Một đặc trưng của không gian kiểu Zygmund và áp dụng

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TÓM TẮT

Trong bài báo này, chúng tôi khảo sát điều kiện để cho một không gian kiểu Zygmund $\mathcal{Z}_\omega$ là một không gian nhỏ, biên ổn định bất biến tự đẳng cấu, trong đó ω là một trọng chuẩn tắc trên hình cầu đơn vị $\mathbb{B}_n$ trong $\mathbb{C}^n$. Kết quả này được áp dụng để nghiên cứu mối quan hệ giữa tính bị chặn và tính compact của các toán tử hợp liên tục $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, từ không gian kiểu Bloch $\mathcal{B}_\omega$ vào không gian kiểu Zygmund $\mathcal{Z}_\omega$, và từ $\mathcal{Z}_\omega$ vào chính nó.

Từ khóa: *Không gian Bloch, không gian Zygmund, toán tử hợp có trọng, tính bị chặn, tính compact.*

A characterization of Zygmund-type spaces and its application

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ABSTRACT

In this paper, we investigate the conditions under which a Zygmund-type space $\mathcal{Z}_\omega$ is an automorphism invariant boundary regular small space, where ω is a normal weight on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. This result is applied to study the relationship between the boundedness and compactness of the weighted composition operators $W_{\psi,\varphi}$, $f \mapsto \psi \cdot (f \circ \varphi)$, from the Bloch-type space $\mathcal{B}_\omega$ to the Zygmund-type space $\mathcal{Z}_\omega$, and from $\mathcal{Z}_\omega$ to itself.

Keywords: *Bloch spaces, Zygmund spaces, weighted composition operators, boundedness, compactness*

1. INTRODUCTION

Given a natural number n, let us consider the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ the space of all holomorphic functions in $\mathbb{B}_n$. The standard basis for $\mathbb{C}^n$ consists of the vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_n = (0, \dots, 0, 1)$.

In the paper, for $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$, we write

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k,$$

and

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}.$$

For $f \in H(\mathbb{B}_n)$, let

$$\nabla_z f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\Big),$$
$$Rf(z) = \langle \nabla f(z), \overline{z}\rangle, \quad z \in \mathbb{B}_n$$

Let $\mathbb{D} = \mathbb{B}_1$ denote the unit disk of $\mathbb{C}$. If $f \in H(\mathbb{D})$, and $\sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty$ then f is said to belong to the Zygmund space. In fact, the $1 - |z|^2$ is a kind of weight function. Later, the weight function was extended to $(1 - |z|^2)^\alpha$, $0 < \alpha < \infty$.

A positive continuous function ω on the interval $[0, 1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that

$$\frac{\omega(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^a} = 0, \tag{W_1}$$

$$\frac{\omega(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^b} = \infty. \tag{W_2}$$

If we say that a function $\omega : \mathbb{B}_n \to [0, \infty)$ is normal, we also assume that it is radial, that is, $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{B}_n$. Strictly positive continuous functions on $\mathbb{B}_n$ are called weights.

We define Bloch-type space $\mathcal{B}_\omega$, Zygmund-type space $\mathcal{Z}_\omega$, respectively, as follows:

$$\mathcal{B}_\omega = \Big\{ f \in H(\mathbb{B}_n): \ \|f\|_{s\mathcal{B}_\omega} < \infty \Big\},$$
$$\mathcal{Z}_\omega = \Big\{ f \in H(\mathbb{B}_n): \ \|f\|_{s\mathcal{Z}_\omega} < \infty \Big\},$$

where

$$f \mapsto \|f\|_{s\mathcal{B}_\omega} = \sup_{z\in\mathbb{B}_n} \omega(z)|\nabla(Rf)(z)|,$$
$$f \mapsto \|f\|_{s\mathcal{Z}_\omega} = \sup_{z\in\mathbb{B}_n} \omega(z)|\nabla(Rf)(z)|$$

are seminorms on $\mathcal{B}_\omega$ and $\mathcal{Z}_\omega$, respectively. The spaces $\mathcal{B}_\omega$, $\mathcal{Z}_\omega$ be endowed with Banach space structures via the norm

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \|f\|_{s\mathcal{B}_\omega},$$
$$\|f\|_{\mathcal{Z}_\omega} = |f(0)| + \|f\|_{s\mathcal{Z}_\omega}.$$

When $\omega(r) = 1 - r$, from [20] we see that $f \in \mathcal{Z}_{1-r} := \mathcal{Z}$ if and only if f belongs to the ball algebra $A(\mathbb{B}_n)$ on $\mathbb{B}_n$ and there exists a constant $C > 0$ such that $|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)| < C|h|$, for all $\zeta \in \partial\mathbb{B}_n$ and $\zeta \pm h \in \partial\mathbb{B}_n$. The space $\mathcal{Z}_\omega$ can be considered as a generalization of the classical Zygmund space which was introduced in [16].

Let $S(\mathbb{B}_n)$ be the set of holomorphic self-maps of $\mathbb{B}_n$. Given $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$. The weighted composition operator with symbols ψ and φ is the linear operator $W_{\psi,\varphi} : E \to F$ defined by

$$W_{\psi,\varphi}(f) := \psi \cdot (f \circ \varphi), \quad \text{for } f \in E,$$

where E, F are Banach spaces of holomorphic functions on $\mathbb{B}_n$. We can regard this operator as a generalization of a multiplication operator and a composition operator.

