Một đặc trưng của không gian kiểu Zygmund và áp dụng

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu một số điều kiện để không gian kiểu Zygmund $\mathcal{Z}_\omega$, với ω là một trọng chuẩn tắc trên hình cầu đơn vị $\mathbb{B}$ trong $\mathbb{C}^n$, trở thành một không gian nhỏ, ổn định biên và bất biến dưới các tự đẳng cấu. Chúng tôi áp dụng kết quả này để phân tích mối quan hệ giữa tính bị chặn và tính compact của các toán tử hợp liên tục $W_{\psi,\varphi}$, từ $\mathcal{B}_\omega$ vào $\mathcal{Z}_\omega$ và trên $\mathcal{Z}_\omega$.

Từ khóa: *Không gian Bloch, không gian Zygmund, toán tử hợp có trọng, tính bị chặn, tính compact.*

A characterization of Zygmund-type spaces and its application

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ABSTRACT

In this paper, we examine the conditions under which a Zygmund-type space $\mathcal{Z}_\omega$, where ω is a normal weight on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$, becomes a small space that is boundary regular and invariant under automorphisms. These results are then applied to analyze the relationship between the boundedness and compactness of weighted composition operators $W_{\psi,\varphi}$, defined by $f \mapsto \psi \cdot (f \circ \varphi)$, acting from the Bloch-type space $\mathcal{B}_\omega$ to the Zygmund-type space $\mathcal{Z}_\omega$, as well as from $\mathcal{Z}_\omega$ into itself.

Keywords: *Bloch spaces, Zygmund spaces, weighted composition operators, boundedness, compactness*

1. INTRODUCTION

Given a natural number n, let us consider the open unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and $H(\mathbb{B})$ the space of all holomorphic functions in $\mathbb{B}$. The vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_n = (0, \dots, 0, 1)$ form the standard basis for $\mathbb{C}^n$.

Throughout this paper, for any $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we use $\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$, to denote their standard inner product, and write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the corresponding Euclidean norm.

For $f \in H(\mathbb{B})$, let

$$\nabla_z f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\Big),$$
$$Rf(z) = \langle \nabla f(z), \overline{z}\rangle, \quad z \in \mathbb{B}.$$

Let $\mathbb{D}$ denote the unit disk of $\mathbb{C}$. If $f \in H(\mathbb{D})$ satisfies $\sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty$ then f is said to belong to the Zygmund space. In this definition, $1 - |z|^2$ acts as a weight function, which was later generalized to $(1 - |z|^2)^\alpha$ for all $\alpha > 0$.

A positive continuous function ω defined on the interval $[0, 1)$ is said to be normal if there exist constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that

$$\frac{\omega(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \qquad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^a} = 0, \tag{W_1}$$

$$\frac{\omega(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \qquad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^b} = \infty. \tag{W_2}$$

If we say that a function $\omega : \mathbb{B} \to [0, \infty)$ is normal, we also assume that it is radial, that is, $\omega(z)=\omega(|z|)$ for every $z\in\mathbb{B}$. Strictly positive continuous functions on $\mathbb{B}$ are referred to as weights.

We define Bloch-type space $\mathcal{B}_\omega$, Zygmund-type space $\mathcal{Z}_\omega$, respectively, as follows:

$$\mathcal{B}_\omega=\Big\{f\in H(\mathbb{B}):\ \|f\|_{s\mathcal{B}_\omega}<\infty\Big\},$$
$$\mathcal{Z}_\omega=\Big\{f\in H(\mathbb{B}):\ \|f\|_{s\mathcal{Z}_\omega}<\infty\Big\},$$

where

$$\|f\|_{s\mathcal{B}_\omega}=\sup_{z\in\mathbb{B}_n}\omega(z)|(Rf)(z)|,$$
$$\|f\|_{s\mathcal{Z}_\omega}=\sup_{z\in\mathbb{B}_n}\omega(z)|\nabla(Rf)(z)|$$

are seminorms on $\mathcal{B}_\omega$ and $\mathcal{Z}_\omega$, respectively. The spaces $\mathcal{B}_\omega$, $\mathcal{Z}_\omega$ are endowed with Banach space structures via the norm

$$\|f\|_{\mathcal{B}_\omega}=|f(0)|+\|f\|_{s\mathcal{B}_\omega},$$
$$\|f\|_{\mathcal{Z}_\omega}=|f(0)|+\|f\|_{s\mathcal{Z}_\omega}.$$

The space $\mathcal{Z}_\omega$ generalizes the classical Zygmund space, which was introduced in [1].

Define $S(\mathbb{B})$ as the set of holomorphic self-maps of $\mathbb{B}$. Given $\psi\in H(\mathbb{B})$ and $\varphi\in S(\mathbb{B})$, the weighted composition operator $W_{\psi,\varphi}:E\to F$ is defined by

$$W_{\psi,\varphi}(f):=\psi\cdot(f\circ\varphi),\quad \text{for } f\in E,$$

where E and F are Banach spaces consisting of holomorphic functions on $\mathbb{B}$. It may be regarded as a generalization of multiplication and composition operators.

