Toán tử hợp có trọng từ không gian kiểu Bloch vào không gian tăng trưởng trên hình cầu đơn vị của không gian Hilbert

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TÓM TẮT

Cho ν, μ là các trọng chuẩn tắc trên hình cầu đơn vị B_X của một không gian Hilbert phức với số chiều tùy ý và ψ là một hàm chỉnh hình trên B_X, φ là một ánh xạ tự chỉnh hình của B_X. Trong bài báo này, chúng tôi nghiên cứu các đặc trưng cho tính bị chặn và tính compact của toán tử hợp có trọng $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, từ không gian kiểu Bloch $\mathcal{B}_\nu(B_X)$ đến không gian tăng trưởng (nhỏ) $\mathcal{H}_\mu^\infty(B_X)$, $\mathcal{H}_\mu^0(B_X)$ thông qua tính chất của ψ, các phiếm hàm đánh giá điểm $\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}$, và các hạn chế của các đại lượng này lên các không gian con m-chiều với $m \geq 2$. Chúng tôi cũng tính được chính xác công thức của chuẩn toán tử $W_{\psi,\varphi}$.

Từ khóa: *Toán tử hợp có trọng, không gian Bloch, không gian tăng trưởng, tính bị chặn, tính compact.*

Weighted composition operators from Bloch-type spaces into growth spaces on the unit ball of a Hilbert space

ABSTRACT

Let ν, μ be normal weights on the unit ball B_X of an Hilbert space X with arbitrary dimension and ψ be a holomorphic function on B_X and φ a holomorphic self-map of B_X. In this work, we characterize the boundedness and the compactness of weighted composition operators $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, from the Bloch-type spaces $\mathcal{B}_\nu(B_X)$ to the (little) growth spaces $\mathcal{H}_\mu^\infty(B_X)$, $\mathcal{H}_\mu^0(B_X)$ via function theoretic properties of the symbol ψ and the point evaluation function

$\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}$, specifically, of the restrictions of functions ψ, φ to the m-dimensional subspaces for some $m \geq 2$. We obtain also the formula of the operator norm of $W_{\psi,\varphi}$.

Keywords: *Bloch spaces, growth spaces, compactness, boundedness, weighted composition operator*

1. INTRODUCTION

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be spaces of holomorphic functions on the unit ball B_X of a Banach space X, ψ be a holomorphic function on B_X and φ a holomorphic self-map of B_X. The weighted composition operator, defined by symbols ψ and φ, maps from $\mathcal{E}_1$ to $\mathcal{E}_2$ and is defined by

$$W_{\psi,\varphi}(f) = M_\psi C_\varphi(f) = \psi \cdot (f \circ \varphi)$$

where M_ψ represents the multiplication operator with symbol ψ and C_φ is the composition operator with symbol φ.

In recent years, there has been significant interest in studying weighted composition operators. A famous theorem developed by Banach asserts that for a compact metric space K, the surjective linear isometries of $C(K)$ are given by $Tf = u(f \circ \varphi)$ where $|u(x)| = 1$ for all $x \in K$, and $\varphi : K \to K$ is a homeomorphism. Inspired by this theorem, ongoing research on the characterization of isometries in Banach spaces of analytic functions has revealed that weighted composition operators define the isometries of many such spaces, including the Hardy space H^p (for $1 \leq p \leq \infty$, $p \neq 2$), the weighted Bergman space, and the disk algebra.[1]

For a comprehensive overview of various aspects of the theory of (weighted) composition operators acting on several spaces of holomorphic functions, we refer to a standard reference. [2] There is extensive literature on weighted composition operators and integral operators between specific holomorphic function spaces. To address these spaces in a unified way, certain frameworks for Banach spaces of holomorphic functions on the unit disk have been introduced.[3,4] For instance, in reference, [3] established certain topological and function-theoretic conditions for the domain space and provided criteria for boundedness and compactness, along with estimates for the operator norm and the essential norm of the weighted composition operators that map to the weighted-type space or the Bloch-type space on the unit disk. In recent years, there has been significant interest in the study of weighted composition operators. More recently, attention has also focused on composition operators and operator-valued multipliers in various vector-valued analytic function spaces, particularly when X is an infinite-dimensional Hilbert space.[5,6,7,8,9,10]

In this paper, we aim to consider the compactness and boundedness of $W_{\psi,\varphi}$ when $\mathcal{E}_1$ is a general Banach space of holomorphic functions and and $\mathcal{E}_2$ is either growth space $\mathcal{H}^\infty_\mu(B_X)$ or the little growth space $\mathcal{H}^0_\mu(B_X)$ determined as follows:

$$\mathcal{H}^\infty_\mu(B_X) = \Big\{f \in \mathscr{H}(B_X) : \sup_{z\in B_X} \mu(z)|f(z)| < \infty\Big\},$$

$$\mathcal{H}^0_\mu(B_X) = \Big\{f \in \mathcal{H}^\infty_\mu(B_X) : \lim_{\|z\|\to 1} \mu(z)|f(z)| = 0\Big\},$$

where $\mathscr{H}(B_X)$ is the space of holomorphic functions on B_X and μ is a normal weight on B_X.

Growth spaces are a significant and intriguing class of Banach spaces of holomorphic functions. They have been investigated in various contexts, with numerous general and specialized references available.[12,13]

Some key properties of these spaces, when B_X is the unit disk $\mathbb{B} \subset \mathbb{C}$, include the following:

- For a normal weight μ, $\mathcal{H}^\infty_\mu(\mathbb{B}) \supset \mathcal{H}^\infty$ if and only if $\lim_{|z|\to 1} \mu(z) = 0$. On the other hand, if $\limsup_{|z|\to 1} \mu(z) > 0$, then $\mathcal{H}^0_\mu = \{0\}$;
- The identity map $I : \mathcal{H}^\infty_\mu(\mathbb{B})) \to (\mathcal{H}^\infty_\mu(\mathbb{B}), \tau_{co})$ is continuous.
- The bidual $[\mathcal{H}^0_\mu(\mathbb{B})]''$ isometrically isomorphic to $\mathcal{H}^\infty_\mu(\mathbb{B})$;
- The point evaluation functionals $\delta^{\mathcal{H}}_z$ on $\mathcal{H}^0_\mu(\mathbb{B})$ are bounded and can be uniquely extended to point evaluation functionals on $\mathcal{H}^\infty_\mu(\mathbb{B})$ with the same norms;
- The operator $\mathcal{B}^0_\mu(\mathbb{B}) \to \mathcal{H}^\infty_\mu(\mathbb{B})$, $f \mapsto f''$, is an isometric isomorphism, where $\mathcal{B}^0_\mu(\mathbb{B})$ is the subspace of the Bloch space $\mathcal{B}_\mu(\mathbb{B})$ of functions with $f(0) = 0$. It is important to note that the Bloch space consists of functions characterized by the growth of their derivatives, making it closely related to growth spaces;
- The maps $z \mapsto \delta^{\mathcal{H}}_z$ is continuous, and $\|\delta^{\mathcal{H}}_z\|$ goes to infinity as $|z| \to 1$.