Theory of (weighted/unweighted) composition operators has been establishing since the last century. The boundedness, compactness, essential norm, and spectral properties are always the highlights of research of composition operators. Book [4] is a good reference for studying the composition operators on classical spaces of analytic functions. Moreover, this theory is also established on the basis of theory of analytic functions (on the unit disk), which is basically a convenient tool.

Composition operators mapping into the classical Zygmund were studied in [1,2,3,8,10,11,18]. Many scholars have discussed similar problems (see [5,6,7,12,13,14,19,21], etc.)

However, for abstract normal weight especially in high dimensions, when investigating and using the properties (for example, discussing weighted/unweighted composition operator of the Zygmund type space, we often encounter some obstacles. This is one of the reasons why the sufficient and necessary conditions for $W_{\psi,\varphi}$ to be bounded or compact between Zygmund-type spaces (normal weight Zygmund spaces) have not been studied much so far. In order to overcome these obstacles, we need a variety of means or techniques.

Motivated by the above-mentioned discussions and the previous investigations, the purpose of this paper is to uncover additional characteristics of Zygmund-type spaces and serve them as technical tools to solve the problem of the relationship between the boundedness, compactness of weighted composition operators from a Bloch-type space $\mathcal{B}_\omega$ into the Zygmund-type space $\mathcal{Z}_\omega$ and from $\mathcal{Z}_\omega$ into itself.

In Section 2, we provide a condition for the normal weight ω that is sufficient for the space $\mathcal{Z}_\omega$ to be an automorphism invariant boundary regular small space. A main motivation for the section is a result of Shapiro [15] (also, Theorem 4.5 of [4]) asserting that the condition $\|\varphi\|_\infty < 1$ is necessary for the composition C_φ to be compact on a "suitably small" Banach space. In [15], four axioms are necessary for appropriately small spaces. Two of these axioms are fundamental, concerning norm naturality and the nontriviality of the spaces. The other two axioms govern the size of the spaces. That is, suitably small spaces are small by the boundary regularity axiom, which ensures continuous extension up to the boundary, but are not "too small" due to the automorphism-invariance axiom. For further details, refer to [15] or [4].

Applying the result in the previous section, in Section 3, we establishes the relation between the boundedness, compactness of weighted composition operators from $\mathcal{B}_\omega$ into $\mathcal{Z}_\omega$ and from $\mathcal{Z}_\omega$ into itself.

Throughout this paper, we use the notions $a \lesssim b$ and $a \asymp b$ for non negative quantities a and b to mean $a \le Cb$ and, respectively, $C^{-1}b \le a \le Cb$ for some inessential constant $C > 0$.

2. A CHARACTERIZATION OF ZYGMUND-TYPE SPACES

This section is devoted to the study of the properties "*small*" and "*automorphism invariant boundary regular*" of the Zygmund-type spaces which will be necessary in establishing one of our main result.

For a normal weight ω on $\mathbb{B}_n$ we use there certain quantities, which will be used in this work:

$$I^1_\omega(z) := \int_0^{|z|} \frac{dt}{\omega(t)},$$
$$I^2_\omega(z) := \int_0^{|z|} \Big(\int_0^t \frac{ds}{\omega(s)}\Big)dt, \quad \forall z \in \mathbb{B}_n.$$

Remark 2.1. Since ω is positive, continuous, $m_{\omega,\delta} := \min_{t\in[0,\delta]} \omega(t) > 0$. Moreover, it follows from (W_1) that ω is strictly decreasing on $[\delta, 1)$, hence, we obtain that $\max_{t\in[0,1)} \omega(t) =: M_\omega < \infty$. Then, it is easy to check that

$$\omega(z)I^1_\omega(z) < R_\omega := \delta\frac{M_\omega}{m_{\omega,\delta}} + 1 - \delta < \infty \quad (2.1)$$

and, hence,

$$\omega(z)I^2_\omega(z) < |z|R_\omega < R_\omega < \infty \quad (2.2)$$

for every $z \in \mathbb{B}_n \setminus \{0\}$.

Proposition 2.1 ([21])**.** For every normal weight ω on $\mathbb{B}_n$ we have

$$\begin{aligned}\mathcal{Z}_\omega = \mathcal{Z}^R_\omega &:= \Big\{f \in H(\mathbb{B}_n) :\ \|f\|_{\mathcal{Z}^R_\omega} < \infty\Big\}\\ = \mathcal{Z}^\nabla_\omega &:= \Big\{f \in H(\mathbb{B}_n) :\ \|f\|_{\mathcal{Z}^\nabla_\omega} < \infty\Big\}\end{aligned}$$

and $\|\cdot\|_{\mathcal{Z}_\omega} \cong \|\cdot\|_{\mathcal{Z}^R_\omega} \cong \|\cdot\|_{\mathcal{Z}^\nabla_\omega}$, where

$$R^{(2)}f = R(Rf),$$
$$|\nabla^{(2)}f(z)| = \Big(\sum_{i,j=1}^n \Big|\frac{\partial^2 f}{\partial z_i \partial z_j}(z)\Big|^2\Big)^{\frac12},$$
$$\|f\|_{\mathcal{Z}^R_\omega} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|R^{(2)}f(z)|,$$
$$\|f\|_{\mathcal{Z}^\nabla_\omega} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|\nabla^{(2)}f(z)|,$$

fő every $f \in \mathcal{Z}_\omega$.