The theory of composition operators, both weighted and unweighted, has its origins in the previous century. The boundedness, compactness, essential norm, and spectral properties are always the highlights of research of composition operators. Book [2] is a good reference for studying the composition operators on classical spaces of analytic functions. Furthermore, the theory relies on the theory of analytic functions on the unit disk, which provides a convenient foundation.

Composition operators mapping into the classical Zygmund were studied in [3–9]. Many scholars have discussed similar problems (see [10–17], etc.)

However, for abstract normal weight especially in high dimensions, when investigating and using the properties (for example, discussing weighted/unweighted composition operator of the Zygmund type space, we often encounter some obstacles. This partly explains why the boundedness and compactness criteria for $W_{\psi,\varphi}$ between Zygmund-type spaces (normal weight cases) have not been extensively investigated to date. In order to overcome these obstacles, we need a variety of means or techniques.

Motivated by the above-mentioned discussions and the previous investigations, the purpose of this paper is to uncover additional characteristics of Zygmund-type spaces and serve them as technical tools to solve the problem of the relationship between the boundedness, compactness of weighted composition operators from a Bloch-type space $\mathcal{B}_\omega$ into the Zygmund-type space $\mathcal{Z}_\omega$ and from $\mathcal{Z}_\omega$ into itself.

Section 2 provides a sufficient condition on the normal weight ω ensuring that $\mathcal{Z}_\omega$ is an automorphism invariant boundary regular small space. A key motivation for this section comes from a result of Shapiro [18] (and Theorem 4.5 in [2]), which asserts that $\|\varphi\|_\infty<1$ is a necessary condition for C_φ to be compact on any “suitably small” Banach space. According to [18], four axioms must be satisfied for a space to qualify as appropriately small. Among these, two axioms are fundamental, concerning norm naturality and space nontriviality, while the other two regulate the size of the spaces. Specifically, the boundary regularity axiom makes the spaces small by ensuring continuous boundary extension, and the automorphism-invariance axiom prevents them from being excessively small. For further details, refer to [18] or [2].

Building on the results obtained in the previous section, Section 3 establishes the connections between the boundedness and compactness of weighted composition operators acting from $\mathcal{B}_\omega$ to $\mathcal{Z}_\omega$, and those acting on $\mathcal{Z}_\omega$ itself.

In this paper, we use the notation $a \lesssim b$ to denote that $a \leq Cb$, and $a \asymp b$ to indicate that $C^{-1}b \leq a \leq Cb$, where $C > 0$ is an inessential constant, with all quantities a and b assumed to be non-negative.

2. A CHARACTERIZATION OF ZYGMUND-TYPE SPACES

This section is devoted to the study of the properties "*small*" and "*automorphism invariant boundary regular*" of the Zygmund-type spaces which will be necessary in establishing one of our main result.

For a normal weight ω on $\mathbb{B}$ we use there certain quantities, which will be used in this work:

$$I^1_\omega(z) := \int_0^{|z|} \frac{dt}{\omega(t)},$$
$$I^2_\omega(z) := \int_0^{|z|} \Big(\int_0^t \frac{ds}{\omega(s)}\Big) dt, \quad z \in \overline{\mathbb{B}}.$$

Remark 2.1. Since ω is positive and continuous, it follows that $m_{\omega,\delta} := \min_{t\in[0,\delta]} \omega(t) > 0$. In addition, by (W_1), ω is strictly decreasing on $[\delta, 1)$, so $\max_{t\in[0,1)} \omega(t) =: M_\omega < \infty$. Consequently, one can easily verify that

$$\omega(z) I^1_\omega(z) < R_\omega := \delta \frac{M_\omega}{m_{\omega,\delta}} + 1 - \delta < \infty \quad (2.1)$$

and, hence,

$$\omega(z) I^2_\omega(z) < |z| R_\omega < R_\omega < \infty \quad (2.2)$$

for every $z \in \mathbb{B} \setminus \{0\}$.

Proposition 2.1 ([17])**.** For every normal weight ω on $\mathbb{B}$ we have

$$\mathcal{Z}_\omega = \mathcal{Z}^R_\omega := \Big\{ f \in H(\mathbb{B}) : \ \|f\|_{\mathcal{Z}^R_\omega} < \infty \Big\}$$
$$= \mathcal{Z}^\nabla_\omega := \Big\{ f \in H(\mathbb{B}) : \ \|f\|_{\mathcal{Z}^\nabla_\omega} < \infty \Big\}$$

and $\|\cdot\|_{\mathcal{Z}_\omega} \cong \|\cdot\|_{\mathcal{Z}^R_\omega} \cong \|\cdot\|_{\mathcal{Z}^\nabla_\omega}$, where

$$R^{(2)} f = R(Rf),$$
$$|\nabla^{(2)} f(z)| = \Bigg(\sum_{i,j=1}^n \Big| \frac{\partial^2 f}{\partial z_i \partial z_j}(z) \Big|^2 \Bigg)^{\frac{1}{2}},$$
$$\|f\|_{\mathcal{Z}^R_\omega} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z) |R^{(2)} f(z)|,$$
$$\|f\|_{\mathcal{Z}^\nabla_\omega} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z) |\nabla^{(2)} f(z)|,$$

for every $f \in \mathcal{Z}_\omega$.

