerically extended to several commuting ones by Mendle.[15] Recently, there have been some works [3, 5, 16, 17] that present some (equivalent or sufficient) conditions for the $*$-SDC property of collections of either complex or real Hermitian matrices, and another one deals with the T-SDC problem for complex symmetric matrices.[6] It is noticed that the $*$- and T-congruences coincide only when the initial matrices are real symmetric, which are also real Hermitian. The two $*$-SDC and T-SDC problems are different, even if the initial matrices are symmetric. For example, Bustamante et. al. [6] show that the two real symmetric matrices

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{S}^2$$

are T-SDC. But they are not $*$-SDC over $\mathbb{C}$.[5]

Several works deal with the normal SDC problem, i.e., the simultaneous diagonalization of several normal matrices via $*$-congruence.[18, 19] However, they sound purely theoretical. There has been no algorithm to detect whether the given normal matrices are $*$-SDC.

Contribution of the paper. In this paper, we solve the normal SDC problem, i.e., the simultaneous diagonalization of several normal matrices via $*$-congruence. We first give a sufficient and necessary condition for a finite family of normal matrices to be simultaneously diagonalizable (via either congruence or similarity). It is noticed that the SDC property of a family of arbitrary square matrices can be checked by splitting the matrices into their Hermitian and skew-Hermitian parts.[5] The SDC property of the matrix family is confirmed if a positive definite matrix exists that solves a system of linear equations defined by the Hermitian and skew-Hermitian parts; see Theorem 7 below. The number of linear equations depends upon the number of Hermitian and skew-Hermitian parts. This may have a big computation complexity. Our (sufficient and necessary) condition in this paper restricts the number of such matrix linear equations.

On the other hand, we develop a MATLAB package to solve the normal SDC problem and its numerical tests.

Auxiliary results. We now recall some existing results on SDC that will be frequently used in this paper.

Lemma 1. [3] *Suppose there is $0 \neq \lambda \in \mathbb{R}^m$ such that $\mathbf{C}(\lambda) \succ 0$, where, without loss of generality, we assume $\lambda_1 \neq 0$. Then $C_1, \dots, C_m \in \mathcal{S}^n$ is SDC if and only if $P^{\mathrm{T}}C_iP$ and $P^{\mathrm{T}}C_jP$ commute for all $2 \leq i \neq j \leq m$, where P is determined such that $P^{\mathrm{T}}\mathbf{C}(\lambda)P = I$ (the identity matrix).*

As shown in the paper of Jiang and Li,[3] the matrix P in Lemma 1 is determined as $P = UD^{1/2}$, where U is orthogonal and $D^{1/2}$ is the square root of the diagonal matrix D in an eigenvalue decomposition of $\mathbf{C}(\lambda)$:

$$D = U^{\mathrm{T}}\mathbf{C}(\lambda)U.$$

The following results can be found in many books on Linear Algebra; their proofs are hence omitted in this paper.

Lemma 2. [4] *(i) Every $A \in \mathcal{H}^n$ can be diagonalized via similarity by a unitary matrix. That is, it can be written as $A = U\Lambda U^*$, where U is unitary, Λ is real diagonal and is uniquely defined up to a permutation of diagonal elements.*

Moreover, if $A \in \mathcal{S}^n$, then U is picked to be real.

(ii) Suppose each of $C_1, \dots, C_m \in \mathbb{F}^{n\times n}$ is $\mathbb{F}$-DS. Then, they are $\mathbb{F}$-SDS if and only if they are commuting.

(iii) Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{m\times m}$. The matrix $M = \mathrm{diag}(A, B)$ is diagonalizable via similarity if and only if so are both A and B.

(iv) A complex symmetric matrix A is diagonalizable via similarity, i.e., $P^{-1}AP$ is diagonal for some invertible matrix $P \in \mathbb{C}^{n\times n}$, if and only if it is complex orthogonally diagonalizable, i.e., $Q^{-1}AQ$ is diagonal for some complex orthogonal matrix $Q \in \mathbb{C}^{n\times n}$: $Q^TQ = I$.

(v) Suppose $A = \mathrm{diag}(\alpha_1 I_{n_1}, \dots, \alpha_k I_{n_k})$, α_i's are distinct. If $AB = BA$ then $B = \mathrm{diag}(B_1, \dots, B_k)$ with $B_i \in \mathbb{F}^{n_i\times n_i}$ for all $i = 1, \dots, k$. Furthermore, B is Hermitian (resp., symmetric) if and only if so are B_i's.