These are just a few of the reasons motivating our research.

In Section 2, we review the key conditions for spaces of holomorphic functions that will be used to establish the boundedness and compactness of these operators, as well as to provide estimates for their essential norms in our context.

To characterize the boundedness and compactness, and building on the ideas from [5,9] with minor modifications, we establish in Section 3 a connection between functions in the growth space $\mathcal{H}^\infty_\mu(B_X)$ and and their restrictions to finite-dimensional subspaces. Specifically, we show that if the restrictions of a function to m-dimensional subspaces (for $m \geq 2$) have uniformly bounded growth norms, then the function belongs to the growth space $\mathcal{H}^\infty_\mu(\mathbb{B}_m)$, and vice versa.

In Section 4, we characterize the boundedness and the compactness of $W_{\psi,\varphi}$ from $\mathcal{B}_\nu(B_X)$ into $\mathcal{H}^\infty_\mu(B_X)$ and into $\mathcal{H}^0_\mu(B_X)$ as well as calculate the operator norms. We will show that these characterizations are completely determined by their behaviour on $\psi^{[m]}$ and on the point evaluation functions $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi^{[m]}(z)}$ and $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z)}$, where $\psi^{[m]}$ and $\varphi^{[m]}$ are the restrictions of ψ and φ, respectively, on the m-dimensional subspaces and $\varphi_{(m)} = (\varphi_1, \ldots, \varphi_m)$, $m \geq 2$.

Throughout this paper, we use the notions $a \lesssim b$ and $a \asymp b$ for non negative quantities a and b to mean $a \leq Cb$ and, respectively, $C^{-1}b \leq a \leq Cb$ for some inessential constant $C > 0$.

2. PRELIMINARIES AND AUXILIARY RESULTS

Let X be a complex Hilbert space of arbitrary dimension, Y a Banach space. Denote by B_X the closed unit ball of X, and use $\mathbb{B}_n$ instead of $B_{\mathbb{C}^n}$. Fix an orthonormal basis $(e_k)_{k\in\Gamma}$ of X. Then any $z \in X$ can be expressed as

$$z = \sum_{k\in\Gamma} z_k e_k, \quad \overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k.$$

2.1. Möbius transformations

The analogues of Möbius transformations on a Hilbert space X are the mappings $\Phi_a : B_X \to B_X$, $a \in B_X$, defined as follows:

$$\Phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \; z \in \mathcal{B}_X \tag{2.1}$$

where $s_a = \sqrt{1 - \|a\|^2}$, $P_a(z) = \frac{\langle z,a\rangle}{\|a\|^2} a$, and $Q_a(z) = z - \frac{\langle z,a\rangle}{\|a\|^2} a$ for $z \in B_X$.

We define $\Phi_0(z) = -z$.

Denote by $Aut(B_X)$ the group of automorphisms of the unit ball B_X.

For details on Möbius transformations, we refer to K. Zhu's book. [14]

2.2. Banach spaces of holomorphic functions

$\mathscr{H}(B_X, Y)$ is denoted by the vector space of Y-valued holomorphic functions on B_X. An element $f \in \mathscr{H}(B_X, Y)$ is named locally bounded holomorphic on B_X if for every $z \in B_X$ there exists a neighbourhood U_z of $0 \in X$ such that $f(U_z)$ is bounded. Put

$$\mathscr{H}_{LB}(B_X, Y) = \{f \in \mathscr{H}(B_X, Y) : f \text{ is locally bounded on } B_X\}.$$

For $f \in \mathscr{H}(B_X)$, its complex gradient and radial derivative are defined by

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_k}(z)\Big)_{k\in\Gamma},$$

$$Rf(z) := \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(z_k e_k) = \langle z, \overline{\nabla f(z)}\rangle,$$

respectively. Thus, $\nabla f(z)$ is the unique element in X representing the linear operator $f'(z) \in X'$, hence,

$$f'(z)(x) = \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(x_k e_k) = \langle x, \overline{\nabla f(z)}\rangle, \quad x \in X.$$

It is clear that

$$|Rf(z)| \leq \|\nabla f(z)\|\|z\|, \quad z \in B_X.$$

Now, let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space.

For each $z \in B_X$, the point-evaluation functional $\delta^{\mathcal{E}}_z$ at z defined by $\delta^{\mathcal{E}}_z(f) := f(z)$ for all $f \in \mathcal{E}$. Thus,

$$|f(z)| \leq \|f\|\|\delta^{\mathcal{E}}_z\|, \quad f \in \mathcal{E}, z \in B_X, \tag{2.2}$$

where $\|\delta^{\mathcal{E}}_z\| = \sup\{|f(z)| : f \in \mathcal{E}, \|f\| \leq 1\}$.

For all $\Phi = (\Phi_j)_{j\in\Gamma} \in Aut(B_X)$, for every $j \geq 1$, $m \geq 2$ and all $f \in \mathcal{E}$, we write

$$\Phi_{(m)} = (\Phi_1, \ldots, \Phi_m),$$

$$f \cdot \Phi_{(m)} = (f \cdot \Phi_1, \ldots, f \cdot \Phi_m).$$

Below, we present a comprehensive list of conditions, some of which will be necessary for characterizing the boundedness, compactness, or determining the essential norm of the operators discussed in this work.

- **(e1)** $\mathcal{E}$ includes the constant functions.
- **(e2)** The closed unit ball $B_{\mathcal{E}}$ is τ_{co}- compact.

