Các hằng số tương đương của chuẩn $\|\cdot\|_p^{B_{N,3}}$ và chuẩn $\|\cdot\|_p^{B_{2N,3}}$ trong không gian các đường cong Bézier bậc ba N mảnh

Hoàng Văn Đức*

Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam

**Tác giả liên hệ chính. Email: hoangvanduc@qnu.edu.vn*

TÓM TẮT

Bài báo này nghiên cứu sự tương đương giữa chuẩn $\|\cdot\|_p^{B_{N,3}}$ và chuẩn $\|\cdot\|_p^{B_{2N,3}}$ trong không gian $B_{N,3}$ của các đường cong Bézier N khúc bậc ba. Các đường cong này là phổ biến nhất trong việc xấp xỉ các đường cong liên tục. Các đường cong Bézier N khúc bậc ba được xác định thông qua các điểm điều khiển. Các chuẩn $\|\cdot\|_p^{B_{N,3}}$ và $\|\cdot\|_p^{B_{2N,3}}$ trong không gian $B_{N,3}$ được tính toán bởi các điểm điều khiển. Để tăng thêm sự tự do trong việc thiết kế đường cong và tránh việc tăng bậc của đường cong, một đường cong Bézier N khúc bậc ba có thể chia tách thành một đường cong Bézier $2N$ khúc bậc ba. Chúng ta sẽ nghiên cứu các hằng số tương đương giữa chuẩn $\|\cdot\|_p^{B_{N,3}}$ và chuẩn $\|\cdot\|_p^{B_{2N,3}}$ trên không gian $B_{N,3}$ của các đường cong Bézier N khúc bậc ba. Vì vậy, chúng ta có thể sử dụng chuẩn $\|\cdot\|_p^{B_{N,3}}$ để xét sự hội tụ của chuỗi các đường cong Bézier từng khúc khúc bậc ba. Kết quả này quan trọng trong việc sử dụng các đường cong Bézier từng khúc bậc ba để tìm quỹ đạo tối ưu.

Từ khóa: *Đường cong Bezier bậc ba, hằng số tương đương, chuẩn, khoảng cách.*

Equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$ on the space of N-piece cubic Bézier curves

Hoang Van Duc*

Faculty of Mathematics and Statistics, Quy Nhon University, Vietnam

* *Corresponding author. Email: hoangvanduc@qnu.edu.vn*

ABSTRACT

This article is concerned with the equivalence relations between the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$ of N-piece cubic Bézier curves. These curves are most common to approximate continuous curves. N-piece cubic Bézier curves are defined through control points. The norms $\|\cdot\|_p^{B_{N,3}}$ and $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$ are determined through control points. In order to give additional freedom for curve design and avoids increasing the degree of the curve, an N-piece cubic Bézier curve can be split to become a $2N-$piece cubic Bézier curve. We will study the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$ of N-piece cubic Bézier curves. So, we can use the norm $\|\cdot\|_p^{B_{N,3}}$ to consider the convergence for sequences of piecewise cubic Bézier curves. This result is important for using piecewise cubic Bézier curves to find optimal trajectories.

Key words: *Cubic Bézier curves, equivalence constants, norm, distance.*

1. INTRODUCTION

In mathematics and engineering, there are many curves which has complex shapes or curves which given by a set of points. To overcome this difficulty, we create a new curve that closely matches an existing one, often to simplify a complex shape or to fit a set of data points. Some methods to approximate the curve include using simpler curves or interpolating polynomials or Bézier curves.

In the method using simpler curves to approximate the curve, we divide the curve into a series of points and connect them with straight lines or circular arcs, with more points resulting in a closer approximation. The main advantage of approximating curves with straight lines or circular arcs is computational simplicity and efficiency, as line segments are defined by fewer parameters than higher-order curves. This method is also useful for data compression and noise reduction by simplifying complex curves. However, approximating a curve with simple curves can lead to inaccuracies, especially in areas of high curvature, and may result in poor extrapolation beyond the measured data range. ([1–9])

Beside, a curve can be approximated by an interpolating polynomial such as a Lagrange polynomial or a Newton polynomial, which fits a curve through a set of known points on the curve. This method improve the approximation as the degree of the polynomial increases. Interpolating polynomials passing exactly through specified points. The main advantages of using interpolating polynomials for approximation include their high accuracy for small datasets, the ability to obtain an explicit function for calculations, and the ease of differentiation and integration. They also provide exact results at the given data points and can be used for data points that are not equally spaced. The main disadvantages of using interpolating polynomials for curve approximation are Runge's phenomenon (oscillations, especially at the endpoints), computational expense for high-degree polynomials, and poor extrapolation properties, where the curve can behave erratically outside the range of the data points (see [10–12]).