In this paper, let us write simply we denote $\mathcal{Z}_\omega$ for the complex $(\mathcal{Z}_\omega, \|\cdot\|_{\mathcal{Z}^R_\omega})$.

Lemma 2.2. Let ω be a normal weight on $\mathbb{B}_n$. Then there exists $C > 0$ such that for every $f \in \mathcal{Z}_\omega$ and for every $z \in \mathbb{B}_n$ we have

$$\begin{aligned}&|Rf(z)| \le CI^1_\omega(z)\|f\|_{\mathcal{Z}_\omega},\\ &|\nabla f(z)| \le C(1 + I^1_\omega(z))\|f\|_{\mathcal{Z}_\omega};\end{aligned} \quad (2.3)$$

and

$$|f(z)| \le |f(0)| + CI^2_\omega(z)\|f\|_{\mathcal{Z}_\omega}. \quad (2.4)$$

Proof. The estimate (2.3) follows from [17] which says there exists $C > 0$ such that for every $f \in \mathcal{B}_\omega$ and for every $z \in \mathbb{B}_n$ we have

$$|f(z)| \le C\big(1 + I^1_\omega(z)\big)\|f\|_{\mathcal{B}_\omega}. \quad (2.5)$$

Then by (2.3) and (2.5) again we obtain (2.4). □

Note that, in fact, by using (2.5) the estimate for $|\nabla f(z)|$ in (2.3) can be replced by

$$|\nabla f(z)| \lesssim \big(1 + I^1_\nu(z)\big)|\nabla f(0)| + I^1_\nu(z)\|f\|_{\mathcal{Z}_\mu}. \quad (2.6)$$

Now, by $Aut(\mathbb{B}_n)$, we denote the automorphism group of $\mathbb{B}_n$ that consists of all biholomorphic mappings of $\mathbb{B}_n$. It is known that every $\varphi \in Aut(\mathbb{B}_n)$ is a unitary transformation of $\mathbb{C}^n$ if and only if $\varphi(0) = 0$ (see [20]).

For any $\alpha \in \mathbb{B}_n \setminus \{0\}$, we define

$$\varphi_\alpha(z) = \frac{\alpha - P_\alpha(z) - s_\alpha Q_\alpha(z)}{1 - \langle z, \alpha\rangle}, \quad z \in \mathbb{B}_n, \tag{2.7}$$

where $s_\alpha = \sqrt{1 - |\alpha|^2}$, $P_\alpha(z) = \frac{\langle z,\alpha\rangle}{|\alpha|^2}\alpha$ is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[\alpha]$ generated by α, and $Q_\alpha(z) = z - \frac{\langle z,\alpha\rangle}{|\alpha|^2}\alpha$ is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [\alpha]$. It is clear that

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a$$
$$Q_a(z) = z - \frac{\langle z, a\rangle}{|a|^2} a, \quad z \in \mathbb{B}_n.$$

When $\alpha = 0$, we simply define $\varphi_\alpha(z) = -z$. It is obvious that each φ_α is a holomorphic mapping from $\mathbb{B}_n$ into $\mathbb{C}^n$. It is well known that each φ_α is a homeomorphism of the closed unit ball $\overline{\mathbb{B}}_n$ onto $\overline{\mathbb{B}}_n$ and every automorphism φ of $\mathbb{B}_n$ is the form $\varphi = \varphi_\alpha U$, where U is a unitary transformation of $\mathbb{C}^n$.

Theorem 2.3. Let ω be normal weight on $\mathbb{B}_n$ such that $I^2_\omega(1) < \infty$. Then the Zygmund-type space $\mathcal{Z}_\omega$ is an automorphism invariant boundary regular small space in the following sense:

(i) Every function in $\mathcal{Z}_\omega$ extends continuously to the closed unit ball,

(ii) $\mathcal{Z}_\omega$ contains all the polynomials,

(iii) Evaluation at each point of $\mathbb{B}_n$ is a bounded linear functional,

(iv) If $\varphi \in Aut(\mathbb{B}_n)$ and $f \in \mathcal{Z}_\omega$ then $f \circ \varphi \in \mathcal{Z}_\omega$.

Remark 2.2. The axioms (i) and (iii) guarantee convergence in the norm of $\mathcal{Z}_\omega$ implies convergence in the sup norm: the identity map from $(\mathcal{Z}_\omega, \|\cdot\|_{\mathcal{Z}_\omega})$ to $(\mathcal{Z}_\omega, \|\cdot\|_\infty)$ is continuous by the closed graph theorem. Moreover, another closed graph theorem argument using the axiom (iii) shows that the axiom (iv) implies that C_φ is bounded on $\mathcal{Z}_\omega$ whenever φ is a conformal automorphism of $\mathbb{B}_n$.

Proof. From the definitions it is easy to see that (ii)-(iii) hold for $\mathcal{Z}_\omega$. Under the condition $\int_0^1 \frac{dt}{\omega(t)} < \infty$ the space $\mathcal{Z}_\omega$ satisfies (i) (see [14]).