Construction of the paper. Section 2 is devoted to the SDC problem for normal matrices, in which we give a sufficient and necessary condition for a family of normal matrices to be SDC. And then, we propose a corresponding algorithm. Section 3 discusses the numerical experiments with respective our SDC algorithm in Section 2. We also give a numerical example illustrating our algorithm. Section 4 presents the conclusion.

2. THE NORMAL SDC PROBLEM

In this section, we deal with the normal SDC problem. Our conditions for a family of normal matrices to be $*$-SDC can be viewed as a generalization of that in Theorem 7 below. For convenience to the readers, we revisit these results as follows.

2.1 SDC and SDS of Hermitian matrices: revisited

We first summarize some existing results of the SDC and SDS of several Hermitian matrices. The following is presented in the book of Horn and Jonhson [4] whose proof does not completely give a nonsingular matrix that simultaneously diagonalizes the given matrices. Our proof leads to an algorithm that may be useful in practice. The idea is to follow that of proving Theorem 9 in the paper of Jiang and Duan **jiang2016** for real symmetric matrices.

Theorem 3. [5] *The matrices* $I, C_1, \ldots, C_m \in \mathcal{H}^n$, $m \geq 1$, *are SDC if and only if they are commuting. Moreover, when this is the case, they are SDC by a unitary matrix, and the resulting diagonal matrices are all real.*

Theorem 4. *Let* $A_1, \ldots, A_m \in \mathbb{F}^{n\times n}$, $m \geq 1$, *be such that each of them is diagonalizable via similarity. Then, these matrices are simultaneously diagonalizable via similarity (shortly, SDS) if and only if they pairwise commute.*

The following are not hard to prove, we omit their proofs.

Lemma 5. *The matrices* $C_1, \ldots, C_m \in \mathcal{H}^n$ *are SDC if and only if for any* $\lambda \in \mathbb{R}^m$ *with a* $\lambda_i \neq 0$, *the matrices* $C_1, \ldots, C_{i-1}, \sum_{t=1}^m \lambda_t C_t, C_{i+1}, \ldots, C_m$ *are SDC.*

Lemma 6. [5] *The matrices* $C_1 = \begin{bmatrix} \hat{C}_1 & 0 \\ 0 & 0_k \end{bmatrix}, \ldots, C_m = \begin{bmatrix} \hat{C}_m & 0 \\ 0 & 0_k \end{bmatrix}$ *are SDC if and only if so are* $\hat{C}_1, \ldots, \hat{C}_m$.

Using Theorem 3, we comprehensively describe the SDC property of a family of Hermitian matrices as follows.

As a consequence of Theorem 3, every commuting collection of Hermitian matrices can be SDC. However, this is just a sufficient but unnecessary condition. For example, the matrices

$$C_1 = \begin{bmatrix} -1 & -2 & 0 \\ -2 & -28 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 20 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

are SDC by

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

but $C_1C_2 \neq C_2C_1$. The following provides some equivalent SDC conditions for Hermitian matrices. It turns out that the SDC property of a family of such matrices is equivalent to the feasibility of a positive semidefinite program (SDP). This also allows us to use SDP solvers, for example, "CVX",[20] ... to check the SDC property of Hermitian matrices.

Theorem 7. [5] *The following conditions are equivalent:*

(i) *Matrices* $C_1, \ldots, C_m \in \mathcal{H}^n$ *are SDC.*

(ii) *There exists a nonsingular matrix* $P \in \mathbb{C}^{n\times n}$ *such that* $P^*C_1P, \ldots, P^*C_mP$ *are commuting.*

(iii) *There exists a positive definite matrix* $X = X^* \in \mathcal{H}^n$ *solves the following system of* $\frac{m(m-1)}{2}$ *linear equations*

$$C_iXC_j = C_jXC_i, \ 1 \leq i < j \leq m. \quad (1)$$

If $C_1, \ldots, C_m$ *are real, then one can pick* P *and* X *to be real.*

2.2 The normal SDC problem

Recall that a square matrix $N \in \mathbb{F}^{n\times n}$ is said to be normal if

$$N^*N = NN^*.$$

It is well-known that (real or complex) Hermitian, unitary, orthogonal matrices are normal, but the converse is not true in general. The readers are referred to, e.g., the work by Grone et al.,[21] for equivalent conditions for a normal matrix.