Approximating a curve with a Bézier curve involves selecting key points on the original curve to serve as control points and endpoints for the Bézier curve(s). For a given curve, this is often achieved by dividing it into segments and approximating each segment with a Bézier curve, using techniques like the de Casteljau's algorithm for subdivision or fitting algorithms like the Adaptive Extension Fitting Scheme to find the optimal control points for a set of segments. The advantages of Béziers' construction were many. Firstly, the curves were created by moving control points, rather than by making complicated mathematical calculations, which made the tool intuitive even for designers who had no mathematical background. Secondly, each curve was uniquely determined by a few control points, making the method ideal from a data storage point of view - each curve required very little memory. Thirdly, the curves were easy to move, stretch and rotate - all that was required was to move, stretch or rotate the control points accordingly (see [1,13,14]).

In 1959, the mathematician Paul de Casteljau built Bézier curve by using de Casteljau's algorithm while working for the French automaker Citroën. He was the first to apply this method to computer-aided design (CAD). However, his work remained a company secret and was not published for many years. So, his contributions were not widely known at the time.

The Bézier curve was publicized by the French engineer Pierre Bézier in 1962. He defined the Bézier curve based on Bernstein polynomials. Pierre Bézier applied Bézier curves for designing the bodywork of Renault cars. Bézier developed the notation, consisting of nodes with attached control handles, with which the curves are represented in computer software.

Bézier curves were adopted as the standard curve of the PostScript language and subsequently were adopted by vector programs such as Adobe Illustrator, CorelDRAW and Inkscape. Most outline fonts, including TrueType and PostScript Type 1, are defined with Bézier curves. Its importance is due to the fact that, Bézier curves are used in many fields of applications, not only mathematics. Bézier curves are used in computer graphics, computer-aided design system, robotic, industry, walking, communication, path-planning and aerospace (see [15–22]). Bézier curves are also used to find plane shape optimization which appears in many fields such as environment design, aerospace, structural mechanics, networks, automotive, hydraulic, oceanology and wind engineering (see [23–28]).

Bézier curves are presented in many books and articles for instance [1,13,14]. A continuous curve can be approximated by a Bézier curve. However, when the curve is long and complex, the degree of the Bézier curve is high. As a result, the computation is more difficult. Then, the most common use of Bézier curves is as N-piece cubic Bézier curves. We will focus uniform N-piece cubic Bézier curves.

From [29], we have the norm $\|\cdot\|_p^{B_m}$ on the space B_m of Bézier curves of degree m and the norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$ of uniform N-piece Bézier curves of degree m. These norms are computed through control points.

A uniform N-piece cubic Bézier curve can be split to become a uniform $2N$-piece cubic Bézier curve. This approach creates extra control points in order to give additional freedom for curve design and avoids increasing the degree of the curve. Splitting piecewise cubic Bézier curves plays an important role to using piecewise uniform cubic Bézier curves. So, we investigate the equivalence constants for the norm $\|\cdot\|_p^{B_{2N,3}}$ and the norm $\|\cdot\|_p^{B_{N,3}}$ on the space $B_{N,3}$.

Theorem 1. *Let $p \in [1,\infty[\cup\{\infty\}$ and let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve. Then*

$$\min\left\{\frac{1}{24^{1/p}},\frac{1}{4}\right\}\|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{2N,3}} \leq 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.$$

2. Preliminaries

In this section, we briefly recall some definitions and notations that will be used through the article.

Definition 2. ([1] chapter 6, p. 141) Given four points P_0, P_1, P_2 and P_3, the *cubic Bézier curve* associated with the four control points $P_0,\dots,P_3$ is defined by

$$B([P_0,\dots,P_3],t) := \sum_{i=0}^{3} P_i b_{i,3}(t) \quad \text{for } t \in [0,1], \qquad (1)$$

where $b_{i,3}(t) = \binom{3}{i} t^i (1-t)^{3-i}$ is the Bernstein polynomial.

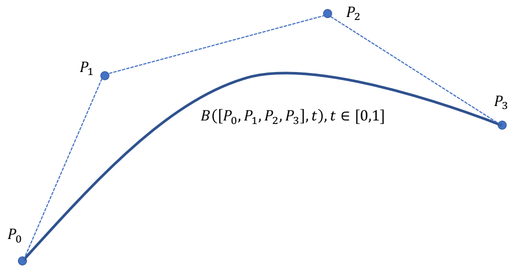

Figure 1. A cubic Bézier curve

The points P_i are called control points for the cubic Bézier curve. The polygon formed by connecting the control points with lines, starting with P_0 and finishing with P_3, is called the control polygon. The convex hull of the control polygon contains the cubic Bézier curve.