To show that (iv) holds, we need to prove that for any conformal automorphism $\varphi = \varphi_a U = (\varphi_1, \dots, \varphi_n)$ of $\mathbb{B}_n$, if $f \in \mathcal{Z}_\omega$ then $f \circ \varphi \in \mathcal{Z}_\omega$ where a is a point of $\mathbb{B}_n$ and U is a unitary transformation of $\mathbb{C}^n$. Without loss of generality, we may assume that $\varphi = \varphi_a$ for some $a \in \mathbb{B}_n$. Note that $\varphi_j \in H(\overline{\mathbb{B}}_n)$, $j = 1, \dots, n$, from (2.7), which implies that $R^{(k)}\varphi_j \in H(\overline{\mathbb{B}}_n)$ and $R^{(k)}\varphi_j$ is bounded in $\overline{\mathbb{B}}_n$ for any positive integer k. Thus,

$$\begin{aligned} M^{(1)}_\varphi &:= \sup_{z\in\mathbb{B}_n} |R\varphi(z)| < \infty, \\ M^{(2)}_\varphi &:= \sup_{z\in\mathbb{B}_n} |R^{(2)}\varphi(z)| < \infty. \end{aligned} \tag{2.8}$$

Let $\lambda \in (0, 1)$ be such that $|R\varphi(z)| \leq 1$ and $|R^{(2)}\varphi(z)| \leq 1$ for $|\varphi(z)| \leq \lambda$. There exists $D_0 > 0$ such that

$$1 \leq D_0 I^1_\omega(\lambda), \quad 1 \leq D_0 I^2_\omega(\lambda). \tag{2.9}$$

Then, thers exists $D_1 > 0$ such that

$$\begin{aligned} &\sup_{|\varphi(z)|\leq\lambda} \omega(\varphi(z))|R\varphi(z)|\big(1 + I^1_\omega(\varphi(z))\big) \\ &\quad\leq D_1 \sup_{|\varphi(z)|\leq\lambda} \omega(\varphi(z))||R\varphi(z)|I^1_\omega(\varphi(z)), \\ &\sup_{|\varphi(z)|\leq\lambda} \omega(\varphi(z))|R^{(2)}\varphi(z)|\big(1 + I^2_\omega(\varphi(z))\big) \\ &\quad\leq D_1 \sup_{|\varphi(z)|\leq\lambda} \omega(\varphi(z))|R^{(2)}\varphi(z)|I^2_\omega(\varphi(z)). \end{aligned} \tag{2.10}$$

Let $D = \max\{D_0 + 1, D_1\}$. For every $f \in \mathcal{Z}_\omega$, by (2.1)$-$(2.4), (2.10), and a standard calculation, we have

$$
\begin{aligned}
&\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\le \omega(z)\big[|R^{(2)}f(\varphi(z))\varphi(z)| \\
&\quad + 2|Rf(\varphi(z))R\varphi(z)| + |f(\varphi(z))R^{(2)}\varphi(z)|\big] \\
&= \frac{\omega(z)}{\omega(\varphi(z))}\omega(\varphi(z))\big[|R^{(2)}f(\varphi(z))\varphi(z)| \\
&\quad + 2|Rf(\varphi(z))R\varphi(z)| + |f(\varphi(z))R^{(2)}\varphi(z)|\big] \\
&\le \frac{\omega(z)}{\omega(\varphi(z))}\Big[1+ \\
&\qquad + C\omega(\varphi(z))\big(2|R\varphi(z)|(1+I^1_\omega(\varphi(z))) \\
&\qquad + |R^{(2)}\varphi(z)|(1+I^2_\omega(\varphi(z)))\big)\Big]\|f\|_{\mathcal{Z}_\omega} \\
&\le \frac{\omega(z)}{\omega(\varphi(z))}\Big[1 \\
&\quad + CD\sup_{|\varphi(z)|\ge\lambda}\omega(\varphi(z))[2M^{(1)}_\varphi + M^{(2)}_\varphi]\Big]\|f\|_{\mathcal{Z}_\omega} \\
&= \frac{\omega(z)}{\omega(\varphi(z))}\Big[1 + CDR_\omega[2M^{(1)}_\varphi + M^{(2)}_\varphi]\Big]\|f\|_{\mathcal{Z}_\omega}
\end{aligned} \tag{2.11}
$$

for every $z\in\mathbb{B}_n$.

(i) First, we consider the case where $a = 0$. Then $|\varphi(z)|\le|z|$ for every $z\in\mathbb{B}_n$. Denote

$$B_\delta := \{z\in\mathbb{B}_n : |\varphi(z)|\le\delta\}.$$

Since μ is decreasing on $[\delta,1)$ we have

$$
\begin{aligned}
\frac{\omega(z)}{\omega(\varphi(z))} &\le \frac{M_\omega}{m_{\omega,\delta}} \quad \forall z\in B_\delta; \\
\frac{\omega(z)}{\omega(\varphi(z))} &< 1 \quad \forall z\in\mathbb{B}_n\setminus B_\delta.
\end{aligned}
$$

Therefore, it follows from (2.11) that

$$
\begin{aligned}
&\sup_{z\in\mathbb{B}_n}\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\le \sup_{z\in B_\delta}\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\quad + \sup_{z\in\mathbb{B}_n\setminus B_\delta}\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\le \Big(\frac{M_\omega}{m_{\omega,\delta}}+1\Big) \\
&\times\Big(1 + CDR_\omega[2M^{(1)}_\varphi + M^{(2)}_\varphi]\Big)\|f\|_{\mathcal{Z}_\omega} < \infty.
\end{aligned} \tag{2.12}
$$

Hence, $f\circ\varphi\in\mathcal{Z}_\omega$.