The third condition of Theorem 7 leads us to a sufficient and necessary condition for the $*$-SDC property of a family of arbitrary square matrices. This can be done by splitting the matrices into their Hermitian and skew-Hermitian parts as follows. For square matrices $A_1, \dots, A_m \in \mathbb{F}^{n\times n}$, their Hermitian and skew-Hermitian parts* are

$$A_i^{\mathfrak{h}} := \frac{A_i + A_i^*}{2}, \;\; A_i^{\mathfrak{s}} := \frac{A_i - A_i^*}{2\mathbf{i}}, \qquad (2)$$

where $\mathbf{i}$ is the imaginary unit, $\mathbf{i}^2 = -1$. Noticing that $A_i^{\mathfrak{h}}$ and $A_i^{\mathfrak{s}}$ are Hermitian and that

$$A_i = A_i^{\mathfrak{h}} + \mathbf{i}A_i^{\mathfrak{s}}, \;\; A_i^* = A_i^{\mathfrak{h}} - \mathbf{i}A_i^{\mathfrak{s}}. \qquad (3)$$

It is not hard to show that $A_1, \dots, A_m$ are $*$-SDC if and only if so are $A_i^{\mathfrak{h}}, A_i^{\mathfrak{s}}, i = 1, \dots, m$.

Lemma 8. [13] *The square matrices $A_1, \dots, A_m \in \mathbb{F}^{n\times n}$ are SDC if and only if so are $A_i^{\mathfrak{h}}, A_i^{\mathfrak{s}}$, $i = 1, \dots, m$.*

Theorem 7 and Lemma 8 lead to a sufficient and necessary condition for a family of arbitrary square matrices to be $*$-SDC, in which, after splitting up the initial matrices into Hermitian and skew-Hermitian, there are $m(m-1)$ matrix equations as in (1). One can apply Theorem 7 and Lemma 8 to normal matrices. Below, we will introduce a smaller number of normal matrix equations; see Theorem 11.

Since any normal matrix is always diagonalizable by a unitary one,[4] it is diagonalizable via both sense similarity and congruence.

It is well-known [4] that any finite family of commuting square matrices can be simultaneously upper triangularized by a unitary matrix. Moreover, if these matrices are normal, then so are the resulting upper triangular matrices, and hence they are diagonal. Theorem 4 thus leads to the following observation.

Lemma 9. [4] *Normal matrices $N_1, \dots, N_m$ are SDC by a unitary matrix if and only if they pairwise commute.*

Consequently, the normal matrices $N_1, \dots, N_m$ are SDS if and only if they are SDC by a unitary matrix.

Proof. Suppose $N_1, \dots, N_m$ pairwise commute. There exists a unitary matrix U such that U^*N_iU is upper triangular for every $i = 1, \dots, m$.[4] Since $U^*N_iU =: T_i$ is normal due to the normality of N_i, T_i must be diagonal. Thus $N_1, \dots, N_m$ are SDC by the unitary matrix U.

Conversely, if $N_1, \dots, N_m$ are SDC by a unitary matrix U, then U^*N_iU's are diagonal and pairwise commute. This implies the commutativity of U^*N_iU's and that of N_i's.

The last part is obvious. □

Lemma 10. *Let M, N be normal matrices and X be a square matrix of the same order n. The following statements are true:*

i) The conditions

$$MXN = NXM \qquad (4)$$
$$MXN^* = N^*XM \qquad (5)$$

hold if and only if all the following conditions hold:

$$M^{\mathfrak{h}} \cdot X \cdot N^{\mathfrak{h}} = N^{\mathfrak{h}} \cdot X \cdot M^{\mathfrak{h}}, \qquad (6)$$
$$M^{\mathfrak{h}} \cdot X \cdot N^{\mathfrak{s}} = N^{\mathfrak{s}} \cdot X \cdot M^{\mathfrak{h}}, \qquad (7)$$
$$M^{\mathfrak{s}} \cdot X \cdot N^{\mathfrak{h}} = N^{\mathfrak{h}} \cdot X \cdot M^{\mathfrak{s}}, \qquad (8)$$
$$M^{\mathfrak{s}} \cdot X \cdot N^{\mathfrak{s}} = N^{\mathfrak{s}} \cdot X \cdot M^{\mathfrak{s}}. \qquad (9)$$

ii) Moreover, with the above materials and if X is Hermitian then (7)&(9) *can be replaced by*

$$M^{\mathfrak{h}} \cdot X \cdot M^{\mathfrak{s}} = M^{\mathfrak{s}} \cdot X \cdot M^{\mathfrak{h}}, \qquad (10)$$
$$N^{\mathfrak{s}} \cdot X \cdot N^{\mathfrak{h}} = N^{\mathfrak{h}} \cdot X \cdot N^{\mathfrak{s}}. \qquad (11)$$