In this paper, let us write simply we denote $\mathcal{Z}_\omega$ for the complex $(\mathcal{Z}_\omega, \|\cdot\|_{\mathcal{Z}^R_\omega})$.

Lemma 2.2. Let ω be a normal weight on $\mathbb{B}$. Then there exists $C > 0$ such that for every $f \in \mathcal{Z}_\omega$ and every $z \in \mathbb{B}$,

$$\begin{aligned} |Rf(z)| &\leq C I^1_\omega(z) \|f\|_{\mathcal{Z}_\omega}, \\ |\nabla f(z)| &\leq C(1 + I^1_\omega(z)) \|f\|_{\mathcal{Z}_\omega}; \end{aligned} \quad (2.3)$$

and

$$|f(z)| \leq |f(0)| + C I^2_\omega(z) \|f\|_{\mathcal{Z}_\omega}. \quad (2.4)$$

Proof. The estimate (2.3) follows from [19] which says there is $C > 0$ such that for every $f \in \mathcal{B}_\omega$ and for all $z \in \mathbb{B}$,

$$|f(z)| \leq C\big(1 + I^1_\omega(z)\big) \|f\|_{\mathcal{B}_\omega}. \quad (2.5)$$

Then by (2.3) and (2.5) again we obtain (2.4). □

Note that, in fact, by using (2.5) the estimate for $|\nabla f(z)|$ in (2.3) can be replced by

$$|\nabla f(z)| \lesssim \big(1 + I^1_\omega(z)\big) |\nabla f(0)| + I^1_\omega(z) \|f\|_{\mathcal{Z}_\mu}. \quad (2.6)$$

Now, by $Aut(\mathbb{B})$, we denote the automorphism group of $\mathbb{B}$ that consists of all biholomorphic mappings of $\mathbb{B}$. It is well known that a mapping $\varphi \in Aut(\mathbb{B})$ is a unitary transformation of $\mathbb{C}^n$ if and only if $\varphi(0) = 0$ (see [20]). For any $\alpha \in \mathbb{B} \setminus \{0\}$, we define

$$\varphi_\alpha(z) = \frac{\alpha - P_\alpha(z) - s_\alpha Q_\alpha(z)}{1 - \langle z, \alpha \rangle}, \quad z \in \mathbb{B}, \quad (2.7)$$

where $s_\alpha = \sqrt{1-|\alpha|^2}$, $P_a(z) = \frac{\langle z,a\rangle}{|a|^2}a$ and $Q_a(z) = z - \frac{\langle z,a\rangle}{|a|^2}a$ for all $z \in \mathbb{B}$.

For $\alpha = 0$, we simply set $\varphi_\alpha(z) = -z$. Clearly, each φ_α is holomorphic from $\mathbb{B}$ into $\mathbb{C}^n$. It is also well known that each φ_α is a homeomorphism of $\overline{\mathbb{B}}$ onto itself, and every automorphism φ of $\mathbb{B}$ can be represented as $\varphi = \varphi_\alpha U$, with U a unitary transformation on $\mathbb{C}^n$.

Theorem 2.3. Let ω be normal weight on $\mathbb{B}$ such that $I^2_\omega(1) < \infty$. Then the space $\mathcal{Z}_\omega$ is an automorphism invariant boundary regular small space in the following sense:

(i) All functions in $\mathcal{Z}_\omega$ can be continuously extended to $\overline{\mathbb{B}}$,

(ii) All polynomials are contained in $\mathcal{Z}_\omega$,

(iii) Evaluation at every point in $\mathbb{B}$ defines a bounded linear functional,

(iv) For any $\varphi \in Aut(\mathbb{B})$ and $f \in \mathcal{Z}_\omega$, we have $f \circ \varphi \in \mathcal{Z}_\omega$.

Remark 2.2. Axioms (i) and (iii) ensure that convergence in the $\mathcal{Z}_\omega$ norm implies convergence in the sup norm; that is, the identity map from $(\mathcal{Z}_\omega, \|\cdot\|_{\mathcal{Z}_\omega})$ to $(\mathcal{Z}_\omega, \|\cdot\|_\infty)$ is continuous by the closed graph theorem. Furthermore, using the closed graph theorem along with axiom (iii), one can show that axiom (iv) ensures that C_φ is bounded on $\mathcal{Z}_\omega$ for any conformal automorphism φ of $\mathbb{B}$.

Proof. It is straightforward from the definitions to verify that (ii) and (iii) hold for $\mathcal{Z}_\omega$. Under the condition $I^1_\omega(1) < \infty$ the space $\mathcal{Z}_\omega$ satisfies (i) (see [15]).