A uniform N-piece cubic Bézier curve is a piecewise cubic Bézier curve which has N pieces, each piece is a cubic Bézier curve and the point at $t = \frac{j}{N}$, $j = 1,\dots,N-1$, is the connecting point of the pieces. We often drop "uniform". Let us consider the definition of the N-piece cubic Bézier curve.

Definition 3. ([1] chapter 7, p. 169) Let N be positive integers and $W_0,\dots,W_{N3}$ be $N3+1$ points in $\mathbb{R}^n$. The *N-piece cubic Bézier curve* with control points $W_0,\dots,W_{N3}$ is formed by

$$\begin{aligned} \beta : [0,1] &\to \mathbb{R}^n \\ t &\mapsto \beta(t) = B\left([W_{j3},\dots,W_{j3+3}], Nt-j\right) \\ &\text{if } t \in \left[\frac{j}{N},\frac{j+1}{N}\right]. \end{aligned}$$

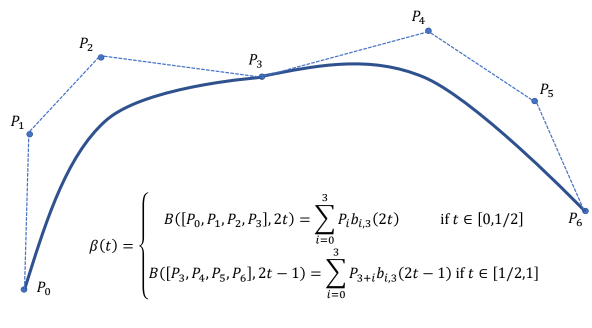

Figure 2. A two-piece cubic Bézier curve

Notation 4.

- The vector space of cubic Bézier curves is denoted by the symbol B_3.
- The vector space of N-piece cubic Bézier curves is denoted by the symbol $B_{N,3}$.

We define some norms and distances through control points on the space of cubic Bézier curves and on the space of N-piece cubic Bézier curves.

Definition 5. Let $p \in [1,\infty]$. The function $\|\cdot\|_p^{B_3} : B_m \to \mathbb{R}$ is defined by: For any $\beta(t) = \sum_{i=0}^{m} W_i b_{i,m}(t) \in B_3$,

$$\|\beta\|_p^{B_3} := \begin{cases} \left(\sum_{i=0}^{3} \|W_i\|_p^p\right)^{1/p} & \text{if } p \in [1,\infty[\\ \max_{i=0,\ldots,3} \{\|W_i\|_\infty\} & \text{if } p = \infty, \end{cases}$$

where $\|\cdot\|_p$ is the p-norm on $\mathbb{R}^n$.

From the properties of the p-norm on $\mathbb{R}^n$ and the Minkowski inequality, it is easily seen that $\|\cdot\|_p^{B_3}$ is a norm on the vector space B_3. Indeed, it is a norm on the space $(\mathbb{R}^n)^{3+1}$ of control polygons. We then have an induced distance on B_3 by $d_p^{B_3}(\beta,\gamma) := \|\beta-\gamma\|_p^{B_3}$.

Definition 6. Let $p \in [1,\infty]$. The function $\|\cdot\|_p^{B_{N,3}} : B_{N,3} \to \mathbb{R}$ is defined by: For any $\beta(t) = \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt-j)$ if $t \in \left[\frac{j}{N}, \frac{j+1}{N}\right]$, $j = 0,\ldots,N-1$,

$$\|\beta\|_p^{B_{N,3}} := \begin{cases} \frac{1}{N^{1/p}} \left(\sum_{j=0}^{N-1} \left(\|\beta^{(j)}\|_p^{B_3}\right)^p\right)^{1/p} & \\ & \text{if } p \in [1,\infty[\\ \max_{j=0,\ldots,N-1} \left\{\|\beta^{(j)}\|_\infty^{B_3}\right\} & \text{if } p = \infty. \end{cases}$$

Using the Minkowski inequality and the properties of the norm $\|\cdot\|_p^{B_3}$ on B_3, it is easy to see that $\|\cdot\|_p^{B_{N,3}}$ is a norm on the vector space $B_{N,3}$. Then we have again an induced distance on $B_{N,3}$ defined by $d_p^{B_{N,3}}(\beta,\gamma) := \|\beta-\gamma\|_p^{B_{N,3}}$.