Proof. The observation is derived from direct computations, see the Appendix 4, using the expansions (2) and (3) for M and N. □

The following is our main theorem.

Theorem 11. *Let $N_1, \dots, N_m \in \mathcal{N}^n$, $m \geq 2$. The following conditions are equivalent:*

i) $N_1, \dots, N_m$ are SDC.

ii) There exists a nonsingular matrix P such that the matrices $P^*N_tP, P^*N_t^*P, t = 1, \dots, m$, pairwise commute.

iii) There exists a positive definite matrix X such that

$$N_iXN_j = N_jXN_i \text{ and}$$
$$N_iXN_j^* = N_j^*XN_i, \;\; 1 \leq i \leq j \leq m. \qquad (12)$$

iv) The matrices $N_t^{\mathfrak{h}}, N_t^{\mathfrak{s}}, t = 1, \dots, m$, are SDC.

Proof. The equivalence of i) and iv) is obvious due to the authors' work.[5, Theorem 3.1]

$i) \Rightarrow ii)$. Suppose $N_1, \dots, N_m$ are SDC by a nonsingular matrix P, that is the matrix P^*N_iP is diagonal, and so is $P^*N_i^*P$, for every $i = 1, \dots, m$. It is then obvious P^*N_iP, $P^*N_i^*P$, $t = 1, \dots, m$, pairwise commute.

$ii) \Rightarrow iii)$. Suppose the $2m$ matrices $P^*N_iP, P^*N_i^*P$, $i = 1, \dots, m$, pairwise commute, for some nonsingular matrix P. Then

$$(P^*N_iP) \cdot (P^*N_jP) = (P^*N_jP) \cdot (P^*N_iP),$$
$$(P^*N_iP) \cdot (P^*N_j^*P) = (P^*N_j^*P) \cdot (P^*N_iP),$$

*In fact, the skew-Hermitian part of A is usually defined as $\frac{A-A^*}{2}$.

for every $i \neq j$. This implies

$$N_i(PP^*)N_j = N_j(PP^*)N_i,$$
$$N_i(PP^*)N_j^* = N_j^*(PP^*)N_i,$$

for every $i \neq j$. The conclusion is obvious with $X = PP^*$.

iii) $\Rightarrow$ *i)*. Let Q be the square root of $X \succ 0$ satisfying (12). Note that $Q = Q^*$. It follows from (12) that the matrices QN_tQ, QN_t^*Q's pairwise commute. This implies that, for $1 \leq t, l \leq m$,

$$N_tQ^2N_l = N_lQ^2N_t, \;\; N_tQ^2N_l^* = N_l^*Q^2N_t.$$

Applying Lemma 10 to each pair of (t, l) and $X = Q^2 = Q^*Q \succ 0$, one obtains the commutativity of the Hermitian matrices

$$QN_t^{\mathfrak{h}}Q, \;\; QN_t^{\mathfrak{s}}Q, \;\; t = 1, \ldots, m.$$

By Lemma 9, these latter matrices are SDC by a unitary matrix V, and hence so are the matrices QN_tQ's due to Lemma 8. This yields $N_1, \ldots, N_m$ are SDC by the nonsingular matrix $U = QV$. □

Algorithm 1. SDC of normal matrices.

INPUT: $N_1, \ldots, N_m \in \mathcal{N}^n$.

OUTPUT: A nonsingular matrix U such that U^*N_iU's are diagonal.

Step 1: If the system (12) has a positive definite solution X, go to the next step.
Otherwise, conclude the initial matrices are not SDC.

Step 2: Compute the square root $X^{\frac{1}{2}}$ of X by using eigenvalue decomposition of X.

