Chia tách đường cong Bézier bậc ba từng mảnh và các hằng số tương đương cho một vài chuẩn

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TÓM TẮT

Bài báo này liên quan đến việc chia tách các đường cong Bézier từng khúc bậc ba và các mối quan hệ tương đương của một số chuẩn được định nghĩa bởi các điểm điều khiển. Các đường cong Bézier từng khúc bậc ba được sử dụng phổ biến nhất để xấp xỉ các đường cong. Chúng được hình thành từ các điểm điều khiển. Các chuẩn $\|\cdot\|_p^{B_{N,3}}$ và $\|\cdot\|_p^{B_{2N,3}}$ trong không gian $B_{N,3}$ được xác định bởi các điểm điều khiển. Với mục tiêu giữ bậc của đường cong và thêm tính linh hoạt trong việc thiết kế đường cong, chúng ta thường chia tách một đường cong Bézier N khúc bậc ba thành một đường cong Bézier $2N$ khúc bậc ba. Chúng ta sẽ tập trung vào các hằng số tương đương cho chuẩn $\|\cdot\|_p^{B_{N,3}}$ và $\|\cdot\|_p^{B_{2N,3}}$ trong không gian $B_{N,3}$ của các đường cong Bézier N khúc bậc ba. Do đó, chúng ta có thể sử dụng chuẩn $\|\cdot\|_p^{B_{N,3}}$ để kiểm tra sự hội tụ của chuỗi các đường cong Bézier từng khúc bậc ba. Kết quả này là quan trọng trong việc áp dụng đường cong Bézier từng khúc bậc ba để tìm ra quỹ đạo tối ưu.

Từ khóa: *Đường cong Bezier bậc ba, hằng số tương đương, chuẩn, chia tách, từng khúc.*

Splitting piecewise cubic Bézier curves and the equivalence constants for some norms

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ABSTRACT

This article is concerned with splitting piecewise cubic Bézier curves and the equivalence relations for some norms defined through control points. Piecewise cubic Bézier curves are most common to approximate curves. These curves are established by control points. The norms $\|\cdot\|_p^{B_{N,3}}$ and $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$ are determined through control points. With the purpose of keeping the degree of the curves and offering additional flexibility for curve design, we often split N-piece cubic Bézier curves to become $2N-$piece cubic Bézier curves. We will concentrate on the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$ of N-piece cubic Bézier curves. So, we can use the norm $\|\cdot\|_p^{B_{N,3}}$ to check the convergence for sequences of piecewise cubic Bézier curves. This result is important for applying piecewise cubic Bézier curves to detect optimal orbits.

Key words: *Cubic Bézier curves, equivalence constants, norm, split, piecewise.*

1. INTRODUCTION

In mathematics and engineering, there are many curves which has complex shapes or curves which given by a set of points. To overcome this difficulty, we create a new curve that closely matches an existing one, often to simplify a complex shape or to fit a set of data points. Some methods to approximate the curve include using simpler curves or interpolating polynomials or Bézier curves.

In the method using simpler curves to approximate the curve, we divide the curve into a series of points and connect them with straight lines or circular arcs, with more points resulting in a closer approximation. The main advantage of approximating curves with straight lines or circular arcs is computational simplicity and efficiency, as line segments are defined by fewer parameters than higher-order curves. This method is also useful for data compression and noise reduction by simplifying complex curves. However, approximating a curve with simple curves can lead to inaccuracies, especially in areas of high curvature, and may result in poor extrapolation beyond the measured data range. ([1–9])

Beside, a curve can be approximated by an interpolating polynomial such as a Lagrange polynomial or a Newton polynomial, which fits a curve through a set of known points on the curve. This method improve the approximation as the degree of the polynomial increases. Interpolating polynomials passing exactly through specified points. The main advantages of using interpolating polynomials for approximation include their high accuracy for small datasets, the ability to obtain an explicit function for calculations, and the ease of differentiation and integration. They also provide exact results at the given data points and can be used for data points that are not equally spaced. The main disadvantages of using interpolating polynomials for curve approximation are Runge's phenomenon (oscillations, especially at the endpoints), computational expense for high-degree polynomials, and poor extrapolation properties, where the curve can behave erratically outside the range of the data points (see [10–12]).