The norms $\|\cdot\|_p^{B_3}$ and $\|\cdot\|_p^{B_{N,3}}$ can be computed more efficiently than, for instance, the L_p- norm.

The purpose of this paragraph is to rewrite an N-piece cubic Bézier curve as a new piecewise cubic Bézier curve such that the number of pieces in the new piecewise cubic Bézier curve is greater than N. We will see that when we split every piece of an N-piece cubic Bézier curve at the middle point of the piece, we get a $2N$-piece cubic Bézier curve.

Let $\beta \in B_3$ be a cubic Bézier curve with control points $W_i \in \mathbb{R}^n$, $i = 0,\ldots,3$. So

$$\begin{aligned} \beta(t) =& B([P_0,P_1,P_2,P_3],t) = \sum_{i=0}^{3} W_i b_{i,3}(t), \\ &=(1-t)^3 W_0 + 3(1-t)^2 t W_1 + 3(1-t)t^2 W_2 + t^3 W_3, \\ &\text{for } t \in [0,1]. \end{aligned}$$

Form the recursive property of Bernstein polynomials, a cubic Bézier curve can be recursively determined as a convex combination of two quadratic Bezier curves as

$$\begin{aligned} \beta(t) =& B([W_0,W_1,W_2,W_3],t) \\ &=(1-t)\left((1-t)^2 W_0 + 2(1-t)tW_1 + t^2 W_2\right) \\ &\quad + t\left((1-t)^2 W_1 + 2(1-t)tW_2 + t^2 W_3\right) \\ &=(1-t)B([W_0,W_1,W_2],t) + tB([W_1,W_2,W_3],t). \end{aligned}$$

Since $b'_{i,n}(x) = n\left(b_{i-1,n-1}(x) - b_{i,n-1}(x)\right)$, The derivative of the cubic Bézier curve with respect to t is another Bezier curve of one degree lower and given by

$$\begin{aligned} &\frac{d}{dt}\beta(t) = \frac{d}{dt}B([W_0,W_1,W_2,W_3],t) \\ &=3(1-t)^2(W_1-W_0) + 6(1-t)t(W_2-W_1) + 3t^2(W_3-W_2) \\ &=3B([W_1-W_0,W_2-W_1,W_3-W_2],t). \end{aligned}$$

By ([2] chapter 9, p. 201), we can split β at any $t_0 \in (0,1)$. When we split β at $t_0 \neq \frac{1}{2}$, we get 2 cubic Bézier curves but the point $\beta\left(\frac{1}{2}\right)$ is not the connecting point of the pieces. In order to get a uniform 2-piece cubic Bézier curve, we split β at $t = \frac{1}{2}$ and obtain a uniform two-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \beta^{(0)}(2t) = \sum_{i=0}^{3} P_i b_{i,3}(2t) & \text{if } t \in \left[0,\frac{1}{2}\right] \\ \beta^{(1)}(2t-1) = \sum_{i=0}^{3} P_{3+i} b_{i,3}(2t-1) & \text{if } t \in \left[\frac{1}{2},1\right], \end{cases} \tag{2}$$

where

$$\begin{cases} P_i = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{i-l}, & i = 0,\ldots,3, \\ P_{3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{3-i+l}, & i = 0,\ldots,3. \end{cases}$$

Then a cubic Bézier curve can be considered as a uniform two-piece cubic Bézier curve.

More generally, let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\ldots,3$, $j = 0,\ldots,N-1$. So

$$\begin{aligned} \beta(t) =& \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt-j) \\ &\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\ldots,N-1. \end{aligned}$$

If we just split a piece of β, we get $N+1$ pieces but some points at $t = \frac{j}{N+1}$, $j = 1,\ldots,N$, are not the connecting points of the pieces. Then we split every piece of β at the middle point of the piece and obtain a uniform $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i} b_{i,3}(2Nt-2j) \\ \qquad \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) \\ \quad = \sum_{i=0}^{3} P_{(2j+1)3+i} b_{i,3}(2Nt-2j-1) \\ \qquad \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ \qquad j = 0,\ldots,N-1, \end{cases} \tag{3}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\ldots,3, j = 0,\ldots,N-1. \end{cases}$$

So, β can be considered as a $2N$-piece cubic Bézier curve. This means that the space $B_{N,3}$ is a subspace of the space $B_{2N,3}$ and the space $B_{N,3}$ inherits the norm $\|\cdot\|_p^{B_{2N,3}}$. We next study the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$ on the space $B_{N,3}$.