To verify that (iv) holds, we need to show that for any conformal automorphism $\varphi = \varphi_a U = (\varphi_1, \ldots, \varphi_n)$ of $\mathbb{B}$, if $f \in \mathcal{Z}_\omega$, then $f \circ \varphi \in \mathcal{Z}_\omega$, where a is a point in $\mathbb{B}$ and U is a unitary transformation of $\mathbb{C}^n$. Without loss of generality, we can assume that $\varphi = \varphi_a$ for some $a \in \mathbb{B}$. It follows from (2.7) that $\varphi_j \in H(\overline{\mathbb{B}})$ for each $j = 1, \ldots, n$. Consequently, $R^{(k)}\varphi_j$ is in $H(\overline{\mathbb{B}})$ and remains bounded on $\overline{\mathbb{B}}$ for any $k \in \mathbb{N}$. Thus,

$$\begin{aligned} M^{(1)}_\varphi &:= \sup_{z\in\mathbb{B}_n} |R\varphi(z)| < \infty, \\ M^{(2)}_\varphi &:= \sup_{z\in\mathbb{B}_n} |R^{(2)}\varphi(z)| < \infty. \end{aligned} \tag{2.8}$$

Let $\lambda \in (0,1)$ be such that $|R\varphi(z)| \le 1$ and $|R^{(2)}\varphi(z)| \le 1$ for $|\varphi(z)| \le \lambda$. There exists $D_0 > 0$ such that

$$1 \le D_0 I^1_\omega(\lambda), \quad 1 \le D_0 I^2_\omega(\lambda). \tag{2.9}$$

Then, thers exists $D_1 > 0$ such that

$$\begin{aligned} &\sup_{|\varphi(z)|\le\lambda} \omega(\varphi(z))|R\varphi(z)|\big(1 + I^1_\omega(\varphi(z))\big) \\ &\quad\le D_1 \sup_{|\varphi(z)|\le\lambda} \omega(\varphi(z))||R\varphi(z)|I^1_\omega(\varphi(z)), \\ &\sup_{|\varphi(z)|\le\lambda} \omega(\varphi(z))|R^{(2)}\varphi(z)|\big(1 + I^2_\omega(\varphi(z))\big) \\ &\quad\le D_1 \sup_{|\varphi(z)|\le\lambda} \omega(\varphi(z))|R^{(2)}\varphi(z)|I^2_\omega(\varphi(z)). \end{aligned} \tag{2.10}$$

Let $D = \max\{D_0+1, D_1\}$. For every $f \in \mathcal{Z}_\omega$, by (2.1)–(2.4), (2.10), and a standard calculation, we have

$$\begin{aligned} &\omega(z)|R^{(2)}(f\circ\varphi)(z)| \\ &\le \omega(z)\big[|R^{(2)}f(\varphi(z))\varphi(z)| \\ &\quad + 2|Rf(\varphi(z))R\varphi(z)| + |f(\varphi(z))R^{(2)}\varphi(z)|\big] \\ &= \frac{\omega(z)}{\omega(\varphi(z))}\omega(\varphi(z))\big[|R^{(2)}f(\varphi(z))\varphi(z)| \\ &\quad + 2|Rf(\varphi(z))R\varphi(z)| + |f(\varphi(z))R^{(2)}\varphi(z)|\big] \\ &\le \frac{\omega(z)}{\omega(\varphi(z))}\Big[1+ \\ &\qquad + C\omega(\varphi(z))\big(2|R\varphi(z)|(1 + I^1_\omega(\varphi(z))) \\ &\qquad + |R^{(2)}\varphi(z)|(1 + I^2_\omega(\varphi(z)))\big)\Big]\|f\|_{\mathcal{Z}_\omega} \\ &\le \frac{\omega(z)}{\omega(\varphi(z))}\Big[1 \\ &\quad + CD\sup_{|\varphi(z)|\ge\lambda} \omega(\varphi(z))[2M^{(1)}_\varphi + M^{(2)}_\varphi]\Big]\|f\|_{\mathcal{Z}_\omega} \\ &= \frac{\omega(z)}{\omega(\varphi(z))}\Big[1 + CDR_\omega[2M^{(1)}_\varphi + M^{(2)}_\varphi]\Big]\|f\|_{\mathcal{Z}_\omega} \end{aligned} \tag{2.11}$$

for every $z \in \mathbb{B}$.

(i) We begin by considering the case when $a = 0$. In this situation, we have $|\varphi(z)| \le |z|$ for all $z \in \mathbb{B}$. Denote

$$B_\delta := \{z \in \mathbb{B} : |\varphi(z)| \le \delta\}.$$

Since μ is decreasing on $[\delta, 1)$ we have

$$\frac{\omega(z)}{\omega(\varphi(z))} \le \frac{M_\omega}{m_{\omega,\delta}}, \quad z \in B_\delta;$$
$$\frac{\omega(z)}{\omega(\varphi(z))} < 1, \quad z \in \mathbb{B} \setminus B_\delta.$$

Therefore, it follows from (2.11) that

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_n} \omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\le \sup_{z\in B_\delta} \omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\quad + \sup_{z\in\mathbb{B}_n\setminus B_\delta} \omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&\le \Big(\frac{M_\omega}{m_{\omega,\delta}} + 1\Big) \\
&\times \Big(1 + CDR_\omega[2M_\varphi^{(1)} + M_\varphi^{(2)}]\Big)\|f\|_{\mathcal{Z}_\omega} < \infty.
\end{aligned} \tag{2.12}$$

Hence, $f\circ\varphi \in \mathcal{Z}_\omega$.