Approximating a curve with a Bézier curve involves selecting key points on the original curve to serve as control points and endpoints for the Bézier curve(s). For a given curve, this is often achieved by dividing it into segments and approximating each segment with a Bézier curve, using techniques like the de Casteljau's algorithm for subdivision or fitting algorithms like the Adaptive Extension Fitting Scheme to find the optimal control points for a set of segments. Béziers' construction has many benefits. Initially, each Bézier curves is presented by a few control points, then it need very little memory. Besides, these curves is intuitive, compact and beautiful. It's easy to compute and design, so the designer without mathematical background can be use them. More, we can easily change, move, turn Bézier curves just by changing, moving, turning their control points. (see [1,13–17]).

In 1959, the mathematician Paul de Casteljau built Bézier curve by using de Casteljau's algorithm while working for the French automaker Citroën. He was the first to apply this method to computer-aided design (CAD). However, his work remained a company secret and was not published for many years. So, his contributions were not widely known at the time. The Bézier curve was publicized by the French engineer Pierre Bézier in 1962. He defined the Bézier curve based on Bernstein polynomials. Pierre Bézier applied Bézier curves for designing the bodywork of Renault cars. Bézier built definitions, symbols, formulas and specially control points of Bézier curves. These things make it easy and convenient to represent curves in computers software.

The computer program language PostScript uses Bézier curves as the standard curve. Many vector graphics editor and design software such as CorelDRAW, Adobe Illustrator and Inkscape apply Bézier curves. Its importance is due to the fact that, Bézier curves are used in many areas of life,not only mathematics. Bézier curves are applied in computer graphics, computer-aided design system, robotic, industry, walking, communication, path-planning and aerospace (see [15–22]). Bézier curves are also used to find plane shape optimization which appears in many fields such as environment design, aerospace, structural mechanics, networks, automotive, hydraulic, oceanology and wind engineering (see [23–28]).

Bézier curves are presented in many books and articles for instance [1,13,14]. A continuous curve can be approximated by a Bézier curve. However, when the curve is long and complex, the degree of the Bézier curve is high. As a result, the computation is more difficult. Then, the most common use of Bézier curves is as N-piece cubic Bézier curves. We will focus uniform N-piece cubic Bézier curves.

From [29], we have the norm $\|\cdot\|_p^{B_m}$ on the space B_m of Bézier curves of degree m and the norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$ of uniform N-piece Bézier curves of degree m. These norms are computed through control points.

With the purpose of keeping the degree of the curves and offering additional flexibility for curve design, we often split N-piece uniform cubic Bézier curves to become $2N-$piece uniform cubic Bézier curves. Splitting piecewise cubic Bézier curves keep huge part of applying these curves. Then, we study the equivalence constants for the norm $\|\cdot\|_p^{B_{2N,3}}$ and the norm $\|\cdot\|_p^{B_{N,3}}$ on the space of piecewise cubic Bézier curves.

Theorem 1. *Let $p \in [1,\infty[\cup\{\infty\}$ and let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve. Then*

$$\min\left\{\frac{1}{24^{1/p}},\frac{1}{4}\right\}\|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{2N,3}} \leq 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.$$

2. Preliminaries

For the readers can follow along easily, we recall some definitions and notations that will be used through the article.

Definition 2. ([1] chapter 6, p. 141) Given four points W_0, W_1, W_2 and W_3, the *cubic Bézier curve* associated with the four control points $W_0,\dots,W_3$ is defined by

$$B([W_0,\dots,W_3],t) := \sum_{i=0}^{3} W_i b_{i,3}(t) \quad \text{for } t \in [0,1], \quad (1)$$

where $b_{i,3}(t) = \binom{3}{i} t^i (1-t)^{3-i}$ is the Bernstein polynomial.