(e3) There are $m \geq 2$ and constant $C > 0$ such that for all $\Phi \in Aut(B_X)$, for all $f \in \mathcal{E}$, $\Phi_j \cdot f \in \mathcal{E}$,

$$\|\Phi_j \cdot f\| \leq C\|f\|, \quad j \in \{1, \dots, m\}.$$

Remark 2.1. It follows from (e1) that $\inf_{z \in B_X} \|\delta_z^{\mathcal{E}}\| > 0$, and in particular, the following equivalent conditions are satisfied:

(e1a) On each compact set, $\|\delta_z^{\mathcal{E}}\|$ is bounded from below by a positive constant;

(e1b) The functions in $\mathcal{E}$ do not all vanish at each point $z \in B_X$.

Indeed, since the function $1 \in \mathcal{E}$, for every $z \in B_X$ we have $\|\delta_z^{\mathcal{E}}\| \geq \frac{1}{\|1\|}$. It is obvious that (e1a) $\Rightarrow$ (e1b). Now, assume that (e1b) holds but (e1a) fails. Then we can find a compact subset K of B_X and a sequence $\{z_n\}_{n \geq 1} \in K$ and $z_0 \in K$ such that $z_n \to z_0$ and $\|\delta_{z_n}^{\mathcal{E}}\| \to 0$. This implies that $f(z_0) = 0$ for all $f \in \mathcal{E}$, which is incompatible with (e1b).

By the uniform boundedness principle, we can easily prove the following:

Proposition 2.1. The mapping $\delta^{\mathcal{E}} : B_X \to \mathbb{C}$, $z \mapsto \|\delta_z^{\mathcal{E}}\|$, is bounded on compact subsets of B_X for every Banach space $\mathcal{E}$ of holomorphic functions on B_X.

3. GROWTH SPACES AND BLOCH-TYPE SPACES

For a normal weight ν on B_X, we write

$$I_\nu^1(z) := \int_0^{\|z\|} \frac{dt}{\nu(t)}.$$

Remark 3.1. Since ν is positive, continuous, $m_{\nu,\delta} := \min_{t \in [0,\delta]} \nu(t) > 0$. Moreover, it follows from (W_1) that ν is strictly decreasing on $[\delta, 1)$, hence, $\max_{t \in [0,1)} \nu(t) =: M_\nu < \infty$. Then, it is easily seen that

$$\nu(z) I_\nu^1(z) < R_\nu := \delta \frac{M_\nu}{m_{\nu,\delta}} + 1 - \delta < \infty. \quad (3.1)$$

for every $z \in B_X$.

We define bounded holomorphic spaces $\mathcal{H}^\infty(B_X)$, growth holomorphic spaces $\mathcal{H}_\mu^\infty(B_X)$, little growth holomorphic spaces $\mathcal{H}_\mu^0(B_X)$, Bloch-type spaces $\mathcal{B}_\nu(B_X)$, and little Bloch-type spaces $\mathcal{B}_{\nu,0}(B_X)$ on the unit ball B_X as follows:

$$\mathcal{H}^\infty(B_X) = \Big\{ f \in \mathscr{H}(B_X) : \sup_{z \in B_X} |f(z)| < \infty \Big\},$$

$$\mathcal{H}_\mu^\infty(B_X) = \Big\{ f \in \mathscr{H}(B_X) : \sup_{z \in B_X} \mu(z)|f(z)| < \infty \Big\},$$

$$\mathcal{H}_\mu^0(B_X) = \Big\{ f \in \mathcal{H}_\mu^\infty(B_X) : \lim_{\|z\| \to 1} \mu(z)|f(z)| = 0 \Big\},$$

$$\mathcal{B}_\nu(B_X) := \Big\{ f \in \mathscr{H}(B_X) : \|f\|_{s\mathcal{B}_\nu(B_X)} := \sup_{z \in B_X} \nu(z)|Rf(z)| < \infty \Big\}.$$

It is easy to check that $\mathcal{H}^\infty(B_X)$, $\mathcal{H}_\mu^\infty(B_X)$ and $\mathcal{B}_\nu(B_X)$ are Banach under following norms

$$\|f\|_\infty := \sup_{z \in B_X} |f(z)|,$$

$$\|f\|_{\mathcal{H}_\mu^\infty} := \sup_{z \in B_X} \mu(z)|f(z)|,$$

$$\|f\|_{\mathcal{B}_\nu(B_X)} := |f(0)| + \|f\|_{s\mathcal{B}_\nu(B_X)},$$

respectively.

Now we consider the holomorphic function

$$g(z) := 1 + \sum_{k > k_0} 2^k z^{n_k}, \quad z \in \mathbb{B}_1, \quad (3.2)$$

where $k_0 = \left[\log_2 \frac{1}{\nu(\delta)}\right]$, $n_k = \left[\frac{1}{1 - r_k}\right]$ with $r_k = \nu^{-1}(1/2^k)$ for every $k \geq 1$. Here, the symbol $[x]$ represents the greatest integer less than or equal to x. By Theorem 2.3 [15], $g(t)$ is increasing on $[0, 1)$ and

$$|g(z)| \leq g(\|z\|) \in \mathbb{R}, \quad z \in \mathbb{B}_1,$$

$$\begin{aligned} 0 < C_1 := \inf_{t \in [0,1)} \nu(t) g(t) \\ \leq \sup_{t \in [0,1)} \nu(t) g(t) \quad (3.3) \\ \leq \sup_{z \in \mathbb{B}_1} \nu(z)|g(z)| =: C_2 < \infty. \end{aligned}$$

Lemma 3.1. Let ν be a normal weight on B_X. Then there is $C > 0$ such that for every $z \in B_X$,

$$|f(z)| \leq \mu(z)^{-1}\|f\|_{\mathcal{H}^\infty_\mu(B_X)},\ f \in \mathcal{H}^\infty_\mu(B_X), \tag{3.4}$$

$$|f(z)| \leq C(1+I^1_\nu(z))\|f\|_{\mathcal{B}_\nu(B_X)},\ f \in \mathcal{B}_\nu(B_X). \tag{3.5}$$

Proof. The inequality (3.4) is obvious. The inequality (3.5) was proved in [9] (Proof of Theorem 3.2). □

Lemma 3.2. For every ν normal weight ν on B_X, we have

(1) $\|\delta_z^{\mathcal{H}^\infty_\nu(B_X)}\| = 1/\nu(z)$;

(2) $\|\delta_z^{\mathcal{B}_\nu(B_X)}\| \asymp 1 + I^1_\nu(z)$.