(ii) Now, we consider the case $a\neq 0$. Take a $\gamma\in Aut(\mathbb{B}_n)$ such that $\gamma(0) = a$. Then $\eta := \varphi\circ\gamma\in Aut(\mathbb{B}_n)$ and $\eta(0) = 0$. By (i), $g := f\circ\eta\in\mathcal{Z}_\omega$. Note that $\gamma^{-1}\in Aut(\mathbb{B}_n)$, as the above, we have $R^{(k)}\gamma^{-1}$ is bounded in $\overline{\mathbb{B}}_n$ for any positive integer k. Then, since $f\circ\varphi = g\circ\gamma^{-1}$, as the estimate (2.12) we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{B}_n}\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&= \sup_{z\in\mathbb{B}_n}\omega(z)|R^{(2)}(g\circ\gamma^{-1})(z)| \\
&\le \Big(\frac{M_\omega}{m_{\omega,\delta}}+1\Big) \\
&\times\Big(1 + CDR_\omega[2M^{(1)}_{\gamma^{-1}} + M^{(2)}_{\gamma^{-1}}]\Big)\|g\|_{\mathcal{Z}_\omega} < \infty.
\end{aligned}
$$

Consequently, $f\circ\varphi\in\mathcal{Z}_\omega$. □

Remark 2.3. The condition $I^2_\omega(1) < \infty$ cannot be omitted. Indeed, consider the weight function $\omega(t) = (1-t)^2$ for $t\in[0,1)$ which satisfies $I^2_\omega(1) = \infty$. Then ir is easy to check that the function $f\in\mathcal{Z}_\omega$ given by $f(z) = \ln(1-z)$ for every $z\in\mathbb{B}_1$ function in $\mathcal{Z}_\omega$ can not extend continuously to $\overline{\mathbb{B}}_1$. This means that the condition (i) is not true for $\mathcal{Z}_\omega$.

3. A RELATION BETWEEN WEIGHTED COMPOSITION OPERATORS $\mathcal{B}_\omega\to\mathcal{Z}_\omega$ AND $\mathcal{Z}_\omega\to\mathcal{Z}_\omega$

In order to conclude the paper we establishes the relation between the boundedness, compactness of weighted composition operators from $\mathcal{B}_\omega$ into $\mathcal{Z}_\omega$ and from $\mathcal{Z}_\omega$ into itself.

Before stating the theorem first let us note that for each $j = 1,\dots,n$ the function id_j given by $id_j(z) := z_j$ belongs to $\mathcal{Z}_\omega$. Then, in the case $\psi\in H^\infty(\mathbb{B}_n)$ with $\|\psi\|_\infty\le 1$ and $W_{\psi,\varphi}:\mathcal{Z}_\omega\to\mathcal{Z}_\omega$ is compact, $W_{\psi,\varphi}(id_j) = \psi\cdot\varphi_j$ hence, $\theta_j := \psi\cdot\varphi_j\in\mathcal{Z}_\omega$, $j = 1,\dots,n$. For each $m\ge 1$, put

$$\theta^m = (\theta^m_1,\dots,\theta^m_n) := \prod_{k=0}^{m-1}(\psi\circ\varphi^k)\cdot\varphi^m,$$

where $\varphi^0 = id$, and $\varphi^k := \underbrace{\varphi\circ\cdots\circ\varphi}_{k\text{ times}}$ for $k\ge 1$.

By Theorem 2.3(i) we can assume that θ^m, $m \geq 0$, are continuous functions on the closed unit the closed unit ball $\overline{\mathbb{B}}_n$.

Theorem 3.1. Let $\psi \in H^\infty(\mathbb{B}_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B}_n)$ and ν, ω be normal weights on $\mathbb{B}_n$ and $\int_0^1 \frac{dt}{\omega(t)} < \infty$.Then the following are equivalent:

(1) $W_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{Z}_\omega$ is compact;

(2) $W_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{Z}_\omega$ is bounded;

(3) $W_{\psi,\varphi} : \mathcal{Z}_\omega \to \mathcal{Z}_\omega$ is compact;

(4) $\psi, \psi \cdot \varphi_j \in \mathcal{Z}_\omega$ for every $j = 1, \ldots, n$ and $\|\varphi\|_\infty < 1$.

In order to prove the theorem we need some lemmas.

Lemma 3.2. Assume that $\varphi(0) = 0$ and $W_{\psi,\varphi} : \mathcal{Z}_\omega \to \mathcal{Z}_\omega$ is compact. Then $\|\theta^m\|_\infty \to 0$.