3. Equivalence constants for the norms $\|\cdot\|_p^{B_{2N,3}}$ and $\|\cdot\|_p^{B_{N,3}}$ on $B_{N,3}$

We first find a constant M such that $\|\cdot\|_p^{B_{2N,3}} \leq M\|\cdot\|_p^{B_{N,3}}$ on $B_{N,3}$. We will consider two cases $p \in [1,\infty[$ and $p = \infty$.

Lemma 7. *Let $p \in [1,\infty]$ and let $\beta \in B_{N,3}$, we have*

$$\|\beta\|_p^{B_{2N,3}} \leq 3^{1/p}\, \|\beta\|_p^{B_{N,3}}.$$

Proof. For any N-piece cubic Bézier curve $\beta \in B_{N,3}$ with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\dots,3$, $j = 0,\dots,N-1$. We have

$$\beta(t) = \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} W_{j3+i} b_{i,3}(Nt-j) \quad \text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\dots,N-1.$$

By (3), β can be considered as a $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i} b_{i,3}(2Nt-2j) & \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) = \sum_{i=0}^{3} P_{(2j+1)3+i} b_{i,3}(2Nt-2j-1) & \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ & j = 0,\dots,N-1, \end{cases}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\dots,3, j = 0,\dots,N-1. \end{cases}$$

Case $p \in [1,\infty[$. Since

$$\begin{aligned}\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p &= \sum_{i=0}^{3} \left\| \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l} \right\|_p^p \\ &\leq 3 \max_{i=0,\dots,3} \|W_{j3+i}\|_p^p \\ &\leq 3 \sum_{i=0}^{3} \|W_{j3+i}\|_p^p = 3\left(\|\beta^{(j)}\|_p^{B_3}\right)^p, \\ &\forall j = 0,\dots,N-1,\end{aligned}$$

and similarly

$$\begin{aligned}\left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p &= \sum_{i=0}^{3} \left\| \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j_0 3+3-i+l} \right\|_p^p \\ &\leq 3\left(\|\beta^{(j)}\|_p^{B_3}\right)^p, \forall j = 0,\dots,N-1,\end{aligned}$$

we obtain

$$\begin{aligned}\|\beta\|_p^{B_{2N,3}} &= \frac{1}{(2N)^{1/p}} \left(\sum_{j=0}^{N} \left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \right)^{1/p} \\ &\leq \frac{1}{(2N)^{1/p}} \left(\sum_{j=0}^{N} 6\left(\|\beta^{(j)}\|_p^{B_3}\right)^p \right)^{1/p} \leq 3^{1/p}\, \|\beta\|_p^{B_{N,3}}.\end{aligned}$$

Case $p = \infty$. Since

$$\begin{aligned}\|\Gamma^{(2j)}\|_\infty^{B_3} &= \max_{i=0,\dots,3} \left\| \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l} \right\|_\infty \\ &\leq \max_{i=0,\dots,3} \|W_{j3+i}\|_\infty \\ &= \|\beta^{(j)}\|_\infty^{B_3}, \ \forall j = 0,\dots,N-1\end{aligned}$$

and similarly

$$\|\Gamma^{(2j+1)}\|_\infty^{B_3} = \max_{i=0,\dots,3} \left\| \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l} \right\|_\infty \leq \|\beta^{(j)}\|_\infty^{B_3}, \quad \forall j = 0,\dots,N-1,$$

we obtain

$$\begin{aligned}\|\beta\|_\infty^{B_{2N,D}} &= \max_{j=0,\dots,N-1} \max\left\{ \|\Gamma^{(2j)}\|_\infty^{B_D}, \|\Gamma^{(2j+1)}\|_\infty^{B_D} \right\} \\ &\leq \max_{j=0,\dots,N-1} \|\beta^{(j)}\|_\infty^{B_D} = \|\beta\|_\infty^{B_{N,D}}.\end{aligned}$$

From the above two cases, we have the proof of the lemma. □

In order to find a constant m such that $m\|\cdot\|_p^{B_{2N,3}} \leq \|\cdot\|_p^{B_{N,3}}$ on the space $B_{N,3}$, we also study two cases $p \in [1,\infty[$ and $p = \infty$.