Proof. (1) It is obvious.

(2) It follows easily from the definition of $\delta_z^{\mathcal{B}_\nu(B_X)}$ and (3.5) that

$$\|\delta_z^{\mathcal{B}_\nu(B_X)}\| \lesssim 1 + I^1_\nu(z).$$

Now we consider the function f_0 defined by

$$f_0(z) = (1+C_2)^{-1}(1+\int_0^{\|z\|} g(t)dt), \quad z \in B_X,$$

where g is defined by (3.2). It is clear that $f_0 \in \mathcal{B}_\nu(B_X)$ and by (3.3), it is easy to see that $\|f_0\|_{\mathcal{B}_\nu(B_X)} \leq 1$. Then, in view of (3.3) again, this yields that

$$\begin{aligned}\|\delta_z^{\mathcal{B}_\nu(B_X)}\| &\geq |f_0(z)| \\ &\geq \max\Big\{\frac{1}{1+C_2}, \frac{C_1}{1+C_2}\Big\}(1 + I^1_\nu(z)).\end{aligned}$$

□

It is easy to prove the following:

Corollary 3.3. $\mathcal{H}^\infty_\nu(B_X)$, $\mathcal{B}_\nu(B_X)$ satisfy the properties (e1), (e2), (e3).

We will demonstrate below that the analysis of growth spaces on $\mathcal{B}_X$ can be reduced to studying functions defined on finite-dimensional subspaces. It is worth noting that similar results for Bloch-type spaces were recently studied in [9].

For each finite subset $F \subset \Gamma$, without loss of generality we may assume that $F = \{1, \dots, m\}$, we denote by $\mathbb{B}_m$ the unit ball of $\operatorname{span}\{e_k,\ k \in F\}$. For each $m \in \mathbb{N}$, we write

$$\mu^{[m]} := \mu|_{\operatorname{span}\{e_1,\dots,e_m\}},$$
$$z_{[m]} := (z_1, \dots, z_m) \in \mathbb{B}_m.$$

For $m \geq 2$, we denote by

$$OS_m := \{x = (x_1, \dots, x_m),\ x_k \in X, \langle x_k, x_j\rangle = \delta_{kj}\}$$

the family of orthonormal systems of order m.

For every $x \in OS_m$ fixed and $f \in \mathscr{H}(B_X)$, we define

$$f_x(z_{[m]}) = f\Big(\sum_{k=1}^m z_k x_k\Big).$$

Then

$$\Big\|\nabla f_x(z_{[m]})\Big\| = \Big\|\nabla f\Big(\sum_{k=1}^m z_k x_k\Big)\Big\|. \tag{3.6}$$

Definition 3.1. We denote

$$\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X) := \{f \in \mathscr{H}(B_X):\ \|f\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)} < \infty\}$$

where

$$\|f\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)} := \sup_{\|x\|=1} \|f(\cdot\, x)\|_{\mathcal{H}^\infty_\mu(\mathbb{B}_1)},$$

with $f(\cdot\, x) : \mathbb{B}_1 \to \mathbb{C}$ given by $f(\cdot\, x)(\lambda) = f(\lambda x)$ for every $\lambda \in \mathbb{B}_1$, and

$$\|f(\cdot\, x)\|_{\mathcal{H}^\infty_\mu(\mathbb{B}_1)} = \sup_{\lambda\in\mathbb{B}_1} \mu(\lambda x)|f(\lambda x)|.$$

It is easy to verify that $\|\cdot\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)}$ is a norm on $\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)$, called the *affine norm*, moreover, $(\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X), \|\cdot\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)})$ is a Banach space.

Proposition 3.4. Let $f \in \mathscr{H}(B_X)$. The following are equivalent:

(1) $f \in \mathcal{H}^\infty_\mu(B_X)$;

(2) $\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty$ for every $m \ge 2$;

(3) $\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty$ for some $m \ge 2$.

Moreover, for each $m \ge 2$

$$\|f\|_{\mathcal{H}^\infty_\mu(B_X)} = \sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)}. \tag{3.7}$$

Proof. (1) $\Rightarrow$ (2): Let $m \ge 2$ and $z_{[m]} := (z_1, \dots, z_m) \in \mathbb{B}_m$. Since $\|\sum_{j=1}^m z_j e_j\| = \|z_{[m]}\|$, we get

$$\begin{aligned} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(B_X)} &= \sup_{z_{[m]}\in\mathbb{B}_m} \mu^{[m]}(z_{[m]})|f_x(z_{[m]})| \\ &\le \sup_{z\in B_X} \mu(z)\Big|f\Big(\sum_{j\in F} z_j e_j\Big)\Big| \\ &\le \|f\|_{\mathcal{H}^\infty_\mu(B_X)} < \infty. \end{aligned} \tag{3.8}$$

In particular, we obtain (2).

(2) $\Rightarrow$ (3): This is evident.

(3) $\Rightarrow$ (1): Assume that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty$$

for some $m \ge 2$. We fix $z \in B_X \setminus \{0\}$. Consider $x = (\frac{z}{\|z\|}, x_2, \dots, x_m) \in OS_m$ and $z_{[m]} := (\|z\|, 0, \dots, 0) \in \mathbb{C}^m$. Then $\|z_{[m]}\| = \|z\|$ and

$$\big|f_x(z_{[m]})\big| = \Big|f\Big(\sum_{k=1}^m z_k x_k\Big)\Big| = |f(z)|.$$

This yields that

$$\begin{aligned} \|f\|_{\mathcal{H}^\infty_\mu(B_X)} &= \sup_{z\in B_X} \mu(z)|f(z)| \\ &\le \sup_{z\in B_X} \mu^{[m]}(z_{[m]})|f_x(z_{[m]})| \\ &\le \sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} < \infty. \end{aligned} \tag{3.9}$$

Thus $f \in \mathcal{H}^\infty_\mu(B_X)$.

