On sylvester matrix rank functions over commutative semirings

ABSTRACT

In this paper, the author introduces two distinct Sylvester matrix rank functions over quasi-selective semirings without zero divisors. Several fundamental properties of Sylvester matrix rank functions over commutative semirings are investigated, and cases are identified in which Sylvester matrix rank functions over commutative semirings do not behave in the same way as over rings. The paper also provides a comparison between Sylvester matrix rank functions and factor rank of matrices, and establishes some characteristic properties of Sylvester matrix rank functions that satisfy the Nullity law over semirings with a symmetry.

Keywords: *Semiring, Matrix, Sylvester matrix rank function, Nullity law.*

1. INTRODUCTION

The Sylvester matrix rank function over rings has been thoroughly studied and many important results have been obtained. However, the study of Sylvester matrix rank functions over semirings remains rather limited, and many of their properties are significantly more restrictive than in the case of rings. For example, Beasley and Guterman showed that over the semiring of nonnegative real numbers, the factor rank of a matrix satisfies the Nullity law, whereas the Sylvester and Frobenius inequalities are not valid, see [1, Example 4.8]. This behavior is different from that over rings, see [2, Corollary 5.5.2].

In recent years, the rank functions of matrices over semirings has attracted considerable attention from mathematicians, and many characteristic properties have been investigated, see [3]-[6]. However, the results obtained concerning Sylvester matrix rank functions over semirings remain limited. In order to enrich the body of research results and fundamental properties of rank functions over semirings, in this paper we conduct a detailed study of some basic properties of Sylvester matrix rank functions over commutative semirings and semirings with an ε - function (semirings with a symmetry). We also provide several examples illustrating cases in which Sylvester matrix rank functions do not behave as they do over rings.

2. PRELIMINARIES

A *semiring* [3] is a set R containing the elements 0_R and 1_R, equipped with two binary operations, addition $(+)$ and multiplication $(.)$, such that:

i) $(R,+,0_R)$ forms a commutative monoid with identity element 0_R;

ii) $(R,.,1_R)$ forms a monoid with identity element 1_R;

iii) $t(u+v)=tu+tv$, $(t+u)v=tv+uv$ for all $t,u,v\in R$;

iv) $0_Ru=u0_R=0_R$ for all $u\in R$.

When there is no ambiguity, we write 0 for 0_R and 1 for 1_R.

A semiring R is said to be *commutative* if $rs=sr$ for all $r,s\in R$. An element $a\in R$ is called *additively invertible* if there exists $b\in R$ such that $a+b=0$. The set of all additively invertible elements of the semiring R is denoted by $V(R)$. The semiring R is called *zerosumfree* if $V(R)=\{0\}$. If $V(R)=R$, then R is a *ring*. A semiring R is called *entire* if for any $r,s\in R$, $rs=0$ implies that $r=0$ or $s=0$.

Recall from [5] that a commutative semiring R satisfying the following conditions is called a *quasi-selective semiring*:

i) For any $a,b\in R$, there exists an element $c \in R$, $c \neq 0$, such that $ac + bc \in \{ac, bc\}$;

ii) For any $a \in R$, $a + a = a$.

Let S and R be semirings. A map $\alpha : S \to R$ is called a *homomorphism* if it satisfies the following conditions:

i) $\alpha(0_S) = 0_R$;

ii) $\alpha(1_S) = 1_R$;

iii) $\alpha(u+v) = \alpha(u) + \alpha(v)$ for all $u, v \in S$;

iv) $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in S$.

For a given semiring R, the set of all matrices of size $m \times n\,(n \times n)$ is denoted by $M_{m \times n}(R)$ $(M_n(R))$, the set of all matrices is denoted by $M(R)$. A matrix $A \in M_n(R)$ is called *idempotent* if $A^2 = A$. The set of all idempotent matrices of size $n \times n$ over R is denoted by $IM_n(R)$, the set of all idempotent matrices over R is denoted by $IM(R)$. A matrix $M \in M_m(R)$ is called *invertible* if there exists a matrix $N \in M_m(R)$ such that $MN = NM = I_m$. Then, we write $N = M^{-1}$. The set of invertible $m \times m$ matrices over R is denoted by $GL_m(R)$. A matrix $P \in M_n(R)$ is called a *permutation matrix* if in each row and each column of P there is exactly one entry equal to 1, and all remaining entries are equal to 0. Note that $P \in GL_n(R)$ and that $P^{-1} = P^T$, the transpose of P.

According to [2, p.13], two idempotent matrices E, F are said to be *isomorphic*, denoted by $E \cong F$, if there exist matrices M, N such that $E = MN$ and $F = NM$.

