Hàm năng lượng tự do của mô hình Ising với từ trường mạnh

TÓM TẮT

Trong bài báo này, chúng tôi trình bày kết quả về miền giải tích của hàm năng lượng tự do cho mô hình Ising chỉ cho phép mỗi spin tương tác với các spin lân cận của nó. Nghiên cứu của chúng tôi dựa trên khai triển cụm, một công cụ mạnh mẽ trong vật lý thống kê, kết hợp những hiểu biết mới từ Fernandez và Procacci về tiêu chí hội tụ của nó.

Từ khóa: *Mô hình Ising, khai triển cụm, hàm năng lượng tự do*

The free energy function of Ising model in a strong magnetic field

ABSTRACT

In this paper, we study the analytic domain of free energy function for Ising model, specifically for cases where only neighboring spins interact. Our study comes from using cluster expansion, a powerful tool in statistical physics, incorporating new insights from Fernandez and Procacci on its convergence criteria.

Keywords: *Ising model, cluster expansion, free energy function.*

1. INTRODUCTION

The Ising model is a mathematical representation of ferromagnetism in statistical mechanics. It consists of discrete variables that represent the magnetic dipole moments of atomic "spins", which can take on one of two states: +1 or -1. These spins are arranged in a graph, typically a crystal lattice, where the local structure repeats periodically in all directions, allowing each spin to interact with its neighbors. One area of interest for mathematicians and physicists is the analytical validity of the free energy function in the Ising model. These issues can be referenced in Friedli and Velenik's book. [1]

The free energy function in mathematics and physics is defined as the logarithm of the partition function. Analyzing the properties of the free energy function is crucial for understanding phase transitions, particularly in the Ising model. This approach is also applicable to more complex mathematical models in statistical mechanics, such as the Potts model, Blume-Capel model, and Curie-Weiss model. To investigate the analytical properties of the free energy function, we typically calculate its analytical domain.

One of the tools used to study the analytical properties of the free energy function is the cluster expansion technique. The cluster expansion represents a power series with respect to auxiliary parameters, often referred to as fugacity. The first application of the cluster expansion was to examine the pressure function at equilibrium as a power series based on the density function derived from the empirical descriptions of gases and liquids. In the theoretical framework of fluids, these parameters correspond to pressures determined respectively from the ideal gas law or the pressure of certain suitable reference components. This problem is discussed in various references. [1,2]

Cluster expansion has extensive applications in various fields, including probability theory and improving the bounds on colored graphs. These applications hinge on representing the partition function specific to each problem, along with the associated reference parameters for their characteristics. Further insights into the applications of cluster expansion can be found in references. [1–4]

A common question that arises in these contexts is when the free energy function behaves as an analytic function in relation to the identified parameters. One method to improve the analytic range of the free energy function in the Ising model involves refining the convergence criteria of the cluster expansion.

Study on the convergence domain of cluster expansion began in the late sixties,[5,6] but it did not receive significant attention until several decades later. The convergence of cluster expansion has been examined using various methods, including: Kirkwood-Salzburg equation,[5] tree-graph boundaries,[7] induction methods,[8] and partition schemes[9] (for a comparison of these methods, see reference[10]). Among these approaches, the partition scheme has yielded the most promising results and has been the focus of several applications and improvements in the paper.[9] This work has been expanded upon in various studies, summarized in reference.[2] Nevertheless, the advancements made by Fernandez-Procacci regarding the convergence criteria remain the most promising for practical applications.

This paper aims to explore the expansion of the analytical domain of the free energy function for the Ising model, building on the advancements made by Fernandez-Procacci in the convergence domain of the cluster expansion. Specifically, we will provide the analytic domain when the external field is not zero (in reference,[1] this model is referred to as the Ising model with a strong field).

2. ISING MODEL IN A STRONG FIELD AND MAIN RESULTS

We consider the set $\Omega = \{-1,1\}^{\mathbb{Z}^d}$. Configurations denote as $\boldsymbol{\sigma} = (\sigma_x)_{x\in\mathbb{Z}^d}$. Let us consider a finite set $\Lambda \subset \mathbb{Z}^d$, configurations $\boldsymbol{\sigma}_\Lambda \in \{-1,1\}^\Lambda := \Omega_\Lambda$ and Hamiltonians with free boundary condition

$$\mathrm{H}^{\varnothing}_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda) := -\beta \sum_{\{i,j\}\in\Lambda} f(i,j)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i, \tag{1}$$

where $\beta \in \mathbb{R}_{\geq 0}$ is the inverse temperature, $h \in \mathbb{R}$ is the external field, and the interaction $f(\cdot,\cdot)$ is defined as

$$f(i,j) = \begin{cases} 1 & \text{if } \|i-j\|_1 = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Let us consider a finite set $\Lambda \subset \mathbb{Z}^d$, configurations $\boldsymbol{\sigma}_\Lambda\boldsymbol{\omega}_{\Lambda^c} \in \Omega$ and Hamiltonians