Proof. Without loss of generality we may assume that $\|\psi\|_\infty \leq 1$. We have two cases to consider:

(i) In the case $|\psi(0)| = 1$, it follows from Theorem 2.3(i) and the maximum modulus principle we have $\psi \equiv 1$. Then $W_{\psi,\varphi} = C_\varphi$, the composition operator on $\mathcal{Z}_\omega$, and hence, the lemma follows from Lemma 2.2 of [15].

(ii) Now we assume that $|\psi(0)| < 1$.

We will prove that $W_{\psi,\varphi}$ has spectral radius $\varrho(W_{\psi,\varphi}) < 1$.

Let $\lambda \neq 0$ be a spectral point of $W_{\psi,\varphi}$. Since $W_{\psi,\varphi}$ is compact, λ must to be an eigenvalue. Fix $f \in \mathcal{Z}_\omega$, an eigenfunction of $W_{\psi,\varphi}$ for the eigenvalue λ. Thus $W_{\psi,\varphi}(f) = \lambda f$ and there is a point $a \in \mathbb{B}_n$ for which $f(a) \neq 0$. Denote $\mathbb{B}^a := \big\{z \in \mathbb{B}_n : \ |z| < \frac{1+|a|}{2}\big\}$. Note that, $|\varphi(z)| < |z|$ for every $z \in \mathbb{B}_n$, since otherwise, the composition operator C_φ would be an isomorphism. Consequently, by $\|\psi\|_\infty \leq 1$, $(\psi \cdot C_\varphi)(B_{\mathcal{Z}_\omega})$ is not relatively compact subset of the unit ball $B_{\mathcal{Z}_\omega}$ of $\mathcal{Z}_\omega$. This means $\psi \cdot C_\varphi$ is not a compact operator. This contradicts the compactness of $W_{\psi,\varphi}$. Then, by the Schwarz Lemma, $\varphi(\mathbb{B}^a)$ is a relatively compact subset of $\mathbb{B}^a$. A second application of the Schwarz Lemma, this time to the (suitably normalized) restriction of φ to $\varphi(\mathbb{B}^a)$, and go on, shows that $\varphi^m(a) \to 0$ as $m \to \infty$.

Now, since $\lim_{m\to\infty} |\psi(\varphi^{m-1}(a))| = |\psi(0)| \neq 1$, by using the fact that, if $0 < a_m < 1$ and $\{a_m\}_{m\geq 1}$ does not converge to 1 then $\prod_{m=1}^\infty a_m = 0$, we obtain

$$\begin{aligned}\lambda^m f(a) &= [W_{\psi,\varphi}]^m(f)(a) \\ &= \Big(\prod_{k=0}^{m-1} \psi(\varphi^k(a))\Big) \cdot f(\varphi^m(a)) \to 0 \cdot f(0)\end{aligned}$$

as $m \to \infty$. Because $f(a) \neq 0$ it therefore must has $|\lambda| < 1$. The compactness of $W_{\psi,\varphi}$ also forces its spectrum to consist of the point 0 along with an at most countable set of eigenvalues which can cluster only at 0. Thus the spectral radius of $W_{\psi,\varphi}$ is the magnitude of the largest eigenvalue of $W_{\psi,\varphi}$, which we have just seen to be < 1. The spectral radius formula now shows that

$$\lim_{m\to\infty} \|[W_{\psi,\varphi}]^m\|^{1/m} = \varrho(W_{\psi,\varphi}) < 1,$$

so in particular, $\lim_{m\to\infty} \|[W_{\psi,\varphi}]^m\| = 0$. Note that $\theta_j^m = [W_{\psi,\varphi}]^m(id_j) \in \mathcal{Z}_\omega$, $j = 1, \ldots, n$. Then,

$$\begin{aligned}\|\theta_j^m\|_{\mathcal{Z}_\omega} &= \|[W_{\psi,\varphi}]^m(id_j)\|_{\mathcal{Z}_\omega} \\ &\leq \|[W_{\psi,\varphi}]^m\| \|id_j\|_{\mathcal{Z}_\omega} \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

On the other hand, it follows from Theorem 2.3(i & iii) that the topology of $\mathcal{Z}_\omega$ is stronger than the sup-norm topology. Consequently, $\|\theta^m\|_\infty \to 0$ as $m \to \infty$. Lemma is proved. □

Lemma 3.3. Let $\psi \in H(\mathbb{B}_n)$, $\varphi \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$. Let $X = \mathcal{B}_\nu$ or $\mathcal{Z}_\nu$. Then the operators $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is compact if and only if for any bounded sequence $\{f_m\} \subset X$ which converges to 0 uniformly on any compact subsets of $\mathbb{B}_n$ as $m \to \infty$, we have $\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\mu} \to 0$ as $m \to \infty$.

The lemma for the case $X = \mathcal{B}_\nu$ has been proven in [9]. For the case $X = \mathcal{Z}_\nu$, it is similar to that of $X = \mathcal{B}_\nu$ and will therefore be omitted.