According to [1], the *factor rank* of a matrix $A \in M_{m \times n}(R)$, denoted by $f(A)$, is the smallest nonnegative integer k such that $A = MN$ for some matrices $M \in M_{m \times k}(R)$ and $N \in M_{k \times n}(R)$. By convention, the factor rank of the zero matrix is defined to be 0. Several basic properties of the factor rank of matrices over semirings have been proved to be completely analogous to those over rings:

$f(AB) \leq \min\{f(A), f(B)\}$ for all $A \in M_{m \times n}(R)$ and $B \in M_{n \times p}(R)$.

$\max\{f(A), f(B)\} \leq f(A \quad B) \leq f(A) + f(B)$ for all $A \in M_{m \times n}(R)$ and $B \in M_{m \times p}(R)$.

Note that if E and F are idempotent matrices sucht that $E \cong F$, then $f(E) = f(F)$.

Recall from [7] that if R is a commutative semiring, then for all matrices $A, B \in M_n(R)$, the equality $AB = I_n$ implies $BA = I_n$. Consequently, by applying [8, Proposition 3.12], we obtain $f(I_n) = n$ for all positive integer n.

Recall from [5] that a family $\{a_1, a_2, \ldots, a_n\} \subset R^m$, consisting of column vectors over a semiring R, is said to be *Gondran-Minoux linearly dependent* if there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in R$, not all equal to 0, and two subsets $I, J \subseteq \{1, 2, \ldots, n\}$ such that

$$I \cap J = \varnothing,\ I \cup J = \{1, 2, \ldots, n\},$$

and $\sum_{i \in I} \lambda_i a_i = \sum_{j \in J} \lambda_j a_j$. If the family $\{a_1, a_2, \ldots, a_n\}$ is not Gondran-Minoux linearly dependent, it is called *Gondran-Minoux linearly independent*. The *Gondran-Minoux column rank* of a matrix $A \in M_{m \times n}(R)$, denoted by $GMc(A)$, is defined as the maximum number of columns of A that are Gondran-Minoux linearly independent.

Recall from [6] that, a semiring R with an ε- function is a semiring together with a bijection $\varepsilon : R \to R$ satisfying the following condition: For any $r, s \in R$, we have $\varepsilon(r+s) = \varepsilon(r) + \varepsilon(s)$, $\varepsilon(rs) = r\varepsilon(s) = \varepsilon(r)s$, $\varepsilon(\varepsilon(r)) = r$.

For a matrix $A = (a_{ij}) \in M(R)$, we define $\varepsilon(A) \in M(R)$ by $\varepsilon(A) = (\varepsilon(a_{ij}))$.

3. MAIN RESULTS

In this section, we investigate some basic properties of the Sylvester matrix rank function over commutative semirings. Throughout this section, all semirings under consideration are assumed to be commutative.

Definition 3.1 ([9]). Let R be a semiring. A mapping $r: M(R) \to \mathbf{R}^{+}$ is called a Sylvester matrix rank function over the semiring R if it satisfies the following conditions:

i) $r(0)=0; r(I_1)=1$;

ii) $r(AB) \le \min\{r(A), r(B)\}$ for all $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$;

iii) $r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r(A)+r(B)$ for all $A \in M_{m\times n}(R)$ and $B \in M_{p\times q}(R)$;

iv) $r\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \ge r(A)+r(B)$ for all $A \in M_{m\times n}(R)$, $B \in M_{p\times q}(R)$ and $C \in M_{m\times q}(R)$.

Remark 3.2. Assume that r is a Sylvester matrix rank function on the semiring R. Then, for every matrix $A \in M_{m\times n}(R)$, since $A = I_m A = A I_n$, it follows that

$$r(A) \le \min\{r(I_m), r(I_n)\} = \min\{m,n\}.$$

For any matrix $U \in GL_m(R)$ and $V \in GL_n(R)$, we have $r(A) = r(UA) = r(AV)$.

Moreover, $r(A \quad 0) = r\begin{pmatrix} A \\ 0 \end{pmatrix} = r(A)$.

If $E \in IM_n(R)$, $F \in IM_m(R)$ satisfy $E \cong F$, then it is straightforward to verify that $r(E) = r(F)$.

Next, we present several classes of semirings on which there exists at least one Sylvester matrix rank function.

Lemma 3.3. Let R be a entire zerosumfree semiring, and let $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$. If $AB = 0$, then there exists a permutaion matrix $P \in M_n(R)$ such that $AP = (A' \quad 0)$ and $P^{-1}B = \begin{pmatrix} 0 \\ B' \end{pmatrix}$. Where $A' \in M_{m\times k}(R)$ has all columns nonzero, $B' \in M_{(n-k)\times p}(R)$, and $0 \le k \le n$.

Proof.