$$\mathrm{H}^{\boldsymbol{\omega}}_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda\boldsymbol{\omega}_{\Lambda^c}) := \mathrm{H}^{\varnothing}_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda) - \beta \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} f(i,j)\sigma_i\omega_j, \tag{3}$$

where the interaction $f(\cdot,\cdot)$ is defined in (2), a configuration $\boldsymbol{\sigma}_\Lambda\boldsymbol{\omega}_{\Lambda^c} \in \Omega$ includes two part $\boldsymbol{\sigma}_\Lambda \in \{-1,1\}^\Lambda := \Omega_\Lambda$, and $\boldsymbol{\omega}_{\Lambda^c} \in \{-1,1\}^{\Lambda^c}$ which is usually called a boundary of the systems, or configurations are frozen outside of finite set Λ, and the term,

$$\beta \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} f(i,j)\sigma_i\omega_j,$$

refers to *the interaction between the internal and external components of the system*.

The *partition function* with free boundary condition in Λ is

$$Z^{\varnothing}_\Lambda(\beta,h) := \sum_{\boldsymbol{\sigma}_\Lambda\in\Omega_\Lambda} \exp\left(-\mathrm{H}^{\varnothing}_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda)\right), \tag{4}$$

the (finite-volume) free energy function (pressure function) with free boundary condition is

$$P^{\varnothing}_\Lambda(\beta,h) := \frac{1}{|\Lambda|}\log Z^{\varnothing}_\Lambda(\beta,h). \tag{5}$$

And the *partition function* with $\boldsymbol{\omega}-$boundary condition in Λ is

$$Z^{\boldsymbol{\omega}}_\Lambda(\beta,h) := \sum_{\boldsymbol{\sigma}_\Lambda\in\Omega_\Lambda} \exp\left(-\mathrm{H}^{\boldsymbol{\omega}}_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda\boldsymbol{\omega}_{\Lambda^c})\right), \tag{6}$$

the (finite-volume) free energy function with $\boldsymbol{\omega}-$boundary condition in Λ is

$$P^{\boldsymbol{\omega}}_\Lambda(\beta,h) := \frac{1}{|\Lambda|}\log Z^{\boldsymbol{\omega}}_\Lambda(\beta,h). \tag{7}$$

The thermodynamic free energy function $p^{\#}$ is obtained through the thermodynamic limit

$$p^{\#}(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}^d} P^{\#}_\Lambda(\beta,h) \tag{8}$$

e.g. in Fisher sense, where $\# := \varnothing$ or $\boldsymbol{\omega}$.

We utilize the fact that thermodynamic pressure is independent of boundary conditions (for reference, please take a look in Theorem 3.8 [1]) and, for the sake of algebraic convenience, we will focus in this section on "plus" boundary conditions: $\omega_i = 1$ for all $i \notin \Lambda$. The interaction between the inside and outside of the system can be described as the following term.

$$\beta \sum_{\substack{i\in\Lambda,\, j\notin\Lambda \\ \|i-j\|=1}} \sigma_i \,.$$

To get the presentation of partition function, we add and subtract 1 to each term in this Hamiltonian and for each $\boldsymbol{\sigma}_\Lambda \in \Omega_\Lambda$, let us introduce the set

$$\Lambda^-(\boldsymbol{\sigma}_\Lambda) \;=\; \{i \in \Lambda :\; \sigma_i = -1\}\,. \tag{9}$$

We obtain

$$\begin{aligned} \mathrm{H}^+_{\Lambda;\beta,h}(\boldsymbol{\sigma}_\Lambda) = &- \beta\,|\mathcal{E}_\Lambda| - h\,|\Lambda| + 2\beta\,\big|\partial_e\Lambda^-(\boldsymbol{\sigma}_\Lambda)\big| \\ &+ 2h\,\big|\Lambda^-(\boldsymbol{\sigma}_\Lambda)\big| \end{aligned} \tag{10}$$

where

$$\begin{aligned} \partial_e\Lambda^-(\boldsymbol{\sigma}_\Lambda) \;=\; \Big\{&\{i,j\} : i \in \Lambda^-(\boldsymbol{\sigma}_\Lambda), \\ & j \notin \Lambda^-(\boldsymbol{\sigma}_\Lambda),\; \|i-j\|_1 = 1\Big\} \end{aligned}$$

and

$$\mathcal{E}^b_\Lambda = \big\{\{i,j\} \subset \mathbb{Z}^d : \{i,j\}\cap\Lambda \neq \varnothing,\; \|i-j\|_1 = 1\big\}\,.$$