(ii) Now, we consider the case $a \ne 0$. Take a $\gamma \in Aut(\mathbb{B})$ such that $\gamma(0) = a$. Then $\eta := \varphi\circ\gamma \in Aut(\mathbb{B})$ and $\eta(0) = 0$. By (i), $g := f\circ\eta \in \mathcal{Z}_\omega$. Note that $\gamma^{-1} \in Aut(\mathbb{B})$, as the above, we have $R^{(k)}\gamma^{-1}$ is bounded in $\overline{\mathbb{B}}$ for any positive integer k. Then, since $f\circ\varphi = g\circ\gamma^{-1}$, as the estimate (2.12) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_n} \omega(z)|R^{(2)}(f\circ\varphi)(z)| \\
&= \sup_{z\in\mathbb{B}_n} \omega(z)|R^{(2)}(g\circ\gamma^{-1})(z)| \\
&\le \Big(\frac{M_\omega}{m_{\omega,\delta}} + 1\Big) \\
&\times \Big(1 + CDR_\omega[2M_{\gamma^{-1}}^{(1)} + M_{\gamma^{-1}}^{(2)}]\Big)\|g\|_{\mathcal{Z}_\omega} < \infty.
\end{aligned}$$

Consequently, $f\circ\varphi \in \mathcal{Z}_\omega$. □

Remark 2.3. The condition $I_\omega^2(1) < \infty$ cannot be omitted. Indeed, consider the weight function $\omega(t) = (1-t)^2$ for $t \in [0, 1)$ which satisfies $I_\omega^2(1) = \infty$. It is easy to see that the function $f(z) = \ln(1-z)$, which belongs to $\mathcal{Z}_\omega$, does not admit a continuous extension to $\overline{\mathbb{D}}$. This means that the condition (i) is not true for $\mathcal{Z}_\omega$.

3. A RELATION BETWEEN WEIGHTED COMPOSITION OPERATORS $\mathcal{B}_\omega \to \mathcal{Z}_\omega$ AND $\mathcal{Z}_\omega \to \mathcal{Z}_\omega$

In order to conclude the paper we establishes the relation between the boundedness, compactness of weighted composition operators from $\mathcal{B}_\omega$ into $\mathcal{Z}_\omega$ and from $\mathcal{Z}_\omega$ into itself.

Before stating the theorem first let us note that for each $j = 1, \dots, n$ the function id_j given by $id_j(z) := z_j$ belongs to $\mathcal{Z}_\omega$. Then, in the case $\psi \in H^\infty(\mathbb{B})$ with $\|\psi\|_\infty \le 1$ and $W_{\psi,\varphi} : \mathcal{Z}_\omega \to \mathcal{Z}_\omega$ is compact, $W_{\psi,\varphi}(id_j) = \psi\cdot\varphi_j$ hence, $\theta_j := \psi\cdot\varphi_j \in \mathcal{Z}_\omega$, $j = 1, \dots, n$. For each $m \ge 1$, put

$$\theta^m = (\theta_1^m, \dots, \theta_n^m) := \prod_{k=0}^{m-1} (\psi\circ\varphi^k)\cdot\varphi^m,$$

where $\varphi^0 = id$, and $\varphi^k := \underbrace{\varphi\circ\cdots\circ\varphi}_{k \text{ times}}$ for $k \ge 1$.

Theorem 2.3(i) allows us to assume that θ^m are continuous on $\overline{\mathbb{B}}$ for every $m \ge 0$.

Theorem 3.1. Let $\psi \in H^\infty(\mathbb{B})$, $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B})$ and ν, ω be normal weights on $\mathbb{B}$ and $I_\omega^1(1) < \infty$ Then the following are equivalent:

(1) $W_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{Z}_\omega$ is compact;

(2) $W_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{Z}_\omega$ is bounded;

(3) $W_{\psi,\varphi} : \mathcal{Z}_\omega \to \mathcal{Z}_\omega$ is compact;

(4) $\psi, \psi\cdot\varphi_j \in \mathcal{Z}_\omega$ for every $j = 1, \dots, n$ and $\|\varphi\|_\infty < 1$.

To establish the theorem, we first require several lemmas.

Lemma 3.2. Assume that $\varphi(0) = 0$ and $W_{\psi,\varphi} : \mathcal{Z}_\omega \to \mathcal{Z}_\omega$ is compact. Then $\|\theta^m\|_\infty \to 0$.

Proof. Without loss of generality we may assume that $\|\psi\|_\infty \le 1$. We have two cases to consider:

(i) When $|\psi(0)| = 1$, it follows from Theorem 2.3(i) and the maximum modulus principle that ψ must be identically equal to 1. Then $W_{\psi,\varphi} = C_\varphi$, the composition operator on $\mathcal{Z}_\omega$, and hence, the lemma follows from Lemma 2.2 of [18].

(ii) Now we assume that $|\psi(0)| < 1$.

We will prove that $W_{\psi,\varphi}$ has spectral radius $\varrho(W_{\psi,\varphi}) < 1$.