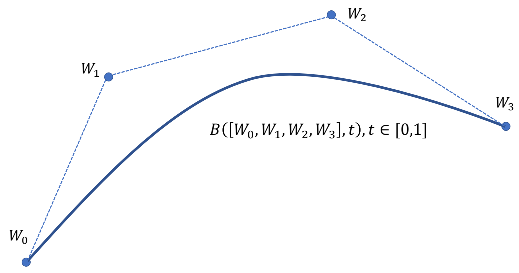

Figure 1. A cubic Bézier curve

The points W_0, W_1, W_2 and W_3 are control points of the cubic Bézier curve. The pentagon $W_0W_1W_2W_3$ is the control polygon of the curve. The cubic Bézier curve lies in its control polygon.

For any cubic Bézier curve

$$\begin{aligned}\beta(t) =& B([W_0,W_1,W_2,W_3],t) = \sum_{i=0}^{3} W_i b_{i,3}(t),\\ &=(1-t)^3 W_0 + 3(1-t)^2 t W_1 + 3(1-t)t^2 W_2 + t^3 W_3,\\ &\text{for } t \in [0,1].\end{aligned}$$

Form the recursive property of Bernstein polynomials, a cubic Bézier curve can be recursively determined as a convex combination of two quadratic Bezier curves as

$$\begin{aligned}\beta(t) =& B([W_0,W_1,W_2,W_3],t)\\ &=(1-t)\left((1-t)^2 W_0 + 2(1-t)tW_1 + t^2 W_2\right)\\ &\quad + t\left((1-t)^2 W_1 + 2(1-t)tW_2 + t^2 W_3\right)\\ &=(1-t)B([W_0,W_1,W_2],t) + tB([W_1,W_2,W_3],t).\end{aligned}$$

Since $b'_{i,n}(x) = n\left(b_{i-1,n-1}(x) - b_{i,n-1}(x)\right)$, The derivative with respect to t of the cubic Bézier curve is another Bezier curve which has a lower degree as follows

$$\begin{aligned}&\frac{d}{dt}\beta(t) = \frac{d}{dt}B([W_0,W_1,W_2,W_3],t)\\ &=3(1-t)^2(W_1-W_0) + 6(1-t)t(W_2-W_1) + 3t^2(W_3-W_2)\\ &=3B([W_1-W_0, W_2-W_1, W_3-W_2],t).\end{aligned}$$

A uniform N-piece cubic Bézier curve is formed by the combination of N cubic Bézier curves and the connecting points of the pieces are the points at $t = \frac{j}{N}$ for $j = 1,\dots,N-1$. For convenient, we will omit the word “uniform".

Definition 3. ([1] chapter 7, p. 169) Let N be positive integers and $W_0,\dots,W_{N3}$ be $N3+1$ points in $\mathbb{R}^n$. The *N-piece cubic Bézier curve* associated with control points $W_0,\dots,W_{N3}$ is defined by

$$\begin{aligned}\beta :& [0,1] \to \mathbb{R}^n\\ & t \mapsto \beta(t) = B\left([W_{j3},\dots,W_{j3+3}], Nt - j\right)\\ & \quad \text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right].\end{aligned}$$

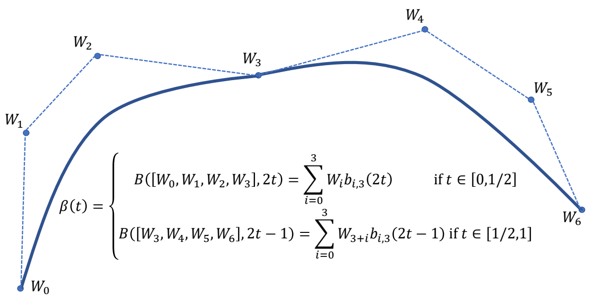

Figure 2. A two-piece cubic Bézier curve

Notation 4.