Step 3: Simultaneously diagonalizing the commuting and Hermitian matrices

$$\frac{1}{2}X^{\frac{1}{2}}(N_i + N_i^*)X^{\frac{1}{2}}, \; \frac{\mathbf{i}}{2}X^{\frac{1}{2}}(N_i^* - N_i)X^{\frac{1}{2}},$$

for $i = 1, \ldots, m$, by applying the Jacobi-like algorithm,[5] to determine a unitary matrix V. Return $U = QV$.

The last step of Algorithm 1 can apply the Jacobi-like algorithm [5, Algorithm 3.1] exploiting the works by Bunse-Gerstner et. al. [14] and by Mendl.[15]

Example 1. The real symmetric matrices

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \;\; C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

which are normal, are $\mathbb{C}$-SDC as shown in the work of Bustamante and collaborators.[6] However, they are not SDC due to Theorem 7. Indeed, we want to check if there is a positive semidefinite matrix $X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succ 0$, which is equivalent to $x > 0$ and $xz > y^2$, such that

$$C_1XC_2 = C_2XC_1 \,(= (C_1XC_2)^*)\,.$$

This is equivalent to

$$\begin{cases} x > 0, & xz > y^2 \\ x + y + z & = 0. \end{cases}$$

But the last condition is impossible since there do not exist $x, z > 0$ such that $xz > y^2 = (x + z)^2$. Thus C_1 and C_2 are not SDC on $\mathbb{R}$. ◇

Example 2. . Let

$$N_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \;\; N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 3\mathbf{i} & -\mathbf{i} & \mathbf{i} \\ -\mathbf{i} & 5\mathbf{i} & -\mathbf{i} \\ \mathbf{i} & -\mathbf{i} & 3\mathbf{i} \end{pmatrix}.$$

Theorem 11 leads to finding a positive definite matrix

$$X = \begin{bmatrix} x & y & z \\ \bar{y} & t & u \\ \bar{z} & \bar{u} & v \end{bmatrix} \succ 0, \;\; x, t, v \in \mathbb{R}, \qquad (13)$$

which is equivalent to that

$$\Leftrightarrow x > 0, xt > |y|^2, \det(X) > 0,$$

such that

$$N_iXN_j = N_jXN_i,$$
$$N_iXN_j^* = N_j^*XN_i,$$

$1 \leq i < j \leq 3$. By directly computing, with the help of the expansion $y = \text{Re}(y) + \mathbf{i}\text{Im}(y)$ and similarly to u, z, the linear system above (in X) is equivalent to

$$v = x, \;\; t = x - y + z, \;\; u = y = \bar{y}, \;\; z = \bar{z}.$$

We then pick $x = 3, \; z = 2, \; y = u = 0, t = 5$ and then

$$X = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{bmatrix} \succ 0$$

makes $X^{\frac{1}{2}}N_1X^{\frac{1}{2}}$, $X^{\frac{1}{2}}N_2X^{\frac{1}{2}}$, $X^{\frac{1}{2}}N_3X^{\frac{1}{2}}$, $X^{\frac{1}{2}}N_1^*X^{\frac{1}{2}}$, $X^{\frac{1}{2}}N_2^*X^{\frac{1}{2}}$, $X^{\frac{1}{2}}N_3^*X^{\frac{1}{2}}$ to be commuting by Theorem 11. Thus three initial matrices are SDC on $\mathbb{R}$, and so are they on $\mathbb{C}$.

We will see Example 3 showing the numerical experiment of computing a square root of X and a nonsingular for $*$-SDC N_1, N_2 and N_3. ◇

3. NUMERICAL TESTS

In this section, we perform some numerical tests illustrating our main algorithm implemented in MATLAB R2022a running on a PC with Intel Core i3 CPU 3.3GHz, 8GB RAM, Windows 10 x64 operating system.