Each $\boldsymbol{\sigma}_\Lambda$ corresponds one to one a term of deviations from the ground state $\Lambda^-(\boldsymbol{\sigma}_\Lambda)$ (the configuration with minimal energy), which is the "all $+1$" configuration. As a consequence, the partition function can be expressed in terms of deviations from the ground state:

$$Z^+_\Lambda(\beta,h) \;=\; \exp(\beta|\mathcal{E}^b_\Lambda| + h|\Lambda|)Z^{\mathrm{LF}}_\Lambda(\beta,h), \tag{11}$$

where *the large field polymers partition function* $Z^{\mathrm{LF}}_\Lambda(\beta,h)$ is given as

$$Z^{\mathrm{LF}}_\Lambda(\beta,h) := \sum_{\Lambda^-\subset\Lambda} \exp\big(-2\beta|\partial_e\Lambda^-| - 2h|\Lambda^-|\big)\,.$$

Before going on with the new expression of partition function $Z^+_\Lambda(\beta,h)$, let us remind the definition of distance,

$$d(i,j) \;:=\; \|i-j\|_1 \;=\; \max_{1\le k\le d} |i_k - j_k|$$

for $i,j \in \Lambda$, and

$$d(S_i,S_j) \;:=\; \inf\{d(\kappa,\ell) : \kappa \in S_i, \ell \in S_j\}$$

for $S_i, S_j \subset \Lambda$. From the definition of the distance, let us declare that two vertices $i,j \in \Lambda^-$ *connected* if and only if $d(i,j) \le 1$, we can decompose Λ^- into maximally connected components (For example, see Figure 1),

$$\Lambda^- = S_1 \cup ... \cup S_n$$

with $d(S_i,S_j) > 1$ for $i \neq j$.

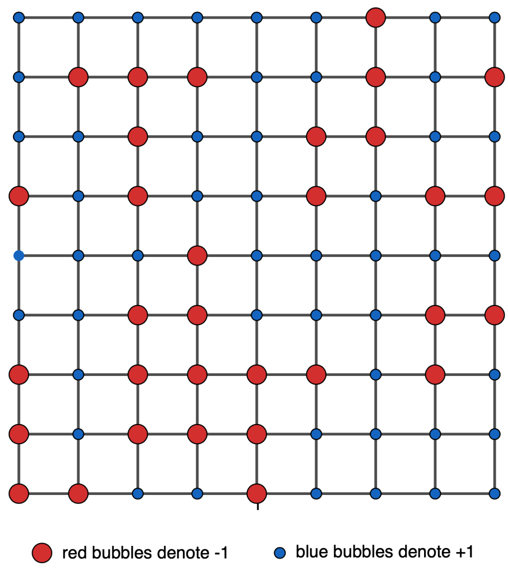

Figure 1. A configuration of the Ising model. Each connected component of the shaded area delimits one of the polymers $S_1,\ldots,S_8$

Before giving a alternative expression of large field polymers partition function, let us introduce the definitions of compatible and incompatible objects as follows:

Definition 2.1. Let us define S, S' to be compatible, and denote $S \sim S'$, if $d(S, S') \geq 2$. Otherwise S, S' are incompatible and we denote $S \nsim S'$.

Denote

$$\zeta(S, S') = \begin{cases} 1 & \text{if} \quad S \sim S' \\ 0 & \text{if} \quad S \nsim S' \end{cases}.$$

Since $|\partial_e \Lambda^-| = \sum_{i=1}^n |\partial_e S_i|$ and $|\Lambda^-| = \sum_{i=1}^n |S_i|$, then the expression of large field polymers partition function can rewrite the following form

$$Z_\Lambda^{\text{LF}}(\beta, h) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(S_1, \ldots, S_n) \in \mathcal{P}_\Lambda^n} \prod_{1 \leq i < j \leq n} \zeta(S_i, S_j) \prod_{i=1}^n w_{\beta,h}(S_i), \tag{12}$$

with

$$\mathcal{P}_\Lambda = \{S \subset \Lambda : S \text{ is non-empty and connected of } \Lambda\}$$

and

$$w_{\beta,h}(S_i) = \exp(-2\beta |\partial_e S_i| - 2h|S_i|). \tag{13}$$

Theorem 2.1. *The pressure with* $+1$*-boundary condition in* Λ *can be expressed as the following form:*

$$P_\Lambda^+(\beta, h) = \beta \frac{|\mathcal{E}_\Lambda|}{|\Lambda|} + h + \frac{1}{|\Lambda|} \log Z_\Lambda^{\text{LF}}(\beta, h), \tag{14}$$

where

$$\log Z_\Lambda^{\text{LF}}(\beta, h) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(S_1 \ldots S_n) \in \mathcal{P}_\Lambda^n} a_n^T(S_1, \ldots, S_n) \prod_{i=1}^n w_{\beta,h}(S_i), \tag{15}$$

with Ursell function $\omega_n^T(\cdot)$ *defined as*

$$a_n^T(S_1, \ldots, S_n) := \sum_{G \in \mathcal{C}[n]} \prod_{\{i,j\} \in E(G)} [\zeta(S_i, S_j) - 1]. \tag{16}$$

Expression (15) *is well-known as cluster expansion.*

Proof. Expression (12) serves as the partition function for a gas of polymers, which is comprised of subsets of Λ, as discussed in Fernandez and al. paper.[10] Consequently, we can apply the theory developed for these systems to prove Expression (14). More detail, see references.[10,11] □

The next theorem establishes a sufficient condition for the existence of the pressure function as $\Lambda \to \mathbb{Z}^d$ in the thermodynamic limit and allows us to verify the analyticity domain of the pressure function.