- The vector space of cubic Bézier curves is denoted by the symbol B_3.
- The vector space of N-piece cubic Bézier curves is denoted by the symbol $B_{N,3}$.

We define some norms and distances through control points on the space of cubic Bézier curves and on the space of N-piece cubic Bézier curves.

Definition 5. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_3} : B_m \to \mathbb{R}$ is defined by: For any $\beta(t) = \sum_{i=0}^{m} W_i b_{i,m}(t) \in B_3$,

$$\|\beta\|_p^{B_3} := \begin{cases} \left(\sum_{i=0}^{3} \|W_i\|_p^p \right)^{1/p} & \text{if } p \in [1, \infty[\\ \max_{i=0,\dots,3} \{\|W_i\|_\infty\} & \text{if } p = \infty, \end{cases}$$

where $\|\cdot\|_p$ is the p-norm on $\mathbb{R}^n$.

Using the Minkowski inequality and the properties of the p-norm on $n-$dimensional Euclidean space $\mathbb{R}^n$, we can easily show that $\|\cdot\|_p^{B_3}$ is a norm on the vector space B_3 of cubic Bézier curves. Indeed, this is a norm of their control polygons. Naturally, we get an induced distance on B_3 by $d_p^{B_3}(\beta, \gamma) := \|\beta - \gamma\|_p^{B_3}$, for $\beta, \gamma \in B_3$.

Definition 6. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_{N,3}} : B_{N,3} \to \mathbb{R}$ is defined by: For any $\beta(t) = \beta^{(j)}(Nt - j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt - j)$ if $t \in \left[\frac{j}{N}, \frac{j+1}{N}\right]$, $j = 0, \dots, N-1$,

$$\|\beta\|_p^{B_{N,3}} := \begin{cases} \frac{1}{N^{1/p}} \left(\sum_{j=0}^{N-1} \left(\|\beta^{(j)}\|_p^{B_3} \right)^p \right)^{1/p} & \text{if } p \in [1, \infty[\\ \max_{j=0,\dots,N-1} \left\{ \|\beta^{(j)}\|_\infty^{B_3} \right\} & \text{if } p = \infty. \end{cases}$$

By $\|\cdot\|_p^{B_3}$ is a norm on B_3 and the Minkowski inequality, it is easily seen that $\|\cdot\|_p^{B_{N,3}}$ is a norm on the vector space $B_{N,3}$ of N-piece cubic Bézier curves. Naturally, we get an induced distance on $B_{N,3}$ determined by $d_p^{B_{N,3}}(\beta, \gamma) := \|\beta - \gamma\|_p^{B_{N,3}}$ for any $\beta, \gamma \in B_{N,3}$.

The norm $\|\cdot\|_p^{B_{N,3}}$ is determined through the control points. It is more convenient than the L_p norm. From[29], we have the equivalence relations between the norms $\|\cdot\|_p^{B_{N,3}}$ and L_p as follows:

For $p \in [1, \infty]$. The inequalities

$$\|\beta\|_{L_p} \leq \|\beta\|_p^{B_{N,3}} \leq 2^{10} \|\beta\|_{L_p}.$$

hold for any $\beta \in B_{N,3}$.

We can present an N-piece cubic Bézier curve as an N-piece Bézier curve of degree 4 (see more[29]). So, The norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{N,4}}$ are two norms on the space $B_{N,3}$ of N-piece cubic Bézier curves. By[29], we get a corollary about equivalence constants between these norms as follows:

For $p \in [1, \infty]$. The inequalities

$$\frac{1}{8} \|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{N,4}} \leq 2 \|\beta\|_p^{B_{N,3}}.$$

hold for any $\beta \in B_{N,m}$.