Let $\lambda \ne 0$ be a spectral point of $W_{\psi,\varphi}$. Since $W_{\psi,\varphi}$ is compact, λ must to be an eigenvalue. Let $f \in \mathcal{Z}_\omega$ be an eigenfunction of $W_{\psi,\varphi}$ corresponding to the eigenvalue λ. Thus $W_{\psi,\varphi}(f) = \lambda f$ and there is a point $a \in \mathbb{B}$ for which $f(a) \ne 0$. Denote $\mathbb{B}^a := \big\{z \in \mathbb{B} : |z| < \frac{1+|a|}{2}\big\}$. Note that, $|\varphi(z)| < |z|$ for every $z \in \mathbb{B}$, since otherwise, the composition operator C_φ would be an isomorphism. Consequently, by $\|\psi\|_\infty \le 1$, $(\psi \cdot C_\varphi)(B_{\mathcal{Z}_\omega})$ is not relatively compact subset of the unit ball $B_{\mathcal{Z}_\omega}$ of $\mathcal{Z}_\omega$. This means $\psi \cdot C_\varphi$ is not a compact operator. This contradicts the compactness of $W_{\psi,\varphi}$. By the Schwarz Lemma, $\varphi(\mathbb{B}^a)$ is relatively compact in $\mathbb{B}^a$. Applying the Schwarz Lemma again to the appropriately normalized restriction of φ on $\varphi(\mathbb{B}^a)$, and continuing this argument, it follows that $\varphi^m(a) \to 0$ as $m \to \infty$.

Now, since $\lim_{m\to\infty} |\psi(\varphi^{m-1}(a))| = |\psi(0)| \ne 1$, by using the fact that, if $0 < a_m < 1$ and $\{a_m\}_{m\ge1}$ does not converge to 1 then $\prod_{m=1}^\infty a_m = 0$, we obtain

$$\begin{aligned}\lambda^m f(a) &= [W_{\psi,\varphi}]^m(f)(a)\\ &= \Big(\prod_{k=0}^{m-1} \psi(\varphi^k(a))\Big) \cdot f(\varphi^m(a)) \to 0 \cdot f(0)\end{aligned}$$

as $m \to \infty$. Because $f(a) \ne 0$ it therefore must has $|\lambda| < 1$. The compactness of $W_{\psi,\varphi}$ ensures that its spectrum consists of 0 along with at most countably many eigenvalues accumulating only at 0. Thus, the spectral radius equals the largest eigenvalue in modulus, which, as shown above, is strictly less than 1. It then follows from the spectral radius formula that

$$\lim_{m\to\infty} \|[W_{\psi,\varphi}]^m\|^{1/m} = \varrho(W_{\psi,\varphi}) < 1,$$

so in particular, $\lim_{m\to\infty} \|[W_{\psi,\varphi}]^m\| = 0$. Note that $\theta_j^m = [W_{\psi,\varphi}]^m(id_j) \in \mathcal{Z}_\omega$, $j = 1,\dots,n$. Then,

$$\begin{aligned}\|\theta_j^m\|_{\mathcal{Z}_\omega} &= \|[W_{\psi,\varphi}]^m(id_j)\|_{\mathcal{Z}_\omega}\\ &\le \|[W_{\psi,\varphi}]^m\|\|id_j\|_{\mathcal{Z}_\omega} \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

On the other hand, Theorem 2.3(i & iii) shows that $\mathcal{Z}_\omega$ has a topology stronger than that of the sup norm. Hence, $\|\theta^m\|_\infty$ tends to zero as $m \to \infty$. The lemma is thus proved. □

Lemma 3.3. Suppose $\psi \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, and μ, ν are normal weights on $\mathbb{B}$. Let $X = \mathcal{B}_\nu$ or $\mathcal{Z}_\nu$. Then $W_{\psi,\varphi} : X \to \mathcal{Z}_\mu$ is compact if and only if, whenever a bounded sequence f_m in X converges to zero uniformly on compact subsets of $\mathbb{B}$, it follows that $\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\mu} \to 0$ as $m \to \infty$.

The lemma for the case $X = \mathcal{B}_\nu$ has been proven in [21]. For the case $X = \mathcal{Z}_\nu$, it is similar to that of $X = \mathcal{B}_\nu$ and will therefore be omitted.

Lemma 3.4. Let $\psi \in H(\mathbb{B})$, $\varphi = (\varphi_1,\dots,\varphi_n) \in S(\mathbb{B})$ and μ, ν be normal weights on $\mathbb{B}$. Assume that $W_{\psi,\varphi} : \mathcal{Z}_\nu \to \mathcal{Z}_\mu$ is bounded. Then

$$\begin{aligned}&\sup_{z\in\mathbb{B}_n} \mu(z)|A_{\psi,\varphi}(z)| < \infty,\\ &\sup_{z\in\mathbb{B}_n} \mu(z)|B_{\psi,\varphi}(z)| < \infty,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}A_{\psi,\varphi}(z) &:= 2R\psi(z)R\varphi(z) + \psi(z)R^{(2)}\varphi(z),\\ B_{\psi.\varphi}(z) &:= \psi(z)((R\varphi_1(z))^2,\dots,(R\varphi_n(z))^2).\end{aligned}$$

Proof. First, choosing $f_0(z) = 1 \in \mathcal{Z}_\nu$, the boundedness of $W_{\psi,\varphi}$ implies that $\psi \in \mathcal{Z}_\mu$.