Theorem 2.2. *If there exists* $a > 0$ *such that*

$$\sum_{k=1}^{\infty} |\mathcal{A}_k| \, \mathrm{e}^{-2d\beta\, k^{(d-1)/d} V(1)^{1/d} - 2h\, k + (2d+1)ak} \leq \mathrm{e}^a - 1 \tag{17}$$

with

$$\mathcal{A}_k := \{S \in \mathcal{P} : 0 \in S, \ |S| = k\}, \tag{18}$$

then the following holds:

(i.) $|\Gamma|_S(\boldsymbol{w}_{\beta,h})$ *converges. Furthermore, for* $S \in \mathcal{P}$,

$$|\Gamma|_S(\boldsymbol{w}_{\beta,h}) \leq \mathrm{e}^{a|S|}.$$

(ii.) The free energy function (14) *converges absolutely and uniformly in* Λ, *and*

$$p^+(\beta, h) = \beta d + h + \sum_{X \subset \mathbb{Z}^d : X \ni 0} \frac{1}{|X|} \Psi(X), \tag{19}$$

where, for each $X \subset \mathbb{Z}^d$, $\Psi(\cdot)$ *is defined as follow:*

$$\Psi(X) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(S_1 \ldots S_n) \in \mathcal{P}^n \\ S_1 \cup \cdots \cup S_n = X}} a_n^T(S_1, \ldots, S_n) \prod_{i=1}^n w_{\beta,h}(S_i) \tag{20}$$

with

$$\mathcal{P} = \left\{ S \subset \mathbb{Z}^d : \ S \text{ is non-empty and connected of } \mathbb{Z}^d \right\}.$$

Proof. Part (i) of Theorem 2.2 follows from Lemma 4.2, which is presented in Subsection 4.2.

Part (ii) of Theorem 2.2 is derived from the Fernandez-Procacci Theorem.[9] We will omit the detailed proof and refer readers to[1,11] for further information. □

The main purpose of this paper is to identify the domain of inverse temperature β and the external magnetic field h for which the pressure function $p(\beta, h)$ is analytic, as stated in the following theorem.

Theorem 2.3. *The pressure function $p(\beta, h)$ is analytic in the domain $\mathcal{D}$ with*

$$\mathcal{D} = \{(\beta, h) \in \mathbb{R} \times \mathbb{R} : \beta \geq 0;\ 2h \geq \phi_1(\bar{a})\}$$

where ϕ_1 is defined in (45) *and $\bar{a}$ in* (48).

3. CLUSTER EXPANSION FOR SUBSET GASES

Subset gases are specific types of polymer gases that are frequently utilized in cluster expansion within statistical mechanics. Their definition requires a countable subset, denoted as, $\mathbb{V}$ (e.g. $\mathbb{Z}^d$) which acts as an underlying "space." Polymers are defined as finite, non-empty subsets of $\mathbb{V}$, represented mathematically as:

$$\mathcal{P}_{\mathbb{V}} = \{S \subset \mathbb{V} : 0 < |S| < \infty\}.$$

with compatibility relation, denoted as $S \sim S'$ depended on the models we are working with. For instance, in Section 2, we stated that $S \sim S'$ if and only if $d(S, S') \geq 2$. In the work of Bissacot, Procacci and Fernandez,[10] it mentioned that $S \sim S'$ if and only if $S \cap S' = \varnothing$. Polymers can now be measure through its cardinality, so it makes sense to talk about large and small polymers. The definition of the gas is completed by a family of activities $\boldsymbol{z} = \{z_S \in \mathbb{C}\}_{S \in \mathcal{P}_{\mathbb{V}}}$. Let us define the partition function for gas polymers as follows:

$$Z(\boldsymbol{z}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(S_1, \ldots, S_n) \in \mathcal{P}_{\mathbb{V}}^n} \prod_{1 \leq i < j \leq n} \zeta(S_i, S_j) \prod_{i=1}^{n} z_{S_i} \tag{21}$$

Using Mayer's trick (which can be found in[11]), we can derive $\log Z(\boldsymbol{z})$ as following form:

$$\log Z(\boldsymbol{z}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{(S_1, \ldots, S_n) \in \mathcal{P}_{\mathbb{V}}^n} a_n^T(S_1, \ldots, S_n) \prod_{i=1}^{n} z_{S_i}, \tag{22}$$

where $a_n^T(\cdot)$ is defined as (16).