When we need more freedom for designing the curve, we will increase the number of piece in piecewise cubic Bézier curves. To solve this problem, we will split piecewise cubic Bézier curves. By ([2] chapter 9, p. 201), a Bézier curve β can be split at any $t_0 \in (0, 1)$ to become two piece Bézier curves. However, when splitting at $t \neq \frac{1}{2}$, the connecting points of the pieces do not coincide with the point $\beta\left(\frac{1}{2}\right)$. This means that the obtained curve is not uniform. So, we need split a cubic Bézier curves at $t = \frac{1}{2}$ to get a uniform two-piece cubic Bézier curve as follows:

For any cubic Bézier curve

$$\beta(t) = B([W_0, W_1, W_2, W_3], t) = \sum_{i=0}^{3} W_i b_{i,3}(t),\ t \in [0, 1],$$

we have

$$\beta(t) = \begin{cases} \beta^{(0)}(2t) = \sum_{i=0}^{3} P_i b_{i,3}(2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta^{(1)}(2t-1) = \sum_{i=0}^{3} P_{3+i} b_{i,3}(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right], \end{cases} \tag{2}$$

where

$$\begin{cases} P_i = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{i-l}, & i = 0, \dots, 3, \\ P_{3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{3-i+l}, & i = 0, \dots, 3. \end{cases}$$

Thus, we can consider a cubic Bézier curve as a uniform two-piece cubic Bézier curve.

More generally, let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0, \dots, 3$, $j = 0, \dots, N-1$. We have

$$\beta(t) = \beta^{(j)}(Nt - j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt - j)$$
$$\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0, \dots, N-1.$$

In order to get a uniform $2N$-piece cubic Bézier curve β, we need split at the middle point of each piece.

$$\beta(t)=\begin{cases}\Gamma^{(2j)}(2Nt-2j)=\sum_{i=0}^{3}P_{2j3+i}b_{i,3}(2Nt-2j)\\ \qquad\qquad \text{if } t\in\left[\frac{2j}{2N},\frac{2j+1}{2N}\right]\\ \Gamma^{(2j+1)}(2Nt-2j-1)\\ \quad =\sum_{i=0}^{3}P_{(2j+1)3+i}b_{i,3}(2Nt-2j-1)\\ \qquad\qquad \text{if } t\in\left[\frac{2j+1}{2N},\frac{2j+2}{2N}\right],\\ \qquad\qquad j=0,\dots,N-1,\end{cases} \tag{3}$$

where

$$\begin{cases}P_{2j3+i}=\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+i-l},\\ P_{(2j+1)3+i}=\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+3-i+l},\\ i=0,\dots,3,\, j=0,\dots,N-1.\end{cases}$$

So, β is also a $2N$-piece cubic Bézier curve and the space $B_{N,3}$ of N-piece cubic Bézier curves is a subspace of the space $B_{2N,3}$ of $2N$-piece cubic Bézier curves. Therefore, the norms $\|\cdot\|_p^{B_{2N,3}}$ and $\|\cdot\|_p^{B_{N,3}}$ are two norms on the space $B_{N,3}$. Next, we will consider the equivalence relations between theses norms.

3. Equivalence constants for the norms $\|\cdot\|_p^{B_{2N,3}}$ and $\|\cdot\|_p^{B_{N,3}}$ on $B_{N,3}$

We first show a constant K such that $\|\cdot\|_p^{B_{2N,3}} \le K\|\cdot\|_p^{B_{N,3}}$ on $B_{N,3}$. We will consider two cases $p\in[1,\infty[$ and $p=\infty$.