Lemma 8. *Let $p \in [1,\infty[$. For any $\beta \in B_{N,3}$, we have*

$$\frac{1}{24^{1/p}}\, \|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{2N,3}}.$$

Proof. For any N-piece cubic Bézier curve $\beta \in B_{N,3}$ with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\dots,3$, $j = 0,\dots,N-1$. We have

$$\beta(t) = \beta^{(j)}(Nt-j) = \sum_{i=0}^{3} P_{j3+i} b_{i,3}(Nt-j) \quad \text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\dots,N-1.$$

By (3), β can be considered as a $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i} b_{i,3}(2Nt-2j) & \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) = \sum_{i=0}^{3} P_{(2j+1)3+i} b_{i,3}(2Nt-2j-1) & \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ & j = 0,\dots,N-1, \end{cases}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\ldots,3, j = 0,\ldots,N-1. \end{cases}$$

We first consider $\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p$, $j = 0,\ldots,N-1$. Set

$$A = \max\left\{\|W_{j3}\|_p, \frac{1}{2}\|W_{j3+1}\|_p, \frac{1}{2}\|W_{j3+2}\|_p, \|W_{j3+3}\|_p\right\}.$$

- Case 1: $A = \|W_{j3}\|_p$.
  We have

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p \geq \|P_{j3}\|_p^p = \|W_{j3}\|_p^p \\ \geq &\frac{1}{6}\sum_{i=3}^{3}\|W_{j3+i}\|_p^p = \frac{1}{6}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 2: $A = \frac{1}{2}\|W_{j3+1}\|_p$.
  We have

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p \geq \|P_{2j3+1}\|_p^p \\ = &\left\|\frac{1}{2}W_{j3} + \frac{1}{2}W_{j3+1}\right\|_p^p \geq \frac{1}{4}\|W_{j3+1}\|_p^p \\ \geq &\frac{1}{12}\sum_{i=3}^{3}\|W_{j3+i}\|_p^p = \frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 3: $A = \frac{1}{2}\|W_{j3+2}\|_p$.
  In this case, we determine $\Gamma^{(2j+1)}$. As similar to Case 2, we get

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(j+1)}\|_p^{B_3}\right)^p \geq \frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

- Case 4: $A = \|W_{j3+4}\|_p$.
  In this case, we determine $\Gamma^{(2j+1)}$. As similar to Case 1, we get

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \geq \frac{1}{6}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p. \end{aligned}$$

From the results of the above four cases, we obtain

$$\begin{aligned} &\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p \\ \geq &\frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p, \ \forall j = 0,\ldots,N-1. \end{aligned}$$

Thus

$$\begin{aligned} &\|\Gamma_{\beta,j_0}\|_p^{B_{2N,D}} \\ = &\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N-1}\left(\|\Gamma^{(2j)}\|_p^{B_3}\right)^p + \left(\|\Gamma^{(2j+1)}\|_p^{B_3}\right)^p\right)^{1/p} \\ \geq &\frac{1}{(2N)^{1/p}}\left(\sum_{j=0}^{N-1}\frac{1}{12}\left(\|\beta^{(j)}\|_p^{B_3}\right)^p\right)^{1/p} = \frac{1}{24^{1/p}}\|\beta\|_p^{B_{N,D}}. \end{aligned}$$

□

Lemma 9. *Let $\beta \in B_{N,3}$, we have*

$$\frac{1}{4}\|\beta\|_\infty^{B_{N,3}} \leq \|\beta\|_\infty^{B_{2N,3}}.$$

Proof. For any $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve with control points $W_{j3+i} \in \mathbb{R}^n$, $i = 0,\ldots,3$, $j = 0,\ldots,N-1$. We have

$$\begin{aligned} \beta(t) = &\beta^{(j)}(Nt-j) = \sum_{i=0}^{3} W_{j3+i}b_{i,3}(Nt-j) \\ &\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 0,\ldots,N-1. \end{aligned}$$

By (3), β can be considered as a $2N$-piece cubic Bézier curve as follows

$$\beta(t) = \begin{cases} \Gamma^{(2j)}(2Nt-2j) = \sum_{i=0}^{3} P_{2j3+i}b_{i,3}(2Nt-2j) \\ \qquad \text{if } t \in \left[\frac{2j}{2N}, \frac{2j+1}{2N}\right] \\ \Gamma^{(2j+1)}(2Nt-2j-1) \\ \quad = \sum_{i=0}^{3} P_{(2j+1)3+i}b_{i,3}(2Nt-2j-1) \\ \qquad \text{if } t \in \left[\frac{2j+1}{2N}, \frac{2j+2}{2N}\right], \\ \qquad j = 0,\ldots,N-1, \end{cases}$$

where

$$\begin{cases} P_{2j3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+i-l}, \\ P_{(2j+1)3+i} = \sum_{l=0}^{i} b_{l,i}\left(\frac{1}{2}\right) W_{j3+3-i+l}, \\ i = 0,\ldots,3, j = 0,\ldots,N-1. \end{cases}$$