Lemma 3.4. Let $\psi \in H(\mathbb{B}_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$. Assume that $W_{\psi,\varphi} : \mathcal{Z}_\nu \to \mathcal{Z}_\mu$ is bounded. Then

$$\begin{aligned} &\sup_{z\in\mathbb{B}_n} \mu(z)|A_{\psi,\varphi}(z)| < \infty, \\ &\sup_{z\in\mathbb{B}_n} \mu(z)|B_{\psi,\varphi}(z)| < \infty, \end{aligned} \tag{3.1}$$

where

$$A_{\psi,\varphi}(z) := 2R\psi(z)R\varphi(z) + \psi(z)R^{(2)}\varphi(z),$$
$$B_{\psi.\varphi}(z) := \psi(z)((R\varphi_1(z))^2, \ldots, (R\varphi_n(z))^2).$$

Proof. First, by taking $f_0(z) = 1 \in \mathcal{Z}_\nu$, it follows from the boundedness of $W_{\psi,\varphi}$ that $\psi \in \mathcal{Z}_\mu$.

At the same time, for each $j \in \{1, \ldots, n\}$, by considering $f_j(z) = z_j$ and $g_j(z) = z_j^2$ for every $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$ we can check that $\psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$.

Then, since

$$\begin{aligned} &R^{(2)}[\psi(z)\varphi_j(z)] \\ &\quad= R^{(2)}\psi(z)\varphi_j(z) + 2R\psi(z)R\varphi_j(z) \\ &\quad\quad+ \psi(z)R^{(2)}\varphi_j(z) \\ &\quad= R^{(2)}\psi(z)\varphi_j(z) + A_{\psi,\varphi_j}(z), \\ &R^{(2)}[\psi(z)\varphi_j^2(z)] \\ &\quad= \varphi_j(z)\big(R^{(2)}\psi(z)\varphi_j(z) + 4R\psi(z)R\varphi_j(z) \\ &\quad\quad+ 2\psi(z)R^{(2)}\varphi_j(z)\big) + 2\psi(z)\big(R\varphi_j(z)\big)^2 \\ &\quad= \varphi_j(z)\Big[2R^{(2)}[\psi(z)\varphi_j(z)] - R^{(2)}\psi(z)\Big] \\ &\quad\quad+ 2B_{\psi,\varphi_j}(z) \end{aligned} \tag{3.2}$$

for every $z \in \mathbb{B}_n$ and every $j = 1, \ldots, n$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{B}_n} \mu(z)|A_{\psi,\varphi_j}(z)| \\ &\quad\le \|\psi \cdot \varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty, \\ &\sup_{z\in\mathbb{B}_n} \mu(z)|B_{\psi,\varphi_j}(z)| \\ &\quad\le \|\psi \cdot \varphi_j^2\|_{\mathcal{Z}_\mu} + 2\|\psi \cdot \varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty \end{aligned}$$

for every $j = 1, \ldots, n$. Consequently, (3.1) is proved. □

Proof of Theorem 3.1. Theorem is trivial if $\|\psi\|_\infty = 0$. Without loss of generality we may assume that $0 < \|\psi\|_\infty \le 1$, since for ortherwise we can consider the $\|\psi\|_\infty^{-1}\psi$ instead of ψ.

(1) ⇒ (2): It is obvious.

(2) ⇒ (3): Suppose $\{f_m\}_{m\ge1}$ is a bounded sequence in $\mathcal{Z}_\omega$ and it converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. By the Weierstrass theorem the sequences $\{Rf_m\}_{m\ge1}$, $\{R^{(2)}f_m\}$ also converge to 0 uniformly on compact subsets of $\mathbb{B}_n$. We now prove that $\|f_m\|_{\mathcal{B}_\omega}$ converges to 0. Given $\varepsilon > 0$. Since $\lim_{t\to1}\omega(t) = 0$ there exists $\varrho \in (\delta, 1)$ such that $\omega(|z|) < \varepsilon$ whenever $\varrho < |z| < 1$. Meanwhile, there exists a positive integer N such that $|f_m(0)| < \varepsilon$, $|Rf_m(z)| < \varepsilon$, $|R^{(2)}f_m(z)| < \varepsilon$ for all $|z| \le \varrho$ and all $m \ge N$. Then, by (2.3)

$$\begin{aligned} \|f_m\|_{\mathcal{B}_\omega} &\le |f_m(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|Rf_m(z)| \\ &\le \varepsilon + \sup_{|z|\le\varrho} \omega(z)|Rf_m(z)| \\ &\quad+ \sup_{\varrho<|z|<1} \omega(z)|Rf_m(z)| \\ &\le \varepsilon + \varepsilon M_\omega + \sup_{\varrho<|z|<1} \omega(z)\Big|Rf_m\Big(\frac{z}{2|z|}\Big) \\ &\quad+ \int_{1/(2|z|)}^1 R^{(2)}f_m(tz)\frac{dt}{t}\Big| \\ &\le \varepsilon + \varepsilon M_\omega + \varepsilon \sup_{m\ge1} \sup_{|w|=1/2} |Rf_m(w)| \\ &\quad+ 2\varepsilon \int_{1/(2|z|)}^{\varrho/|z|} |R^{(2)}f_m(tz)||z|dt \\ &\quad+ 2 \sup_{\varrho<|z|<1} \omega(z) \int_{\varrho/|z|}^1 |R^{(2)}f_m(tz)||z|dt \\ &\le \varepsilon + \varepsilon M_\nu + \varepsilon \sup_{m\ge1} \sup_{|w|=1/2} |Rf_m(w)| \\ &\quad+ 2\varepsilon\|f_m\|_{\mathcal{Z}_\omega} \int_{1/2}^{\varrho} \frac{dt}{\omega(t)} \\ &\quad+ 2\|f_m\|_{\mathcal{Z}_\omega} \sup_{\varrho<|z|<1} \omega(z) \int_\delta^{|z|} \frac{dt}{\omega(t)} \end{aligned}$$