Lemma 7. *Let $p\in[1,\infty]$ and let $\beta\in B_{N,3}$, we have*

$$\|\beta\|_p^{B_{2N,3}} \le 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.$$

Proof. For any N-piece cubic Bézier curve β associated with control points $W_{j3+i}\in\mathbb{R}^n,\ i=0,\dots,3,\ j=0,\dots,N-1$. We have

$$\beta(t)=\beta^{(j)}(Nt-j)=\sum_{i=0}^{3}W_{j3+i}b_{i,3}(Nt-j)$$
$$\text{if } t\in\left[\frac{j}{N},\frac{j+1}{N}\right], j=0,\dots,N-1.$$

By (3), we split β to become a $2N$-piece cubic Bézier curve as follows

$$\beta(t)=\begin{cases}\Gamma^{(2j)}(2Nt-2j)=\sum_{i=0}^{3}P_{2j3+i}b_{i,3}(2Nt-2j)\\ \qquad\qquad \text{if } t\in\left[\frac{2j}{2N},\frac{2j+1}{2N}\right]\\ \Gamma^{(2j+1)}(2Nt-2j-1)\\ \quad =\sum_{i=0}^{3}P_{(2j+1)3+i}b_{i,3}(2Nt-2j-1)\\ \qquad\qquad \text{if } t\in\left[\frac{2j+1}{2N},\frac{2j+2}{2N}\right],\\ \qquad\qquad j=0,\dots,N-1,\end{cases}$$

where

$$\begin{cases}P_{2j3+i}=\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+i-l},\\ P_{(2j+1)3+i}=\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+3-i+l},\\ i=0,\dots,3,\, j=0,\dots,N-1.\end{cases}$$

Case $p\in[1,\infty[$. Since

$$\begin{aligned}\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p =&\sum_{i=0}^{3}\left\|\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+i-l}\right\|_p^p\\ \le& 3\max_{i=0,\dots,3}\|W_{j3+i}\|_p^p\\ \le& 3\sum_{i=0}^{3}\|W_{j3+i}\|_p^p=3\left(\|\beta^{(j)}\|_p^{B_3}\right)^p,\\ &\forall j=0,\dots,N-1,\end{aligned}$$

and similarly

$$\begin{aligned}\left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p =&\sum_{i=0}^{3}\left\|\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j_03+3-i+l}\right\|_p^p\\ \le& 3\left(\|\beta^{(j)}\|_p^{B_3}\right)^p,\forall j=0,\dots,N-1,\end{aligned}$$

we obtain

$$\begin{aligned}&\|\beta\|_p^{B_{2N,3}}\\ =&\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N}\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p+\left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p\right)^{1/p}\\ \le&\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N}6\left(\|\beta^{(j)}\|_p^{B_3}\right)^p\right)^{1/p}\le 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.\end{aligned}$$

Case $p=\infty$. Since

$$\begin{aligned}\|\Gamma^{(2j)}\|_\infty^{B_3}=&\max_{i=0,\dots,3}\left\|\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+i-l}\right\|_\infty\\ \le&\max_{i=0,\dots,3}\|W_{j3+i}\|_\infty\\ =&\|\beta^{(j)}\|_\infty^{B_3},\ \ \forall j=0,\dots,N-1\end{aligned}$$

and similarly

$$\begin{aligned}\|\Gamma^{(2j+1)}\|_\infty^{B_3}=&\max_{i=0,\dots,3}\left\|\sum_{l=0}^{i}b_{l,i}\left(\frac{1}{2}\right)W_{j3+i-l}\right\|_\infty\le\|\beta^{(j)}\|_\infty^{B_3},\\ &\forall j=0,\dots,N-1,\end{aligned}$$

we obtain

$$\begin{aligned}\|\beta\|_\infty^{B_{2N,D}}=&\max_{j=0,\dots,N-1}\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_D},\|\Gamma^{(2j+1)}\|_\infty^{B_D}\right\}\\ \le&\max_{j=0,\dots,N-1}\|\beta^{(j)}\|_\infty^{B_D}=\|\beta\|_\infty^{B_{N,D}}.\end{aligned}$$

From the above two cases, we have the proof of the lemma. □

In order to show a constant k such that $k\|\cdot\|_p^{B_{2N,3}}\le\|\cdot\|_p^{B_{N,3}}$ on the space $B_{N,3}$, we also study two cases $p\in[1,\infty[$ and $p=\infty$.