On the other hand, it is clear that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{H}^\infty_{\mu^{[m]}}(\mathbb{B}_m)} \le \|f\|_{\mathcal{H}^\infty_\mu(B_X)} \quad \text{for every } m \ge 2. \tag{3.10}$$

Hence, we obtain (3.7) from (3.8), (3.9) and (3.10). □

Remark 3.2. In the case $m = 1$, the proposition is not true. Indeed, we consider $\mu(z) := 1 - \|z\|^2$, and $f : B_X \to \mathbb{C}$ given by

$$f(z) := \sum_{n=1}^\infty \Big\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n \Big\rangle, \quad z \in B_X.$$

Then $f \in \mathscr{H}(B_X)$ because

$$\begin{aligned} &\sum_{n=1}^\infty \Big|\Big\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n \Big\rangle\Big|^2 \\ &\qquad \le \sum_{n=1}^\infty \frac{1}{n^2} + \|z\|^2 + \sum_{n=1}^\infty \frac{2}{n^{3/2}} < \infty. \end{aligned}$$

For each $x = \sum_{n=1}^\infty \langle x, e_n\rangle e_n \in OS_1$ and for every $z_{[1]} := z_k \in \mathbb{B}_1$ for some $k \ge 1$, we have

$$f_x(z_{[1]}) = f(z_k x_k) = \frac{1}{k} - \frac{z_k x_k}{\sqrt{k}},$$

and thus, since $|f_x(z_{[1]})| \le 2$, we get

$$\begin{aligned} &\sup_{x\in OS_1} \|f_x\|_{\mathcal{H}^\infty_\mu(\mathbb{B}_1)} \\ &\qquad = \sup_{x\in OS_1} (1 - \|z_{[1]}\|^2)|f_x(z_{[1]})| \le 2. \end{aligned}$$

However, since

$$\begin{aligned} &(1-\|z\|^2)|f(z)| \\ &\qquad = (1 - \|z\|^2)\Big|\sum_{n=1}^\infty \Big\langle \frac{e_n}{n} - \frac{z}{\sqrt{n}}, e_n \Big\rangle\Big| \\ &\qquad \to \sum_{n=1}^\infty \frac{1}{n} \quad \text{as } z \to 0, \end{aligned}$$

we obtain that $f \notin \mathcal{H}^\infty_\mu(B_X)$.

By employing a similar argument as in the proof of Proposition 2.3 in [9], we can easily obtain the following result, for which the proof will be omitted.

Proposition 3.5. $\mathcal{H}^\infty_\mu(B_X) = \mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)$. Moreover,

$$\begin{aligned} \|f\|_{\mathcal{H}^\infty_\mu(B_X)} &\le \|f\|_{\mathcal{H}^\infty_{\mu,\mathrm{aff}}(B_X)} \\ &\lesssim \|f\|_{\mathcal{H}^\infty_\mu(B_X)}, \quad f \in \mathcal{H}^\infty_\mu(B_X). \end{aligned}$$

4. BOUNDEDNESS AND COMPACTNESS CRITERIA

Consider the weighted composition operator $W_{\psi,\varphi}$ from $\mathcal{B}_\nu(B_X)$ into $\mathcal{H}_\mu^\infty(B_X)$ and into $\mathcal{H}_\mu^0(B_X)$ defined by

$$(W_{\psi,\varphi}f)(z) := \psi(z) \cdot (f \circ \varphi)(z), \quad z \in B_X.$$

The component operators are the multiplication operator $M_\psi f = \psi \cdot f$ and the composition operator $C_\varphi f = f \circ \varphi$, which correspond to the cases when the composition symbol φ is the identity function on $\mathbb{B}$ and the multiplication symbol ψ is the constant function 1, respectively.

Theorem 4.1. The following are equivalent:

(1) $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded;

(2) $M_{\psi,\varphi,\mu}^{[m]} := \sup_{z\in\mathbb{B}_m} \mu^{[m]}(z)|\psi^{[m]}(z)|\|\delta_{\varphi^{[m]}(z)}^{\mathcal{B}_\nu(B_X)}\| < \infty$ for some $m \geq 2$;

(3) $M_{\psi,\varphi,\mu} := \sup_{z\in B_X} \mu(z)|\psi(z)|\|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| < \infty.$

In this case, we have

$$\|W_{\psi,\varphi}\| = M_{\psi,\varphi,\mu}. \tag{4.1}$$

Proof. (3) $\Rightarrow$ (2): It is clear.

(1) $\Rightarrow$ (3): Suppose $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded. Fix $z \in B_X$. For each $f \in \mathcal{B}_\nu(B_X)$ with $\|f\|_{\mathcal{B}_\nu(B_X)} \leq 1$, we have

$$\mu(z)|\psi(z)||f(\varphi(z))| \leq \|W_{\psi,\varphi}f\|_{\mathcal{H}_\mu^\infty(B_X)}$$

$$\leq \|W_{\psi,\varphi}\|\|f\|_{\mathcal{B}_\nu(B_X)} \leq \|W_{\psi,\varphi}\|.$$

By definition of $\delta^{\mathcal{B}_\nu(B_X)}$ (see Proposition 2.1), by taking the supremum over all f within the closed unit ball of $\mathcal{B}_\nu(B_X)$, we obtain:

$$\mu(z)|\psi(z)|\|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\| \leq \|W_{\psi,\varphi}\|.$$

Taking the supremum over all $z \in B_X$ yields

$$M_{\psi,\varphi,\mu} \leq \|W_{\psi,\varphi}\| < \infty. \tag{4.2}$$

(2) $\Rightarrow$ (1): Assume $M_{\psi,\varphi,\mu}^{[m]} < \infty$ for some $m \geq 2$. Let $f \in \mathcal{B}_\nu(B_X)$ with $\|f\|_{\mathcal{B}_\nu(B_X)} \leq 1$. We write $z_x := \sum_{k=1}^m z_k x_k$ for each $x \in OS_m$. It should be noted that $\|z_x\| = \|z_{[m]}\|$ and hence $\mu^{[m]}(z_{[m]}) = \mu^{[m]}(z_x)$. Then

$$\begin{aligned}&\|(W_{\psi,\varphi}(f))_x\|_{\mathcal{H}_{\mu^{[m]}}^\infty(\mathbb{B}_m)}\\ &= \sup_{z_x\in\mathbb{B}_m} \mu^{[m]}(z_x)|\psi^{[m]}(z_x)(f\circ\varphi)_x(z_{[m]})|\\ &\leq M_{\psi,\varphi,\mu}^{[m]} < \infty\end{aligned}$$

for every $x \in OS_m$. By (3.7), $W_{\psi,\varphi}$ is bounded because

$$\begin{aligned}&\|W_{\psi,\varphi}(f)\|_{\mathcal{H}_\mu^\infty(B_X)}\\ &= \sup_{x\in OS_m} \|(W_{\psi,\varphi}(f))_x\|_{\mathcal{H}_{\mu^{[m]}}^\infty(\mathbb{B}_m)}\\ &\leq M_{\psi,\varphi,\mu}^{[m]} < \infty.\end{aligned}$$