To study the convergence of cluster expansion, we typically examine it through the convergence conditions of the formal power series in infinite volume as below (see[9] for a full explanation): For each $S \in \mathcal{P}_{\mathbb{V}}$,

$$|\Gamma|_S(\boldsymbol{\rho}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(S_1, \ldots, S_n) \in \mathcal{P}_{\mathbb{V}}^n} |a_{n+1}^T(S, \ldots, S_n)| \prod_{i=1}^{n} \rho_{S_i} \tag{23}$$

with $\boldsymbol{\rho} \in [0, \infty)^{\mathcal{P}_{\mathbb{V}}}$.

We will now examine some convergent conditions of cluster expansion as given in equation (23). As a consequence of Theorem 2.3.2,[11] the cluster expansion is convergent under Fernandez-Procacci criterion known as the best convergent condition, which is given in the following theorem.

Theorem 3.1 (Fernández-Procacci criterion)**.** *Suppose that for some $\boldsymbol{\xi} \in [0, \infty)^{\mathcal{P}_{\mathbb{V}}}$ there exists $\boldsymbol{\mu} \in [0, \infty)^{\mathcal{P}_{\mathbb{V}}}$ such that*

$$\xi_{S_0} \psi_{S_0}^{\mathrm{FP}}(\boldsymbol{\mu}) \leq \mu_{S_0}, \quad \textit{for each } S_0 \in \mathcal{P}_{\mathbb{V}} \tag{24}$$

with

$$\psi_{S_0}^{\mathrm{FP}}(\boldsymbol{\mu}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{(S_1, \ldots, S_n) \in \mathcal{P}_{\mathbb{V}}^n \\ S_0 \nsim S_i,\ S_i \sim S_j,\ 1 \leq i,j \leq n}} \prod_{i=1}^{n} \mu_{S_i} \tag{25}$$

Then the series $|\Gamma|_S(\boldsymbol{\xi})$ defined in (23) *is convergent. Furthermore, for each $S \in \mathcal{P}_{\mathbb{V}}$,*

$$\xi_S |\Gamma|_S(\boldsymbol{\xi}) \leq \mu_S.$$

To apply the Fernández-Procacci criterion, most of the models, we substitute $\mu_\gamma = \xi_\gamma e^{a|\gamma|}$ to obtain

$$1+\sum_{n\geq 1}\sum_{\substack{\{S_1,\ldots,S_n\}\subset\mathcal{P}_{\mathbb{V}}\\ S_0\cap S_i\neq\varnothing,\ S_i\cap S_j=\varnothing,\ 1\leq i<j\leq n}}\prod_{i=1}^{n}\xi_i e^{a|S_i|} \ \leq\ e^{a|S_0|} \tag{26}$$

From the constraint in the sum, $S_0\cap S_i\neq\varnothing,\ S_i\cap S_j=\varnothing,\ 1\leq i<j\leq n$, this means that each of the polymers $S_1,\ldots,S_n$ must intersect different points in S_0 to avoiding overlapping. Consequently, we can conclude that: (i) $n\leq|S_0|$, and (ii) there are n different points in S_0 touched by $S_1\cup\ldots\cup S_n$. The selection of these points can be done in $\binom{|S_0|}{n}$ ways. Hence the left-hand side of (26) is less than or equal to

$$\begin{aligned}1+\sum_{n=1}^{|S_0|}\binom{|S_0|}{n}\left[\sup_{x\in S_0}\sum_{\substack{S\in\mathcal{P}_{\mathbb{V}}\\ S\ni x}}\xi_S e^{a(S)}\right]^n\\ =\left[1+\sup_{x\in S_0}\sum_{\substack{S\in\mathcal{P}_{\mathbb{V}}\\ S\ni x}}\xi_S e^{a(S)}\right]^{|S_0|}\end{aligned} \tag{27}$$

This leads us to the following sufficient condition for (26)

$$\sup_{x\in S_0}\sum_{\substack{S\in\mathcal{P}_{\mathbb{V}}\\ S\ni x}}\xi_S e^{a|S|}\ \leq\ e^{a}-1. \tag{28}$$

This condition, in fact, coincides with the known (but forgotten) Gruber-Kunz condition[5] except that the later involves a sharp inequality sign. The condition is useful for numerous applications including contour ensembles of low-temperature phases, geometrical objects of high-temperature expansions, random sets of the Fortuin-Kasteleyn representation of the Potts model, ….

4. PROOFS

4.1. Alternative Gruber-Kunz condition

In this section, we will utilize the Fernandez-Procacchi criterion given in Section 3 along with a new compatible relation presented in Section 2 to derive the improved of Gruber-Kunz condition which is presented as following proposition.