First, we will consider $\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\}$, $j = 0,\ldots,N-1$. Set

$$A = \max\left\{\|P_{j3}\|_\infty, \frac{1}{2}\|P_{j3+1}\|_\infty, \frac{1}{2}\|P_{j3+2}\|_\infty, \|P_{j3+3}\|_\infty\right\}.$$

- Case 1: $A = \|W_{j3}\|_\infty$.
  We have

$$\begin{aligned} &\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \\ \geq &\|\Gamma^{(2j)}\|_\infty^{B_3} \geq \|P_{2j3}\|_\infty = \|W_{j3}\|_\infty = \|\beta^{(j)}\|_\infty^{B_3}. \end{aligned}$$

- Case 2: $A = \frac{1}{2}\|W_{j3+1}\|_\infty$.
  We have

$$\begin{aligned} &\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \\ \geq &\|\Gamma^{(2j)}\|_\infty^{B_3} \geq \|P_{2j3+1}\|_\infty = \left\|\frac{1}{2}P_{j3} + \frac{1}{2}P_{j3+1}\right\|_\infty \\ \geq &\frac{1}{4}\|W_{j3+1}\|_\infty = \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3}. \end{aligned}$$

- Case 3: $A = \frac{1}{2}\|W_{j3+2}\|_\infty$.
  In this case, we determine $\Gamma^{(2j+1)}$. This case is similar to Case 2. Thus, we get

$$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \geq \|\Gamma^{(2j+1)}\|_\infty^{B_3} \geq \frac{1}{4}\|\beta^{(j_0)}\|_\infty^{B_3}.$$

- Case 4: $A = \|W_{j3+3}\|_\infty$.
  In this case, we estimate $\Gamma^{(2j+1)}$. This case is similar to Case 1. Then, we get

$$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \geq \|\Gamma^{(2j+1)}\|_\infty^{B_3} \geq \|\beta^{(j)}\|_\infty^{B_3}.$$

From the results of the above four cases, we obtain

$$\max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \geq \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3}. \quad (4)$$

Thus

$$\|\beta\|_\infty^{B_{2N,D}} = \max_{j=0,\dots,N-1} \max\left\{\|\Gamma^{(2j)}\|_\infty^{B_3}, \|\Gamma^{(2j+1)}\|_\infty^{B_3}\right\} \geq \max_{j=0,\dots,N-1} \frac{1}{4}\|\beta^{(j)}\|_\infty^{B_3} = \frac{1}{4}\|\beta\|_\infty^{B_{N,D}}.$$

□

Combining the above two propositions, we have the following theorem.

Theorem 1. *Let $p \in [1,\infty[\cup\{\infty\}$ and let $\beta \in B_{N,3}$ be an N-piece cubic Bézier curve. Then*

$$\min\left\{\frac{1}{24^{1/p}}, \frac{1}{4}\right\}\|\beta\|_p^{B_{N,3}} \leq \|\beta\|_p^{B_{2N,3}} \leq 3^{1/p}\,\|\beta\|_p^{B_{N,3}}.$$

Proof. Using Lemmas 7, 8 and 9, we get the proof of this theorem. □

From the above theorem, we obtain the following corollary:

$$\min\left\{\frac{1}{24^{1/p}}, \frac{1}{4}\right\} d_p^{B_{N,3}}(\beta-\gamma) \leq d_p^{B_{2N,3}}(\beta-\gamma) \leq 3^{1/p}\, d_p^{B_{N,3}}(\beta-\Delta),$$

for any $\beta, \gamma \in B_{N,3}$.

4. Conclusion

This article presents the norm $\|\cdot\|_p^{B_{N,3}}$ of piecewise cubic Bézier curves which is defined by control points. This norm is more convenient to compute than the l_p norm. An N-piece cubic Bézier curve can be split and reparametrized to become a $2N$-piece cubic Bézier curve. This way creates extra control points in order to give additional freedom for curve design and avoids increasing the degree of the curve. We also show the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm $\|\cdot\|_p^{B_{2N,3}}$. These equivalence constants do not depend on the number of pieces. Thus, we can use the norm $\|\cdot\|_p^{B_{N,3}}$ to consider the convergence for sequences of piecewise cubic Bézier curves. This result is important for using piecewise cubic Bézier curves to find optimal trajectories.