(4) $\Rightarrow$ (2): For $z \in B_X$, we have

$$\begin{aligned}\mu(z)|\psi(z)||f(\varphi(z))| &\leq \mu(z)|\psi(z)|\|\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)}\|\\ &\leq M_{\psi,\varphi,\mu} < \infty.\end{aligned}$$

Consequently,

$$\|W_{\psi,\varphi}f\|_{\mathcal{H}_\mu^\infty(B_X)} \leq M_{\psi,\varphi,\mu} < \infty. \tag{4.3}$$

Finally, from (4.2), (4.3) we deduce (4.1). □

We next characterize the compactness of $W_{\psi.\varphi}$. As shown in [10], we can demonstrate the following:

Lemma 4.2 ([10], Lemma 2.10)**.** Let $\mathcal{E}, \mathcal{F}$ be two Banach spaces of holomorphic functions on B_X. Assume that

(1) $\delta_z^{\mathcal{E}}$ are continuous for every $z \in B_X$;

(2) The closed unit ball of $\mathcal{E}$ is τ_{co}-compact.

(3) $T : (\mathcal{E}, \tau_{co}) \to (\mathcal{F}, \tau_{co})$ is continuous.

Then, T is compact if and only if for each bounded sequence $\{f_n\}$ in $\mathcal{E}$ satisfying $f_n \rightrightarrows 0$ on compact sets, then $\|Tf_n\|_{\mathcal{F}} \to 0$.

Theorem 4.3. Assume that the operator $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}_\mu^\infty(B_X)$ is bounded. Then, the following are equivalent:

(1) There exists $m \geq 2$ such that

$$\lim_{r\to 1} \sup_{\|\varphi_{(m)}(z)\|>r} \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z)}\| = 0, \tag{4.4}$$

where $\varphi_{(m)} := (\varphi_1, \ldots, \varphi_m)$.

(2) $W_{\psi,\varphi}$ is compact.

Proof. First, we show that $\psi \in \mathcal{H}^\infty_\mu(B_X)$. Indeed, by the boundedness of $W_{\psi,\varphi}$ and Theorem 4.1, we have $M_{\psi,\varphi,\mu} < \infty$. Then, by Remark 2.1, $\inf_{z\in B_X} \|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi(z)}\| =: \alpha > 0$. Consequently,

$$\alpha\mu(z)|\psi(z)| < M_{\psi,\varphi,\mu}, \quad z \in B_X.$$

This means $\psi \in \mathcal{H}^\infty_\mu(B_X)$.

(2) $\Rightarrow$ (1): Suppose $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}^\infty_\mu(B_X)$ is compact. Fix $m \geq 2$. It is obvious that (4.4) holds if $\varphi_{(m)}(B_X)$ is relatively compact in B_X. So assume $\overline{\varphi_{(m)}(B_X)} \cap \partial B_X \neq \varnothing$. Then we can find sequence $\{z^n\}_{n\geq 1} \subset B_X$ such that $\|\varphi_{(m)}(z^n)\| \to 1$. By the definition of $\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}$, with $\varepsilon > 0$ is given we can find a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{B}_\nu(B_X)$ with $\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq 1$ for every $n \geq 1$ satisfying

$$|f_n(\varphi_{(m)}(z^n))| > \|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\| - \varepsilon. \tag{4.5}$$

By the condition (e2), without loss of generality, we may assume that $f_n \rightrightarrows 0$ in $\mathcal{B}_\nu(B_X)$ on compact subsets of B_X and $\{f_n\}_{n\geq 1}$ is uniformly bounded on compact sets.

For each $n \geq 1$, denote $a^n := \varphi(z^n)$ and consider the automorphism $\Phi_{a^n} \in Aut(B_X)$ defined by (2.1). For each $j \in \{1, \ldots, m\}$, put

$$G_{a^n,j} := (a^n_{(m)})_j \cdot f_n - ((\Phi_{a^n})_{(m)})_j \cdot f_n.$$

By (e3), $G_{a^n,j} \in \mathcal{B}_\nu(B_X)$. It is an easy calculation to show that for every $w \in B_X$,

$$|G_{a^n,j}(w)| = |(a^n_{(m)})_j \cdot f_n(w) - ((\Phi_{a^n})_{(m)}(w))_j| \leq \frac{3\sqrt{1 - \|a^n_{(m)}\|^2}}{1 - \|w\|}|f_n(w)|.$$

Then, by (2.2),

$$|G_{a^n,j}(w)| \leq \frac{3\sqrt{1 - \|a^n_{(m)}\|^2}}{1 - \|w\|}\|\delta_w\|,$$

consequently, by Proposition 2.1, and since $\|a^n_{(m)}\| = \|\varphi_{(m)}(z^n)\| \to 1$ as $n \to \infty$, for each $j \in \{1, \ldots, m\}$, $G_{a^n,j} \rightrightarrows 0$ on compact subsets of B_X Now by the condition (e3), there exists $C > 0$ such that for all $j \in \{1, \ldots, m\}$, we have

$$\begin{aligned}
&\|G_{a^n,j}\|_{\mathcal{B}_\nu(B_X)}\\
&\leq \|a^n_{(m)}\|\|f_n\|_{\mathcal{B}_\nu(B_X)} + \|((\Phi_{a^n})_{(m)})_j \cdot f_n\|_{\mathcal{B}_\nu(B_X)}\\
&\leq (C+1)\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq C+1 \quad \forall n \geq 1.
\end{aligned}$$