It is well known that a matrix N is normal if and only if it can written as $N = A + \mathbf{i}B$ with $A^* = A$, $B^* = -B$ and $AB = BA$. Notice furthermore that $A = A^*$ has only real eigenvalues, while B is skew-Hermitian, and hence its eigenvalues are all purely imaginary, As an existing result,[5] A and B are $*$-SDC by a unitary matrix. This leads us to set up a collection of normal matrices that are for sure $*$-SDC as follows. Fix a unitary matrix Q, and pick m diagonal matrices D_i whose diagonal elements are real in $(1, 1)$, and m diagonal ones S_i whose diagonal elements are purely imaginary in $(1, 1)$. Then the corresponding normal matrices are constructed as

$$N_i = Q(D_i + \mathbf{i}S_i)Q^*,\ i = 1, \dots, m,$$

which are $*$-SDC by Q. . The first stage of Algorithm 3.2 is implemented with the CVX toolbox [19] calling SDPT3 version 4.0 [36] that solves the following semidefinite program:

$$\min\{s \mid X \succeq 0, s \geq \epsilon, N_iXN_j = N_jXN_i, \\ N_iXN_j^* = N_j^*XN_i, 1 \leq i < j \leq m\}, \quad (14)$$

where the tolerance $\epsilon > 0$ is given. We then exploit the MATLAB function `sqrtm.m`, which executes the algorithm proposed by Deadman and collaborators,[22] to compute the square root Q of X. For the second stage, we thank the works of Mendl [15] for executing the Jacobi-like algorithm. In our experiment, we pick ϵ as the floating-point relative accuracy $\texttt{eps}(\frac{3}{2})$ of MATLAB for detecting the SDC property as in (12), while we keep their tolerance [15] for the last stage to be `eps` to the power of $\frac{3}{2}$. We have performed the tests with the collections of at most 20 normal matrices (of common sizes $5, 10, \dots, 30$, respectively). All experiments give the backward errors approximately bounded above by 10^{-8}.

Example 3. We continue Example 2 with finding a nonsingular matrix U that $*$-SDC N_1, N_2 and N_3. We first numerically compute the square root of X as

$$X^{\frac{1}{2}} \simeq \begin{bmatrix} 1.6180 & 0 & 0.6180 \\ 0 & 2.2361 & 0 \\ 0.6180 & 0 & 1.6180 \end{bmatrix}.$$

Noticing that N_1, N_2 are real symmetric and N_3 is complex symmetric. Furthermore, for a nonsingular matrix P. P^*N_3P does not need to be normal. So, we cannot apply the extended Jacobi-like algorithm.[15] However, we can apply the SDP-SDC method [5] to the matrices $X^{\frac{1}{2}}N_1X^{\frac{1}{2}}, X^{\frac{1}{2}}N_2X^{\frac{1}{2}}, X^{\frac{1}{2}}\frac{1}{2}(N_3 + N_3^*)X^{\frac{1}{2}}$ and $X^{\frac{1}{2}}\frac{\mathbf{i}}{2}(N_3^* - N_3)X^{\frac{1}{2}}$, which are all Hermitian and are commuting, to obtain the nonsingular matrix

$$V \simeq \begin{bmatrix} -0.4082 & -0.7071 & 0.5774 \\ 0.8165 & 0 & 0.5774 \\ -0.4082 & 0.7071 & 0.5774 \end{bmatrix}$$

that simultaneously diagonalizes the latter matrices above. Finally, a nonsingular that simultaneously diagonalizes the initial matrices N_1, N_2, N_3 is

$$U = X^{\frac{1}{2}}V \simeq \begin{bmatrix} -0.9129 & -0.7071 & 1.2910 \\ 1.8257 & 0 & 1.2910 \\ -0.9129 & 0.7071 & 1.2910 \end{bmatrix},$$

where

$$U^*N_1U \simeq \text{diag}(0, 0, 15),$$
$$U^*N_2U \simeq \text{diag}(-10, 0, 5),$$
$$U^*N_3U \simeq \text{diag}(30\mathbf{i}, 2\mathbf{i}, 15\mathbf{i}).$$

4. CONCLUSION

We have provided a sufficient and necessary condition for a finite family of normal matrices to be simultaneously diagonalizable via $*$-congruence. A corresponding MATLAB package has been developed, and some numerical tests have also been performed.