Proposition 4.1. *If there exists $a>0$ such that*

$$\sup_{x\in\mathbb{Z}^d}\sum_{\substack{x\in S\\ S\in\mathcal{P}}}w_{\beta,h}(S)e^{a|[S]_1|}\ \leq\ e^{a}-1. \tag{29}$$

then $|\Gamma|_S(\boldsymbol{w}_{\beta,h})$ converges. Furthermore, for every $S\in\mathcal{P}$,

$$|\Gamma|_S(\boldsymbol{w}_{\beta,h})\leq e^{a|S|}.$$

Proof. Let us start with following readily from Theorem 3.1 that $|\Gamma|_S(\boldsymbol{\rho})$ converges if for each $S\in\mathcal{P}$,

$$1+\sum_{n\geq 1}\sum_{\substack{(S_1,\ldots,S_n)\in\mathcal{P}^n\\ S\nsim S_i,\ S_i\sim S_j,\ 1\leq i,j\leq n}}\prod_{i=1}^{n}w_{\beta,h}(S_i)e^{a(S_i)}\ \leq\ e^{a(S)} \tag{30}$$

where we take $\mu_S=w_{\beta,h}(S)e^{a(S)}$. It is easy to see that:

$$S\nsim S'\iff d(S,S')\leq 1\iff S\cap[S']_1=\varnothing,$$

with

$$[S]_1\ :=\ \{j\in\mathbb{Z}^d: d(j,S)\leq 1\},$$

and

$$S\sim S'\Rightarrow d(S,S')>1\Rightarrow S\cap S'=\varnothing.$$

It implies that the left-hand side of convergent condition 30 can be bounded as follows

$$1+\sum_{n\geq 1}\sum_{\substack{\{S_1,\ldots,S_n\}\subset\mathcal{P}\\ [S_0]_1\cap S_i\neq\varnothing,\ S_i\cap S_j=\varnothing,\ 1\leq i,j\leq n}}\prod_{i=1}^{n}w_{\beta,h}(S_i)e^{a(S_i)}. \tag{31}$$

Then we can replace the convergent condition 30 by

$$1+\sum_{n\geq 1}\sum_{\substack{\{S_1,\ldots,S_n\}\subset\mathcal{P}\\ [S]_1\cap S_i\neq\varnothing,\ S_i\cap S_j=\varnothing,\ 1\leq i,j\leq n}}\prod_{i=1}^{n}w_{\beta,h}(S_i)e^{a|[S_i]_1|}\leq e^{a|[S]_1|} \tag{32}$$

By an argument analogous to that used to derive the Gruber-Kunz condition (28), starting with the constraint in the sum, $[S]_1\cap S_i\neq\varnothing,\ S_i\cap S_j=\varnothing,\ 1\leq i<j\leq n$, this means that each of the

polymers $S_1,\ldots,S_n$ must intersect different points in $[S]_1$ to avoiding overlapping. Consequently, we can conclude that: (i) $n \leq |[S]_1|$, and (ii) there are n different points in $[S]_1$ touched by $S_1 \cup \ldots \cup S_n$. The selection of these points can be done in $\binom{|[S]_1|}{n}$ ways. Hence the left-hand side of (26) is less than or equal to

$$1+\sup_{x\in\mathbb{Z}^d}\sum_{\substack{x\in S\\ S\in\mathcal{P}}} w_{\beta,h}(S)\mathrm{e}^{a|[S]_1|} \;\leq\; \mathrm{e}^{a}. \tag{33}$$

The proof has been completed. □

4.2. Proof of Theorem 2.3

Before proceeding with further calculations, let us establish a weaker condition for the convergence of the power series $|\Gamma|_S(\boldsymbol{w}(\beta,h))$ in the following lemma. This condition arises from a bound on the weight $w_{\beta,h}(\cdot)\mathrm{e}^{a|[S]_1|}$, as outlined in this lemma, along with the Gruber-Kunz condition. This bound is particularly useful for estimating the parameters β and h.

Lemma 4.2. *For each $S\in\mathcal{P}$,*

$$w_{\beta,h}(S) \leq \mathrm{e}^{-\beta d V_d(1)^{1/d}[|S|]^{(d-1)/d}-h|S|}, \tag{34}$$

and

$$\mathrm{e}^{a|[S]_1|} \leq \mathrm{e}^{a(2d+1)|S|}. \tag{35}$$

Furthermore, if there exists $a>0$ such that

$$\sum_{k=1}^{\infty} |\mathcal{A}_k|\,\mathrm{e}^{-2d\beta\, k^{(d-1)/d}V(1)^{1/d}-2h\,k+(2d+1)ak} \leq \mathrm{e}^{a}-1 \tag{36}$$

with

$$\mathcal{A}_k \;:=\; \{S\in\mathcal{P}:\; 0\in S,\; |S| \;=\; k\}, \tag{37}$$

then $|\Gamma|_S(\boldsymbol{w}_{\beta,h})$ converges and

$$|\Gamma|_S(\boldsymbol{w}_{\beta,h}) \leq \mathrm{e}^{a|S|},$$

for each $S\in\mathcal{P}$.