Therefore, since $W_{\psi,\varphi}$ is compact, By (2.2) and Lemma 4.2, $\|\psi\cdot((G_{a^n})_j \circ \varphi)\|_{\mathcal{H}^\infty_\mu(B_X)} \to 0$ as $n \to \infty$ for every $j \in \{1, \ldots, m\}$. Note that $\Phi_{a^n}(a^n) = 0$. Therefore, by (4.5), we have

$$\begin{aligned}
&\mu(z^n)|\psi(z^n)|\|\varphi_{(m)}(z^n)\|(\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\| - \varepsilon)\\
&\leq \mu(z^n)|\psi(z^n)|\|\varphi_{(m)}(z^n)\||f_n(\varphi_{(m)}(z^n))|\\
&= \mu(z^n)|\psi(z^n)|\sqrt{\sum_{j=1}^m |(G_{a^n,j})(\varphi_{(m)}(z^n))|^2}\\
&= \sqrt{\sum_{j=1}^m \|\psi\cdot((G_{a^n,j})\circ\varphi)\|^2_{H^\infty_\mu(B_X)}}\\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lim_{n\to\infty}\mu(z^n)|\psi(z^n)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(m)}(z^n)}\|\\
&\quad < \varepsilon \lim_{n\to\infty} \mu(z^n)|\psi(z^n)| \leq \varepsilon\|\psi\|_{H^\infty_\mu(B_X)}.
\end{aligned}$$

This implies that (4.4) holds because ε is arbitrary.

(1) $\Rightarrow$ (2): Assume that (4.4) holds for some $m \geq 2$. By Lemma 4.2, it suffices to prove that if $\{f_n\}_{n\geq 1} \subset \mathcal{B}_\nu(B_X)$, $\|f_n\|_{\mathcal{B}_\nu(B_X)} \leq 1$ for all $n \geq 1$ and $f_n \rightrightarrows 0$ on compact subsets of B_X then $\|W_{\psi,\varphi}f_n\|_{H^\infty_\mu(B_X)} \to 0$ as $n \to \infty$.

Let $\{f_n\}_{n\geq 1}$ be such a sequence. Given $\varepsilon > 0$. Then we can choose a number $r \in (0, 1)$ such that $\mu(z)|\psi(z)|\|\delta_{\varphi_{(m)}(z)}\| < \varepsilon$ whenever $\|\varphi_{(m)}(z)\| > r$. Since $|f_n(w)| \leq \|\delta^{\mathcal{B}_\nu(B_X)}_w\|$ for all $w \in B_X$, if $\|\varphi_{(m)}(z)\| >$

r, then $\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon$. Thus $\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon$ when $\|\varphi(z)\| > r$, because $\|\varphi(z)\| \geq \|\varphi_{(m)}(z)\| > r$ for every $z \in B_X$.

Now, we consider the case $\|\varphi(z)\| \leq r$. Then $\|\varphi_{(m)}(z)\| \leq r$. Note that

$$B[\varphi_{(m)}, r] := \{\varphi_{(m)}(y) : \|\varphi_{(m)}(y)\| < r, y \in B_X\} \subset \mathbb{B}_m \subset \mathbb{C}^m$$

is relatively compact for every $0 \leq r < 1$, by the hypothesis, $f_n \to 0$ uniformly on $\overline{B[\varphi_{(m)}, r]}$. Then, there exists $N \in \mathbb{N}$ such that $|f_n(w)| < \varepsilon/\|\psi\|_{H^\infty_\mu(B_X)}$ for all $n \geq N$, $w \in \overline{B[\varphi_{(m)}, r]}$. Thus,

$$\mu(z)|\psi(z)||f_n(\varphi_{(m)}(z)| < \varepsilon \quad \text{if } \|\varphi(z)\| \leq r.$$

□

We will now examine the boundedness and compactness of the operator $W_{\psi,\varphi}$ mapping into $\mathcal{H}^0_\mu(B_X)$.

Theorem 4.4. The following are equivalent:

(1) $\psi \in \mathcal{H}^0_\mu(B_X)$, and there is $m \geq 2$, such that or every $0 \leq r < 1$, $k \geq 1$:

$$\varphi_{(m)}(rB_X) \text{ is relatively compact;} \tag{4.6}$$

$$\lim_{\|z\|\to 1} \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\| = 0; \tag{4.7}$$

(2) $W^0_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{H}^0_\mu(B_X)$ is compact.

Proof. (1) ⇒ (2): Suppose (1) holds. Fix $f \in \mathcal{B}_\nu(B_X)$. We show that $W_{\psi,\varphi}f = \psi \cdot (f \circ \varphi) \in \mathcal{H}^\infty_\mu(B_X)$. Since $\mu(z)|\psi(z)|f(\varphi_{(k)}(z)) \to \mu(z)|\psi(z)|f(\varphi(z))$ as $k \to \infty$ for each $z \in B_X$, and $\mathcal{H}^0_\mu(B_X)$ is closed in $\mathcal{H}^\infty_\mu(B_X)$, it suffices to show that $\psi \cdot (f \circ \varphi_{(k)}) \in \mathcal{H}^0_\mu(B_X)$ for every $k \geq 1$. Given $k \geq 1$. By the hypothesis (1), for given $\varepsilon > 0$ there is $r \in (0, 1)$ such that

$$\begin{aligned}\mu(z)|\psi(z)||f(\varphi_{(k)}(z))| &\leq \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\|\|f\|_{\mathcal{B}_\nu(B_X)} \\ &\leq \varepsilon\|f\|_{\mathcal{B}_\nu(B_X)} \quad \text{for } \|z\| > r.\end{aligned} \tag{4.8}$$

On the other hand, assumption (1) implies that

$$\begin{aligned}\sup\nolimits_{\|z\|\leq r}\mu(z)|\psi(z)||f(\varphi_{(k)}(z))| \\ \leq \mu(z)|\psi(z)|\|\delta^{\mathcal{B}_\nu(B_X)}_{\varphi_{(k)}(z)}\|\|f\|_{\mathcal{B}_\nu(B_X)} < \infty.\end{aligned} \tag{4.9}$$

Consequently, $\psi\cdot(f\circ\varphi_{(k)}) \in \mathcal{H}^\infty_\mu(B_X)$. Moreover, by (4.8), $\psi \cdot (f \circ \varphi_{(k)}) \in \mathcal{H}^0_\mu(B_X)$.