Lemma 8. *Let $p\in[1,\infty[$ and let $\beta\in B_{N,3}$, we have*

$$\frac{1}{24^{1/p}}\,\|\beta\|_p^{B_{N,3}}\le\|\beta\|_p^{B_{2N,3}}.$$

Proof. For any N-piece cubic Bézier curve β associated with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\dots,3$, $j = 0,\dots,N-1$. We have

$$\beta(t) = \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} P_{j3+i} b_{i,3}(Nt-j)$$
$$\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\dots,N-1.$$

By (3), we split β to become a $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i} b_{i,3}(2Nt-2j) \\ \qquad \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) \\ \quad = \sum_{i=0}^{3} P_{(2j+1)3+i} b_{i,3}(2Nt-2j-1) \\ \qquad \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ \qquad j = 0,\dots,N-1, \end{cases}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\dots,3, j = 0,\dots,N-1. \end{cases}$$

We first consider $\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p$, $j = 0,\dots,N-1$. Set

$$A = \max\left\{\|W_{j3}\|_p, \frac{1}{2}\|W_{j3+1}\|_p, \frac{1}{2}\|W_{j3+2}\|_p, \|W_{j3+3}\|_p\right\}.$$

- Case 1: $A = \|W_{j3}\|_p$.
  We have
$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p \geq \|P_{j3}\|_p^p = \|W_{j3}\|_p^p \\ \geq &\frac{1}{6}\sum_{i=3}^{3}\|W_{j3+i}\|_p^p = \frac{1}{6}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 2: $A = \frac{1}{2}\|W_{j3+1}\|_p$.
  We have
$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p \geq \|P_{2j3+1}\|_p^p \\ = &\left\|\frac{1}{2}W_{j3} + \frac{1}{2}W_{j3+1}\right\|_p^p \geq \frac{1}{4}\|W_{j3+1}\|_p^p \\ \geq &\frac{1}{12}\sum_{i=3}^{3}\|W_{j3+i}\|_p^p = \frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 3: $A = \frac{1}{2}\|W_{j3+2}\|_p$.
  We will estimate $\Gamma^{(2j+1)}$ in this case. This case is similar to Case 2. Then, we get
$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(j+1)}\|_p^{B_3}\right)^p \geq \frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 4: $A = \|W_{j3+4}\|_p$.
  We will estimate $\Gamma^{(2j+1)}$ in this case. This case is similar to Case 1. Then, we get
$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \geq \frac{1}{6}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

From the results of the above four cases, we obtain

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p, \ \forall j = 0,\dots,N-1. \end{aligned}$$

Thus

$$\begin{aligned} &\|\Gamma_{\beta,j_0}\|_p^{B_{2N,D}} \\ = &\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N-1}\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p\right)^{1/p} \\ \geq &\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N-1}\frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p\right)^{1/p} = \frac{1}{24^{1/p}}\|\beta\|_p^{B_{N,D}}. \end{aligned}$$

□

Lemma 9. *Let $\beta \in B_{N,3}$, we get*

$$\frac{1}{4}\|\beta\|_\infty^{B_{N,3}} \leq \|\beta\|_\infty^{B_{2N,3}}.$$

Proof. For any N-piece cubic Bézier curve β associated with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\dots,3$, $j = 0,\dots,N-1$. We have

$$\beta(t) = \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt-j)$$
$$\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\dots,N-1.$$

By (3), we split β to become a $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i} b_{i,3}(2Nt-2j) \\ \qquad \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) \\ \quad = \sum_{i=0}^{3} P_{(2j+1)3+i} b_{i,3}(2Nt-2j-1) \\ \qquad \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ \qquad j = 0,\dots,N-1, \end{cases}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\dots,3, j = 0,\dots,N-1. \end{cases}$$