Proof. We observe that

$$|S| \;\leq\; |[S]_1| \;\leq\; (2d+1)\,|S|\,, \tag{38}$$

Then inequality (35) holds.

To prove the inequality (34), we begin with the fact that the smallest ratio of area to volume is achieved by a d-dimensional sphere. Denoting the volume and surface area of a sphere of radius R, respectively,

$$V_d(R) = V_d(1)\,R^d$$
$$S_d(R) = dV_d(1)\,R^{d-1}$$

we obtain

$$\begin{aligned}|\partial_e S| \geq S_d(R) &= dV_d(1)^{1/d}[V_d(R)]^{(d-1)/d}\\ &= dV_d(1)^{1/d}|S|^{(d-1)/d}.\end{aligned} \tag{39}$$

As a consequence of inequalities (34), (34), and the condition (36), we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{Z}^d}\sum_{\substack{x\in S\\ S\in\mathcal{P}}} \mathrm{e}^{-2\beta|\partial_e S|-2h|S|+a|[S]_1|}\\ &\leq \sum_{k=1}^{\infty}|\mathcal{A}_k|\,\mathrm{e}^{-2d\beta\, k^{(d-1)/d}V(1)^{1/d}-2h\,k+(2d+1)ak}\\ &\leq \mathrm{e}^{a}-1\end{aligned} \tag{40}$$

with $\mathcal{A}_k$ defined as in (37). Thus, the alternative Gruber-Kunz condition is satisfied, completing the proof as a consequence of Proposition 4.1. □

The next two lemmas are crucial for determining the number of elements in $\mathcal{A}_k$.

Lemma 4.3 (Lemma 3.60[1])**.** *Let G be connected graph with n edges. Starting from an arbitrary vertex of G, there is a path in G crossing each edge of G exactly twice and ending at the starting vertex.*

Proof. The proof uses a induction argument. For $n=1$ the result is trivial. Assume that the claim holds for $k=n-1$ and consider a connected graph G with n-edges. Let i_0 be the starting vertex in G and consider a vertex $j_0\in V(G)$ such that $\{i_0,j_0\}\in E(G)$. There are two possibilities for $E(G)\setminus\{i_0,j_0\}$:

• It equals $E(G_1)$ where G_1 is a connected graph. In this case, the desired path is obtained by concatenating $\{i_0, j_0\}$ with the path in G_1 starting from j_0 and satisfying the inductive hypothesis, followed by a final step $\{i_0, j_0\}$.

• It equals $E(G_1) \cup E(G_2)$ where G_1 and G_2 are each connected but they are mutually disconnected. The initial site i_0 is a vertex of G_1 and j_0 a vertex of G_2. In this case the desired path is obtained by the following concatenation: First $\{i_0, j_0\}$, second the inductive path in G_2 starting and ending at j_0, third $\{i_0, j_0\}$ again, and fourth the inductive path in G_1 starting and ending at i_0.

In both cases, we obtain a path that starts and ends at i_0, with exactly two visits to each edge of G. □

Lemma 4.4. *For $k \geq 1$,*

$$|\mathcal{A}_k| \;\leq\; (2d)^{2k-2}. \tag{41}$$

Proof. For each set $S \in \mathcal{P}$ such that $0 \in S$ and $|S| = k$, there exists a spanning tree that contains $k-1$ edges. Consequently, by applying Lemma (4.3), the number of connected sets with k elements that include 0 is bounded above by the number of paths of length $2k-2$ starting from 0. This latter quantity is certainly less than $(2d)^{2k-2}$. □

We now start with the proof of our main result in this paper. The infinite volume free energy $p(\beta, h)$ inherits the analyticity of the positive series $|\Gamma|_S(\boldsymbol{w}(\beta, h))$ (for more details, see references[1,11]). Thus we can confirm that all values of the external field h that meet the condition (36) are sufficient for the analyticity of $p(\beta, h)$.

Proof of Theorem 2.3. We start with the condition (36), and apply Lemma 4.4 to obtain

$$\sum_{k=1}^{\infty} |\mathcal{A}_k| \, \mathrm{e}^{-2d\beta V(1)^{1/d} k^{(d-1)/d} - 2hk + (2d+1)ak}$$
$$\leq \sum_{k=1}^{\infty} (2d)^{2k-2} \mathrm{e}^{[-2h+(2d+1)a]k}$$
$$= \frac{\mathrm{e}^{(2d+1)a-2h}}{1-(2d)^2\mathrm{e}^{(2d+1)a-2h}}, \tag{42}$$

for $h > 0$ satisfying

$$4d^2\mathrm{e}^{(2d+1)a-2h} \;<\; 1. \tag{43}$$

Inequality (42) implies that (36) is valid for h whenever there exists a constant $a > 0$ such that

$$\frac{\mathrm{e}^{(2d+1)a-2h}}{1-(2d)^2\mathrm{e}^{(2d+1)a-2h}} \leq \mathrm{e}^a - 1 \tag{44}$$