We also obtain from (4.8) and (4.9) that $W^0_{\psi,\varphi}$ is bounded.

The compactness of the operator $W^0_{\psi,\varphi}$ can now be established by following a similar argument as in the proof of Theorem 4.3 and using condition (4.6).

(2) ⇒ (1): First, since $W^0_{\psi,\varphi}$ is bounded and $1 \in \mathcal{B}_\nu(B_X)$ it is easy to check that $\psi \in H^0_\mu(B_X)$.

In order to prove (4.6), first we have to show the following claim:

$$\frac{1}{2}\|z-w\| \leq \|\delta^{\mathcal{H}^\infty_\mu(B_X)}_z - \delta^{\mathcal{H}^\infty_\mu(B_X)}_w\|, \; z, w \in B_X. \tag{4.10}$$

Indeed, by direct calculation, it is easy to check that

$$\begin{aligned}&\frac{1}{2}\|z - w\| \\ &\leq \sqrt{1 - \frac{(1 - \|z\|^2)(1 - \|w\|^2)}{|1 - \langle z, w\rangle|^2}} \\ &= \varrho_X(z, w),\end{aligned}$$

where ϱ_X is the pseudohyperbolic metric in B_X (see [16] p.99). On the other hand, we also have

$$\varrho_X(z, w) = \sup_{\substack{f\in\mathcal{H}^\infty(B_X) \\ \|f\|_\infty\leq 1}} \varrho(f(z), f(w))$$

(see (3.4) in [5]), where $\varrho(x, y) = \left|\frac{x-y}{1-\overline{x}y}\right|$ is the pseudohyperbolic metric in $\mathbb{B}_1$. Note that, since the function $\eta \mapsto \frac{\eta}{1-\overline{f(z)}f(w)}$ is holomorphic from $\mathbb{B}_1$ into $\mathbb{B}_1$ and $f(z) - f(w) \mapsto 0$, it follows from Schwarz's lemma that

$$\begin{aligned}\varrho_X(z,w) &\le \sup_{\substack{f\in\mathcal{H}^\infty(B_X)\\ \|f\|_\infty\le 1}} |f(z)-f(w)| \\ &\le \sup_{\substack{f\in\mathcal{H}^\infty(B_X)\\ \|f\|_\infty\le 1}} |\delta_z^{\mathcal{H}_\mu^\infty(B_X)}(f)-\delta_w^{\mathcal{H}_\mu^\infty(B_X)}(f)| \\ &= \|\delta_z^{\mathcal{H}_\mu^\infty(B_X)}-\delta_w^{\mathcal{H}_\mu^\infty(B_X)}\|.\end{aligned}$$

Hence, (4.10) is proved.

Next, we prove (4.6). For $r\in(0,1)$, the set $V_r := \{\delta_z^{\mathcal{H}_\mu^\infty(B_X)} : \|z\|\le r\}\subset(\mathcal{H}_\mu^\infty(B_X))'$ is bounded. Then, since $W_{\psi,\varphi}$ is compact, the set

$$(W_{\psi,\varphi})^*(V_z)=\left\{\psi(z)\delta_{\varphi(z)}^{\mathcal{B}_\nu(B_X)} : \|z\|\le r\right\}$$

is relatively compact in $[\mathcal{B}_\nu(B_X)]'$.

We know that, for every $K\subset[\mathcal{B}_\nu(B_X)]'$ and every bounded subset $D\subset\mathbb{C}$, if the set $\{t\eta : t\in D, \eta\in A\}$ is relatively compact in $\mathcal{B}_\nu(B_X)$ then $A\subset[\mathcal{B}_\nu(B_X)]'$ is also relatively compact. With this in mind, since the set $\{\psi(z) : \|z\|\le r\}$ is bounded, the set $\{\delta_z^{\mathcal{B}_\nu(B_X)}, \|z\|\le r\}$ is relatively compact. Then, it follows from (4.10) that $\varphi(rB_X)$ is relatively compact, so is $\varphi_{(m)}(rB_X)$ for $m\ge 2$.

Finally, we prove (4.7). Assume that there exist $m\ge 1$, $\varrho>0$ and $\{z^n\}_{n\ge1}\subset B_X$, $\|z^n\|\to 1$ such that $\mu(z^n)|\psi(z^n)|\|\delta_{\varphi_{(m)}(z^n)}\|>\varrho$ for all $n\ge 1$. Then, we may choose $\{f_n\}_{n\ge1}\subset\mathcal{B}_\nu(B_X)$ such that $\|f_n\|_{\mathcal{B}_\nu(B_X)}\le 1$ and $|f_n(\varphi_{(m)}(z^n))|>\|\delta_{\varphi_{(m)}(z^n)}^{\mathcal{B}_\nu(B_X)}\|-\varrho/2$ for every $n\ge 1$. Thus

$$\begin{aligned}&\mu(z^n)|\psi(z^n)||f(\varphi_{(m)}(z^n))| \\ &\qquad > \varrho-\varrho/2\mu(z^n)|\psi(z^n)|.\end{aligned}$$

Therefore, since $\psi\in\mathcal{H}_\mu^0(B_X)$, $W_{\psi,\varphi}^0 f_n\notin\mathcal{H}_\mu^0(B_X)$. This contradicts the boundedness of $W_{\psi,\varphi}^0$. □

Remark 4.1. In the case where $\dim X<\infty$, and by following the proof of Theorem 4.4, the following statements are equivalent:

(1) $\lim_{\|z\|\to1}\mu(z)|\psi(z)|\|\delta_{\varphi_{(k)}(z)}^{\mathcal{B}_\nu(B_X)}\|=0$ for every $k\ge1$ and $\psi\in\mathcal{H}_\mu^\infty(B_X)$;

(2) $W_{\psi,\varphi}^0 : \mathcal{B}_\nu(B_X)\to\mathcal{H}_\mu^0(B_X)$ is compact;

(3) $W_{\psi,\varphi}^0 : \mathcal{B}_\nu(B_X)\to\mathcal{H}_\mu^0(B_X)$ is bounded.

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