At the same time, for each $j \in \{1,\dots,n\}$, by considering $f_j(z) = z_j$ and $g_j(z) = z_j^2$ for every $z = (z_1,\dots,z_n) \in \mathbb{B}$ we can check that $\psi \cdot \varphi_j, \psi \cdot \varphi_j^2 \in \mathcal{Z}_\mu$.

Then, since

$$\begin{aligned}
&R^{(2)}[\psi(z)\varphi_j(z)]\\
&\quad = R^{(2)}\psi(z)\varphi_j(z) + 2R\psi(z)R\varphi_j(z)\\
&\qquad + \psi(z)R^{(2)}\varphi_j(z)\\
&\quad = R^{(2)}\psi(z)\varphi_j(z) + A_{\psi,\varphi_j}(z),\\
&R^{(2)}[\psi(z)\varphi_j^2(z)]\\
&\quad = \varphi_j(z)\big(R^{(2)}\psi(z)\varphi_j(z) + 4R\psi(z)R\varphi_j(z)\\
&\qquad + 2\psi(z)R^{(2)}\varphi_j(z)\big) + 2\psi(z)\big(R\varphi_j(z)\big)^2\\
&\quad = \varphi_j(z)\Big[2R^{(2)}[\psi(z)\varphi_j(z)] - R^{(2)}\psi(z)\Big]\\
&\qquad + 2B_{\psi,\varphi_j}(z)
\end{aligned} \tag{3.2}$$

for every $z \in \mathbb{B}$ and every $j = 1, \dots, n$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_n} \mu(z)|A_{\psi,\varphi_j}(z)|\\
&\quad \le \|\psi\cdot\varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty,\\
&\sup_{z\in\mathbb{B}_n} \mu(z)|B_{\psi,\varphi_j}(z)|\\
&\quad \le \|\psi\cdot\varphi_j^2\|_{\mathcal{Z}_\mu} + 2\|\psi\cdot\varphi_j\|_{\mathcal{Z}_\mu} + \|\psi\|_{\mathcal{Z}_\mu} < \infty
\end{aligned}$$

for every $j = 1, \dots, n$. Consequently, (3.1) is proved. □

Proof of Theorem 3.1. Theorem is trivial if $\|\psi\|_\infty = 0$. Without loss of generality we may assume that $0 < \|\psi\|_\infty \le 1$, since for ortherwise we can consider the $\|\psi\|_\infty^{-1}\psi$ instead of ψ.

(1) ⇒ (2): It is obvious.

(2) ⇒ (3): Assume that $\{f_m\}_{m\ge1}$ is a bounded sequence in $\mathcal{Z}_\omega$ which converges to zero uniformly on compact subsets of $\mathbb{B}$. Then, by the Weierstrass theorem, both $\{Rf_m\}_{m\ge1}$ and $\{R^{(2)}f_m\}_{m\ge1}$ converge uniformly to zero on compact subsets of $\mathbb{B}$ as well. We will prove that $\|f_m\|_{\mathcal{B}_\omega}$ converges to zero. Given any $\varepsilon > 0$, since $\omega(t) \to 0$ as $t \to 1$, we can choose $\varrho \in (\delta, 1)$ so that $\omega(|z|) < \varepsilon$ whenever $\varrho < |z| < 1$. In addition, there exists an integer N such that for all $m \ge N$, we have $|f_m(0)| < \varepsilon$, $|Rf_m(z)| < \varepsilon$, and $|R^{(2)}f_m(z)| < \varepsilon$ for all $|z| \le \varrho$. Therefore, by (2.3),

$$\begin{aligned}
\|f_m\|_{\mathcal{B}_\omega} &\le |f_m(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|Rf_m(z)|\\
&\le \varepsilon + \sup_{|z|\le\varrho} \omega(z)|Rf_m(z)|\\
&\quad + \sup_{\varrho<|z|<1} \omega(z)|Rf_m(z)|\\
&\le \varepsilon + \varepsilon M_\omega + \sup_{\varrho<|z|<1} \omega(z)\Big|Rf_m\Big(\frac{z}{2|z|}\Big)\\
&\quad + \int_{1/(2|z|)}^1 R^{(2)}f_m(tz)\frac{dt}{t}\Big|\\
&\le \varepsilon + \varepsilon M_\omega + \varepsilon \sup_{m\ge1}\sup_{|w|=1/2} |Rf_m(w)|\\
&\quad + 2\varepsilon \int_{1/(2|z|)}^{\varrho/|z|} |R^{(2)}f_m(tz)||z|dt\\
&\quad + 2\sup_{\varrho<|z|<1} \omega(z) \int_{\varrho/|z|}^1 |R^{(2)}f_m(tz)||z|dt\\
&\le \varepsilon + \varepsilon M_\nu + \varepsilon \sup_{m\ge1}\sup_{|w|=1/2} |Rf_m(w)|\\
&\quad + 2\varepsilon\|f_m\|_{\mathcal{Z}_\omega} \int_{1/2}^{\varrho} \frac{dt}{\omega(t)}\\
&\quad + 2\|f_m\|_{\mathcal{Z}_\omega} \sup_{\varrho<|z|<1} \omega(z) \int_\delta^{|z|} \frac{dt}{\omega(t)}\\
&\le \varepsilon + \varepsilon M_\omega + \varepsilon \sup_{m\ge1}\sup_{|w|=1/2} |Rf_m(w)|\\
&\quad + 2\varepsilon\|f_m\|_{\mathcal{Z}_\omega} \int_{1/2}^{\varrho} \frac{dt}{\omega(t)} + 2\varepsilon\|f_m\|_{\mathcal{Z}_\omega}\\
&\le \varepsilon K \quad \text{for all } m \ge N.
\end{aligned}$$