Or, it is equivalent with

$$2h \geq \ln\left[1 + 4d^2(\mathrm{e}^a - 1)\right] - \ln(\mathrm{e}^a - 1) + 2da + a$$
$$:= \phi_1(a) \tag{45}$$

and from the condition (43), we have

$$2h > 2\ln(2d) + 2da + a := \phi_2(a).$$

Then, for each $a > 0$,

$$2h \geq \max[\phi_1(a), \phi_2(a)] = \phi_1(a). \tag{46}$$

Since, for $a > 0$, we have

$$\ln\left[1 + 4d^2(\mathrm{e}^a - 1)\right] - \ln(\mathrm{e}^a - 1) \geq \ln(4d^2).$$

As a consequence of inequality (46), to optimize the domain for the field h, we can take,

$$2h \geq \min_{a>0} \phi_1(a). \tag{47}$$

Elementary analysis shows that $\phi_1(a)$ has a unique minimum at

$$\bar{a} \;=\; \ln\left[\frac{(2d+1)(8d^2-1)+1+\sqrt{\Delta}}{8d^2(2d+1)}\right], \tag{48}$$

with

$$\sqrt{\Delta} = [(2d+1)(8d^2-1)+1]^2 - 16(2d+1)^2(4d^4-d^2).$$

Hence,

$$2h \geq \phi_1(\bar{a}). \tag{49}$$

The proof of Theorem 2.3 is completed. □

4.3. Comparison with Friedli and Velenik result.

In last part, we compare our estimations with the results provided in Friedli and Velenik's book[1] which is one of newest result on this topic. To describe the latter, let us denote

$$\begin{aligned} \eta(h,d) &= \sum_{k=1}^{\infty}(2d)^{2k}\mathrm{e}^{(2d+1-2h)k} \\ H^{+} &= \{h : \mathrm{Re}\, h \geq \bar{h}\} \end{aligned} \quad (50)$$

with

$$\bar{h} := \inf\{h : \eta(h,d) < 1, h > 0\}.$$

Friedli and Velenik state that if $h \in H^{+}$, then the cluster expansion in equation (15) converges absolutely. By performing a simple computation, we can estimate that

$$2\bar{h} = \ln(8d^2) + 2d + 1. \quad (51)$$

As a consequence of inequality (47), then $a = \ln 2$ into the function $\phi_1(a)$, we have

$$\begin{aligned} 2\bar{h} &= \ln(8d^2) + 2d + 1 \\ &\geq \ln(4d^2+1) + d2\ln 2 + \ln 2 \geq \min_{a>0} \phi_1(a). \end{aligned} \quad (52)$$

Inequality (52) show that our estimate is wider than Friedli and Velenik's bound[1] (more detail, see Figure 2). Let us examine the ratio between our bound and Friedli and Velenik's bound[1], as defined below:

$$r(d) := \frac{\phi_1(\bar{a})}{\log(8d^2) + 2d + 1},$$

where ϕ_1 is defined in (45) and $\bar{a}$ in (48). In Figure 3, we observe that the rate R decays exponentially, approaching zero as d tends to infinity.

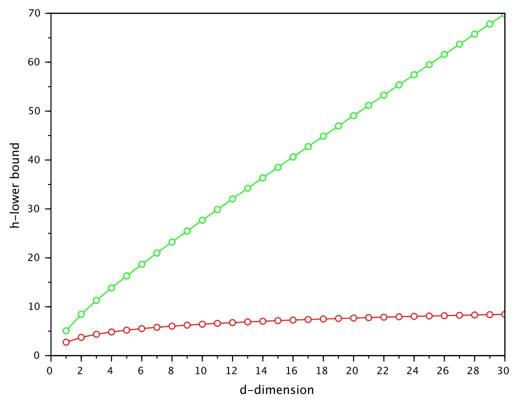

Figure 2. A comparison with Friedli and Velenik result (green line presented for Friedli and Velenik result and red line presented for our result)

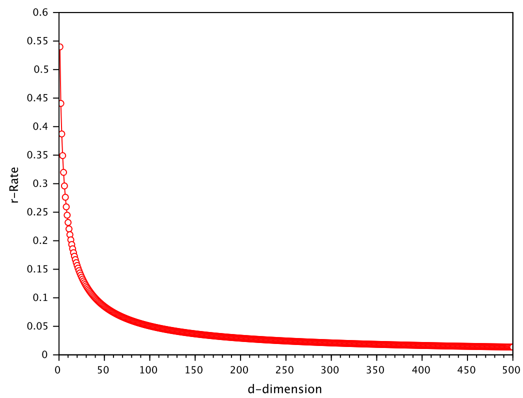

Figure 3. The rate between our bound and Friedli and Velenik bound

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