First, we will consider $\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$, $j = 0,\dots,N-1$. Set

$$A = \max\left\{\|P_{j3}\|_\infty, \frac{1}{2}\|P_{j3+1}\|_\infty, \frac{1}{2}\|P_{j3+2}\|_\infty, \|P_{j3+3}\|_\infty\right\}.$$

- Case 1: $A = \|W_{j3}\|_\infty$.
  We have
  $$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$$
  $$\geq \|\Gamma^{(2j)}\|_\infty^{B_3} \geq \|P_{2j3}\|_\infty = \|W_{j3}\|_\infty = \|\beta^{(j)}\|_\infty^{B_3}.$$

- Case 2: $A = \frac{1}{2}\|W_{j3+1}\|_\infty$.
  We have
  $$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$$
  $$\geq \|\Gamma^{(2j)}\|_\infty^{B_3} \geq \|P_{2j3+1}\|_\infty = \left\|\frac{1}{2}P_{j3} + \frac{1}{2}P_{j3+1}\right\|_\infty$$
  $$\geq \frac{1}{4}\|W_{j3+1}\|_\infty = \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3}.$$

- Case 3: $A = \frac{1}{2}\|W_{j3+2}\|_\infty$.
  We will estimate $\Gamma^{(2j+1)}$ in this case. This case is similar to Case 2. Thus, we get
  $$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$$
  $$\geq \|\Gamma^{(2j+1)}\|_\infty^{B_3} \geq \frac{1}{4}\|\beta^{(j_0)}\|_\infty^{B_3}.$$

- Case 4: $A = \|W_{j3+3}\|_\infty$.
  We estimate $\Gamma^{(2j+1)}$ in this case. This case is similar to Case 1. Then, we get
  $$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$$
  $$\geq \|\Gamma^{(2j+1)}\|_\infty^{B_3} \geq \|\beta^{(j)}\|_\infty^{B_3}.$$

From the results of the above four cases, we obtain

$$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \geq \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3}. \quad (4)$$

Thus

$$\|\beta\|_\infty^{B_{2N,D}} = \max_{j=0,\ldots,N-1} \max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$$
$$\geq \max_{j=0,\ldots,N-1} \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3} = \frac{1}{4}\|\beta\|_\infty^{B_{N,D}}.$$

□

By the above results, we obtain the following theorem.

Theorem 1. *Let $p \in [1,\infty[\cup\{\infty\}$ and let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve. Then*

$$\min\left\{\frac{1}{24^{1/p}}, \frac{1}{4}\right\}\|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{2N,3}} \leq 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.$$

Proof. Using Lemmas 7, 8 and 9, we get the proof of Theorem 1. □

By the above theorem, we obtain the corollary as follows:

$$\min\left\{\frac{1}{24^{1/p}}, \frac{1}{4}\right\} d_p^{B_{N,3}}(\beta - \gamma)$$
$$\leq d_p^{B_{2N,3}}(\beta - \gamma) \leq 3^{1/p}\, d_p^{B_{N,3}}(\beta - \Delta),$$

for any $\beta, \gamma \in B_{N,3}$.

4. CONCLUSION

This article presents the norm $\|\cdot\|_p^{B_{N,3}}$ of piecewise cubic Bézier curves which is defined by control points. This norm is more convenient to compute than the l_p norm. An N-piece cubic Bézier curve can be split and reparametrized to become a $2N$-piece cubic Bézier curve. This way creates extra control points in order to give additional freedom for curve design and avoids increasing the degree of the curve. We also show the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$. These equivalence constants do not depend on the number of pieces. So, the norm $\|\cdot\|_p^{B_{N,3}}$ can be applied to check the convergence for sequences of piecewise cubic Bézier curves. This result is important for applying piecewise cubic Bézier curves to detect optimal orbits.

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