References

1. D. Marsh. *Applied geometry for computer graphics and CAD*, 2nd edition, Springer-Verlag London, London, 2005.
2. Danaila, Ionut and Joly, Pascal and Kaber, Sidi Mahmoud and Postel, Marie. An introduction to scientific computing, Springer, New York,2007.
3. R. Hassin. Approximation algorithms for hitting objects with straight lines, Discrete Applied Mathematics, **1991**, *30*, 29-42.
4. C. M. Williams. Bounded straight-line approximation of digitized planar curves and lines, Computer Graphics and Image Processing, **1981**, *16*(4), 370-381.
5. S Arthur. *Linear Approximation*, American Mathematical Society, Providence, Rhode Island, 1963.
6. L. N. Trefethen. *Approximation Theory and Approximation Practice, Extended Edition*, Society for Industrial and Applied Mathematics, Philadelphia, 2019.
7. E. Bodansky, A. Gribov. Approximation of Polylines with Circular Arcs, Springer, Berlin, **2024**, *3088*, 193-198.
8. S.M. Thomas, Y.T. Chan. A simple approach for the estimation of circular arc center and its radius, Computer Vision, Graphics, and Image Processing, Academic Press, **1989**, *45*, 362-370.
9. C. Ichoku, B. Deffontaines, J. Chorowicz, Segmentation of Digital Plain Curves: A Dynamic Focusing Approach, Pattern Recognition Letters, Elsevier BV, **1996**, *17*, 741–750.
10. R. L. Burden, D. J. Faires. *Numerical Analysis, Cengage Learning*, Boston, 2010.
11. S. C. Chapra, R. P. Canale. *Numerical Methods for Engineers*, McGraw-Hill Education, New York, 2015.
12. T. Heister, L. G. Rebholz, F. Xue. *Numerical Analysis: An Introduction*, De Gruyter, Berlin, 2019.
13. J. Gallier. *Curves and surfaces in geometric modeling*, Morgan Kaufmann, San Francisco, California, 2000.
14. T. W. Sederberg. Applications to computer aided geometric design, *American Mathematical Society*, **1998**, *53*, 67–89.
15. J. E. McIntyre. *Guidance, flight mechanics and trajectory optimization*, National Aeronautics and Space Administration, Washington, D.C., 1968.

16. F. Bullo, J. Cortés, S. Martínez. *Distributed control of robotic networks*, Princeton University Press, Princeton, New Jersey, 2009.

17. M. Farber. *Invitation to topological robotics*, European Mathematical Society (EMS), Zürich, 2008.

18. J. von Neumann. *Collected works. Vol. VI: Theory of games, astrophysics, hydrodynamics and meteorology*, The Macmillan Co., New York, 1963.

19. K. J. Worsley. Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics, *Advances in Applied Probability*, **1995**, *27*, 943–959.

20. M. Farber. Topological complexity of motion planning, *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, **2003**, *29*, 211–221.

21. M. Farber, S. Tabachnikov, S. Yuzvinsky. Topological robotics: motion planning in projective spaces, *International Mathematics Research Notices*, **2003**, *34*, 1853–1870.

22. R. M. Murray, S. S. Sastry. Nonholonomic motion planning: steering using sinusoids, *Institute of Electrical and Electronics Engineers. Transactions on Automatic Control* , **1993**, *38*(5), 700–716.

23. J. Sokolowski, J.-P. Zolésio. *Introduction to shape optimization*, Springer-Verlag, Berlin, 1992.

24. J. Haslinger, R. A. E. Mäkinen. *Introduction to shape optimization*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pennsylvania, 2003.

25. G. Allaire. *Shape optimization by the homogenization method*, Springer-Verlag, New York, 2002.

26. B. Mohammadi, O. Pironneau. *Applied shape optimization for fluids*, 2^{nd} edition, Oxford University Press, Oxford, 2010.

27. F. de Gournay, G. Allaire, F. Jouve. Shape and topology optimization of the robust compliance via the level set method, *ESAIM. Control, Optimisation and Calculus of Variations*, **2008**, *14*(1), 43–70.

28. D. Bucur, G. Buttazzo. *Variational methods in some shape optimization problems*, Scuola Normale Superiore, Pisa, 2002.

29. H. V. Duc. Equivalence constants for some norms on the space of $N-$piece cubic Bezier curves, Quy Nhon University Journal of Science, **2023**, *17*(5), 91-101.