Let $A = (a_{ij})$ and $B = (b_{ij})$. Since $AB = 0$, for every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,p\}$ we have $\sum_{l=1}^{n} a_{il} b_{lj} = 0$. Because R is an entire zerosumfree semiring, it follows that $a_{il} b_{lj} = 0$ for all $l \in \{1,2,...,n\}$, and hence, for each $l \in \{1,2,...,n\}$, $a_{il} = 0$ or $b_{lj} = 0$.

Consequently, if the l-th column of the matrix A is nonzero, then the l-th row of the matrix B must be the zero row. Suppose that A has exactly k nonzero columns, with $0 \le k \le n$. Then there exists a permutation matrix $P \in M_n(R)$ such that $AP = (A' \quad 0)$, where $A' \in M_{m\times k}(R)$ has all columns nonzero.

Set $P^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where $B_1 \in M_{k\times p}(R)$ and $B_2 \in M_{(n-k)\times p}(R)$. We have

$$0 = AB = APP^{-1}B = A' B_1.$$

Since the matrix A' has all columns nonzero, it follows that $B_1 = 0$. Therefore, $P^{-1}B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}$. □

Theorem 3.4. Let R be an entire zerosumfree semiring. Then the factor rank is a Sylvester matrix rank function.

Proof.

It is clear that $f(0) = 0$ and $f(I_1) = 1$. Moreover, for any matrices $G \in M_{m\times n}(R)$ and $H \in M_{n\times p}(R)$, we have

$$f(GH) \le \min\{f(G), f(H)\}.$$

For any matrices $A \in M_{m\times n}(R)$, $B \in M_{p\times q}(R)$, and $C \in M_{m\times q}(R)$, suppose that $f\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = s$. Then there exist matrices $E \in M_{(m+p)\times s}(R)$, $F \in M_{s\times(n+q)}(R)$ such that $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = EF$.

Set $E=\begin{pmatrix}E_1\\E_2\end{pmatrix}$, $F=\begin{pmatrix}F_1 & F_2\end{pmatrix}$, where $E_1\in M_{m\times s}(R)$, $E_2\in M_{p\times s}(R)$, $F_1\in M_{s\times n}(R)$, and $F_2\in M_{s\times q}(R)$. Then $E_2F_1=0$. Since R is an entire zerosumfree semiring, by Lemma 3.3 there exists a permutation matrix $P\in M_s(R)$ such that $E_2P=\begin{pmatrix}E_3 & 0\end{pmatrix}$, $P^{-1}F_1=\begin{pmatrix}0\\F_3\end{pmatrix}$, where $E_3\in M_{p\times x}(R)$, $F_3\in M_{y\times n}(R)$, and $x+y=s$.

Write $E_1P=\begin{pmatrix}E_4 & E_5\end{pmatrix}$ with $E_4\in M_{m\times x}(R)$ and $E_5\in M_{m\times y}(R)$. Then $A=E_1F_1=(E_1P)(P^{-1}F_1)$, and hence $A=E_5F_3$, which implies $f(A)\le y$.

Similarly, write $P^{-1}F_2=\begin{pmatrix}F_4\\F_5\end{pmatrix}$, where $F_4\in M_{x\times q}(R)$ and $F_5\in M_{y\times q}(R)$. Then $B=E_2F_2=(E_2P)(P^{-1}F_2)$, and hence $B=E_3F_4$, which implies $f(B)\le x$.

Thus, $f\begin{pmatrix}A & C\\0 & B\end{pmatrix}=s=x+y\ge f(A)+f(B)$.

Moreover, if $C=0$, then $f(A)+f(B)\le f\begin{pmatrix}A & 0\\0 & B\end{pmatrix}\le f(A)+f(B)$, and consequently, $f\begin{pmatrix}A & 0\\0 & B\end{pmatrix}=f(A)+f(B)$. □

Next, we provide another Sylvester matrix rank function, different from the factor rank, on the class of entire zerosumfree semirings.

Lemma 3.5. Let R be a semiring and let $A\in M_{m\times n}(R)$, $B\in M_{p\times q}(R)$, and $C\in M_{m\times q}(R)$. Then $GMc\begin{pmatrix}A & C\\0 & B\end{pmatrix}\ge GMc(A)+GMc(B)$.

Proof.

Suppose that $GMc(A)=r$ and $GMc(B)=s$. Without loss of generality, assume that $\{A^1,A^2,...,A^r\}$ and $\{B^1,B^2,...,B^s\}$ are Gondran-Minoux linearly independent systems of column vectors of the matrices A and B, respectively. Consider the following system of column vectors of the matrix $\begin{pmatrix}A & C\\0 & B\end{pmatrix}$:

$$X=\left\{\begin{pmatrix}A^1\\0\end{pmatrix},...,\begin{pmatrix}A^r\\0\end{pmatrix},\begin{pmatrix}C'^{r+1}\\B'^{r+1}\end{pmatrix},...,\begin{pmatrix}C'^{r+s}\\B'^{r+s}\end{pmatrix}\right\},$$

where $B'^{r+j}=B^j$ and $C'^{r+j}=C^j$ for all $j\in\{1,...,s\}$.