$$\leq \varepsilon + \varepsilon M_\omega + \varepsilon \sup_{m\geq 1} \sup_{|w|=1/2} |Rf_m(w)|$$
$$+ 2\varepsilon \|f_m\|_{\mathcal{Z}_\omega} \int_{1/2}^{\varrho} \frac{dt}{\omega(t)} + 2\varepsilon \|f_m\|_{\mathcal{Z}_\omega}$$
$$\leq \varepsilon K \quad \text{for all } m \geq N.$$

Then, the boundedness of $W_{\psi,\varphi}$ implies that $\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\omega} \lesssim \|f_m\|_{\mathcal{B}_\omega} \to 0$ as $m \to \infty$. Therefore, $W_{\psi,\varphi}$ is compact by Lemma 3.3.

(3) ⇒ (4): Without loss of generality, we may assume that $\|\psi\|_\infty < 1$.

(i) First we consider the case $\varphi(0) = 0$. Assume the contrary, that $\|\varphi\|_\infty = 1$. Then there is a rotation ζ, $z \mapsto e^{i\alpha}z$, such that $\widetilde{\varphi} := \zeta \circ \varphi$ has a fixed point $z_0 \in \mathbb{B}_n$. We may choose ζ such that $\psi(z_0) \neq 0$. Put

$$\widetilde{\psi} := \frac{\psi}{\psi(z_0)}.$$

Then, for every $m \geq 1$ we obtain that

$$(\widetilde{\theta})^m := \Big(\prod_{k=0}^{m-1} \widetilde{\psi} \circ (\widetilde{\varphi})^k\Big)(\widetilde{\varphi})^m$$

has a fixed point z_0, hence, $\|(\widetilde{\theta})^m\|_\infty \geq 1$. It follows from Lemma 3.2 that the operator $W_{\widetilde{\psi},\widetilde{\varphi}}$, and hence, $W_{\psi,\widetilde{\varphi}}$ is cannot be compact.

Note that $W_{\psi,\widetilde{\varphi}} = W_{\psi,\varphi} \circ C_\zeta$ where the composition operator C_ζ is an isomorphism of $\mathcal{Z}_\omega$. This implies that $W_{\psi,\varphi}$ is not compact. This contradicts the hypothesis.

(ii) We now consider the case $\varphi(0) = a \neq 0$. Let γ be the conformal automorphism of $\mathbb{B}_n$ taking a to 0, and set $\eta = \gamma \circ \varphi$. It follows from Theorem 2.3(iv & iii) that C_γ is a bounded operator on $\mathcal{Z}_\omega$, hence, $W_{\psi,\eta}$ is compact on $\mathcal{Z}_\omega$ because $W_{\psi,\eta} = W_{\psi,\varphi} \circ C_\gamma$. Finally, it follows from the case (i) that $\|\eta\|_\infty < 1$, and hence, $\|\varphi\|_\infty < 1$.

(4) ⇒ (1): Let $\{f_m\}_{m\geq 1}$ be a bounded sequence in $\mathcal{B}_\nu$ converging to 0 uniformly on compact subsets of $\mathbb{B}_n$. By Cauchy integral formula again, it is clear that

$$\sup_{|\varphi_k(z)|\leq \lambda} |R_{\varphi(z)} f_m(\varphi(z))| \to 0,$$
$$\sup_{|\varphi_k(z)|\leq \lambda} |\nabla^{(2)}_{\varphi(z)} f_m(\varphi(z))| \to 0 \quad \text{as } m \to \infty.$$

with $\lambda = \|\varphi\|_\infty < 1$. Then by $\psi \in \mathcal{Z}_\omega$, (3.1) and a standard calculation we have

$$\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\omega}$$
$$\leq |f_m(0)| + \omega(z)|R^{(2)}[\psi(z)]| f_m(\varphi(z))|$$
$$+ \omega(z)|A_{\psi,\varphi}(z)| Rf_m(\varphi(z))|$$
$$+ \omega(z)|B_{\psi,\varphi}(z)| R^{(2)} f_m(\varphi(z))|$$
$$\leq |f_m(0)| + \|\psi\|_{\mathcal{Z}_\omega} \sup_{|\varphi(z)|\leq \|\varphi\|_\infty} |f_m(\varphi(z))|$$
$$+ \sup_{z\in\mathbb{B}_n} \omega(z)|A_{\psi,\varphi}(z)| \sup_{|\varphi(z)|\leq \|\varphi\|_\infty} |Rf_m(\varphi(z))|$$
$$+ \sup_{z\in\mathbb{B}_n} \omega(z)|B_{\psi,\varphi}(z)| \sup_{|\varphi(z)|\leq \|\varphi\|_\infty} |R^{(2)} f_m(\varphi(z))|$$
$$\to 0$$

as $m \to \infty$. By Lemma 3.3, $W_{\psi,\varphi}$ is compact. □

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