Then, the boundedness of $W_{\psi,\varphi}$ implies that $\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\omega} \lesssim \|f_m\|_{\mathcal{B}_\omega} \to 0$ as $m \to \infty$. Therefore, $W_{\psi,\varphi}$ is compact by Lemma 3.3.

(3) ⇒ (4): Without loss of generality, we may assume that $\|\psi\|_\infty < 1$.

(i) We first consider the case where $\varphi(0) = 0$. Suppose, for the sake of contradiction, that $\|\varphi\|_\infty = 1$. Then there is a rotation ζ, $z \mapsto e^{i\alpha}z$, such that $\widetilde{\varphi} := \zeta\circ\varphi$ has a fixed point $z_0 \in \mathbb{B}_n$. We may choose ζ such that $\psi(z_0) \ne 0$. Put

$$\widetilde{\psi} := \frac{\psi}{\psi(z_0)}.$$

Then, for every $m \ge 1$ we obtain that

$$(\widetilde{\theta})^m := \Big(\prod_{k=0}^{m-1} \widetilde{\psi}\circ(\widetilde{\varphi})^k\Big)(\widetilde{\varphi})^m$$

has a fixed point z_0, hence, $\|(\widetilde{\theta})^m\|_\infty \geq 1$. It follows from Lemma 3.2 that the operator $W_{\widetilde{\psi},\widetilde{\varphi}}$, and hence, $W_{\psi,\widetilde{\varphi}}$ is cannot be compact.

Note that $W_{\psi,\widetilde{\varphi}} = W_{\psi,\varphi} \circ C_\zeta$ where the composition operator C_ζ is an isomorphism of $\mathcal{Z}_\omega$. This implies that $W_{\psi,\varphi}$ is not compact. This contradicts the hypothesis.

(ii) We now consider the case $\varphi(0) = a \neq 0$. Let γ be the conformal automorphism of $\mathbb{B}_n$ taking a to 0, and set $\eta = \gamma \circ \varphi$. It follows from Theorem 2.3(iv & iii) that C_γ is a bounded operator on $\mathcal{Z}_\omega$, hence, $W_{\psi,\eta}$ is compact on $\mathcal{Z}_\omega$ because $W_{\psi,\eta} = W_{\psi,\varphi} \circ C_\gamma$. Finally, it follows from the case (i) that $\|\eta\|_\infty < 1$, and hence, $\|\varphi\|_\infty < 1$.

(4) $\Rightarrow$ (1): Suppose $\{f_m\}_{m\geq 1}$ is a bounded sequence in $\mathcal{B}_\nu$ converging uniformly to zero on compact subsets of $\mathbb{B}$. Using the Cauchy integral formula again, we see that

$$\sup_{|\varphi_k(z)|\leq\lambda} |R_{\varphi(z)} f_m(\varphi(z))| \to 0,$$
$$\sup_{|\varphi_k(z)|\leq\lambda} |\nabla^{(2)}_{\varphi(z)} f_m(\varphi(z))| \to 0 \quad \text{as } m \to \infty.$$

with $\lambda = \|\varphi\|_\infty < 1$, Then, by $\psi \in \mathcal{Z}_\omega$, (3.1) and a standard calculation, we have

$$\begin{aligned}
&\|W_{\psi,\varphi}(f_m)\|_{\mathcal{Z}_\omega} \\
&\leq |f_m(0)| + \omega(z)|R^{(2)}[\psi(z)]||f_m(\varphi(z))| \\
&\quad + \omega(z)|A_{\psi,\varphi}(z)|Rf_m(\varphi(z))| \\
&\quad + \omega(z)|B_{\psi,\varphi}(z)|R^{(2)} f_m(\varphi(z))| \\
&\leq |f_m(0)| + \|\psi\|_{\mathcal{Z}_\omega} \sup_{|\varphi(z)|\leq\|\varphi\|_\infty} |f_m(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{B}_n} \omega(z)|A_{\psi,\varphi}(z)| \sup_{|\varphi(z)|\leq\|\varphi\|_\infty} |Rf_m(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{B}_n} \omega(z)|B_{\psi,\varphi}(z)| \sup_{|\varphi(z)|\leq\|\varphi\|_\infty} |R^{(2)} f_m(\varphi(z))| \\
&\quad \to 0
\end{aligned}$$

as $m \to \infty$. By Lemma 3.3, $W_{\psi,\varphi}$ is compact. □

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