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Appendix

Proof of Lemma 10. i) By applying (3) to M and N, one has

$$\begin{aligned} MXN &= (M^{\natural}XN^{\natural} - M^{\mathfrak{s}}XN^{\mathfrak{s}}) \\ &\quad + i(M^{\natural}XN^{\mathfrak{s}} + M^{\mathfrak{s}}XN^{\natural}), \\ NXM &= (N^{\natural}XM^{\natural} - N^{\mathfrak{s}}XM^{\mathfrak{s}}) \\ &\quad + i(N^{\mathfrak{s}}XM^{\natural} + N^{\natural}XM^{\mathfrak{s}}), \\ MXN^* &= (M^{\natural}XN^{\natural} + M^{\mathfrak{s}}XN^{\mathfrak{s}}) \\ &\quad - i(M^{\natural}XN^{\mathfrak{s}} - M^{\mathfrak{s}}XN^{\natural}), \\ N^*XM &= (N^{\natural}XM^{\natural} + N^{\mathfrak{s}}XM^{\mathfrak{s}}) \\ &\quad - i(N^{\mathfrak{s}}XM^{\natural} - N^{\natural}XM^{\mathfrak{s}}). \end{aligned}$$

Substituting the above identities into (4)-(5) one obtains that

$$\begin{aligned} M^{\natural}XN^{\natural} - M^{\mathfrak{s}}XN^{\mathfrak{s}} &= N^{\natural}XM^{\natural} - N^{\mathfrak{s}}XM^{\mathfrak{s}}, \\ M^{\natural}XN^{\mathfrak{s}} + M^{\mathfrak{s}}XN^{\natural} &= N^{\mathfrak{s}}XM^{\natural} + N^{\natural}XM^{\mathfrak{s}}, \\ M^{\natural}XN^{\natural} + M^{\mathfrak{s}}XN^{\mathfrak{s}} &= N^{\natural}XM^{\natural} + N^{\mathfrak{s}}XM^{\mathfrak{s}}, \\ M^{\natural}XN^{\mathfrak{s}} - M^{\mathfrak{s}}XN^{\natural} &= N^{\mathfrak{s}}XM^{\natural} - N^{\natural}XM^{\mathfrak{s}}. \end{aligned}$$

Adding side-by-side the first and the third (resp., the second and the fourth) equations one has

$$M^{\natural}XN^{\natural} = N^{\natural}XM^{\natural}, \;\; M^{\natural}XN^{\mathfrak{s}} = N^{\mathfrak{s}}XM^{\natural}.$$

Subtracting side-by-side the first and the third (resp., the second and the fourth) equations one has

$$M^{\mathfrak{s}}XN^{\mathfrak{s}} = N^{\mathfrak{s}}XM^{\mathfrak{s}}, \;\; M^{\mathfrak{s}}XN^{\natural} = N^{\natural}XM^{\mathfrak{s}},$$

Conversely, from (2), the identities (6)-(9) are equivalent to

$$\begin{aligned} (M+M^*)X(N+N^*) &= (N+N^*)X(M+M^*), \\ (M+M^*)X(N-N^*) &= (N-N^*)X(M+M^*), \\ (M-M^*)X(N+N^*) &= (N+N^*)X(M-M^*), \\ (M-M^*)X(N-N^*) &= (N-N^*)X(M-M^*), \end{aligned}$$

respectively. Expanding the above identities leads to that

$$\begin{aligned} & MXN + MXN^* + M^*XN + M^*XN^* = \\ & NXM + NXM^* + N^*XM + N^*XM^*, \\ & MXN - MXN^* + M^*XN - M^*XN^* = \\ & NXM + NXM^* - N^*XM - N^*XM^*, \\ & MXN + MXN^* - M^*XN - M^*XN^* = \\ & NXM - NXM^* + N^*XM - N^*XM^*, \\ & MXN - MXN^* - M^*XN + M^*XN^* = \\ & NXM - NXM^* - N^*XM + N^*XM^*. \end{aligned}$$

By adding side-by-side the above identities, we have

$$MXN = NXM.$$

Similarly, by adding the first and the third, then subtracting the second and the fourth identities, side-by-side, we additionally obtain

$$MXN^* = N^*XM.$$

ii) This is an immediate consequence of the first part with noting that $X^* = X$ and

$$\begin{aligned} M + M^* &= U(\Lambda_M + \overline{\Lambda_M})U^*, \\ M - M^* &= U(\Lambda_M - \overline{\Lambda_M})U^*, \\ N + N^* &= V(\Lambda_N + \overline{\Lambda_N})V^*, \\ N - N^* &= V(\Lambda_N - \overline{\Lambda_N})V^*, \end{aligned}$$

where $M = U\Lambda_M U^*$, $N = V\Lambda_N V^*$ are eigenvalue decomposition of the normal matrices M and N (Λ_M, Λ_N are complex diagonal and U, V are unitary matrices). □