Assume that the vectors in X are Gondran-Minoux linearly dependent. Then there exist sets I_1,I_2,J_1,J_2 such that $J_1\cup J_2=\{r+1,...,r+s\}$, $I_1\cup I_2=\{1,...,r\}$, $I_1\cap I_2=\varnothing$, $J_1\cap J_2=\varnothing$ and $\alpha_1,\alpha_2,...,\alpha_{r+s}\in R$ not all zero, satisfying

$$\sum_{i\in I_1}\alpha_i\begin{pmatrix}A^i\\0\end{pmatrix}+\sum_{j\in J_1}\alpha_j\begin{pmatrix}C'^j\\B'^j\end{pmatrix}$$
$$=\sum_{u\in I_2}\alpha_u\begin{pmatrix}A^u\\0\end{pmatrix}+\sum_{v\in J_2}\alpha_v\begin{pmatrix}C'^v\\B'^v\end{pmatrix}.$$

This implies that $\sum_{j\in J_1}\alpha_jB'^j=\sum_{v\in J_2}\alpha_vB'^v$. Since $\{B^1,B^2,...,B^s\}$ is Gondran-Minoux linearly independent, it follows that $\alpha_u=0$ for all $u\in\{r+1,...,r+s\}$.

Consequently, $\sum_{i\in I_1}\alpha_i\begin{pmatrix}A^i\\0\end{pmatrix}=\sum_{u\in I_2}\alpha_u\begin{pmatrix}A^u\\0\end{pmatrix}$, which implies $\sum_{i\in I_1}\alpha_iA^i=\sum_{u\in I_2}\alpha_uA^u$. Since $\{A^1,A^2,...,A^r\}$ is Gondran-Minoux linearly independent, we conclude that $\alpha_i=0$ for all $i\in\{1,...,r\}$. Thus, $\alpha_i=0$ for all $i\in\{1,...,r+s\}$. This contradicts the assumption that $\alpha_1,\alpha_2,...,\alpha_{r+s}$ are not all zero. Therefore, the set X is Gondran-Minoux linearly independent, and hence

$$GMc\begin{pmatrix}A & C\\0 & B\end{pmatrix}\ge r+s=GMc(A)+GMc(B).\ \square$$

Theorem 3.6. Let R be an entire quasi-selective semiring. Then the Gondran-Minoux column rank is a Sylvester matrix rank function on R.

Proof.

Applying Lemma 3.5 and [5, Corollary 2.11] yields the desired result.

Remark 3.7. Entire quasi-selective semirings are entire zerosumfree semirings. Hence, both the factor rank and the Gondran-Minoux column rank of matrices define Sylvester matrix rank functions on such semirings. Moreover, these rank functions are mutually distinct (see Remark 3.12 and 3.14 below).

Next, we investigate some basic properties of Sylvester matrix rank functions on semirings.

Proposition 3.8. Let R be a semiring and let r be a Sylvester matrix rank function on R. Then,

i) $r\left(A \quad B\right) \le r\left(A\right)+r\left(B\right)$ for all $A \in M_{m\times n}\left(R\right)$, $B \in M_{m\times p}\left(R\right)$.

ii) $r\left(A+B\right) \le r\left(A\right)+r\left(B\right)$ for all $A,B \in M_{m\times n}\left(R\right)$.

Proof.

i) We have $\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} I_m & I_m \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and hence $r\left(A \quad B\right) \le r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r\left(A\right)+r\left(B\right)$.

ii) Observe that

$$\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I_m & I_m \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & I_n \\ 0 & I_n \end{pmatrix},$$

and that $r\left(A+B\right) = r\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}$. Therefore, $r\left(A+B\right) \le r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r\left(A\right)+r\left(B\right)$. □

The following result can be proved in a completely analogous way to the ring case.

Proposition 3.9. Let R and S be semirings, and let $g: R \to S$ be a homomorphism. If there exists a Sylvester matrix rank function on S, then there also exists a Sylvester matrix rank function on R.

Definition 3.10. Let r be a Sylvester matrix rank function on a semiring R. A matrix $A \in M_n\left(R\right)$ is called r*-full* if $r\left(A\right) = n$.

Proposition 3.11. Let r be a Sylvester matrix rank function on a semiring R. Assume that r satisfies the condition: For any matrix $A \in M_{m\times n}\left(R\right)$, if $r\left(A\right) = k$, then there exist an idempotent matrix $E \in IM_k\left(R\right)$ and matrices $B \in M_{m\times k}\left(R\right)$, $C \in M_{k\times n}\left(R\right)$ such that $A = BEC$.

Then, for any idempotent matrix M, there exists an r-full idempotent matrix N such that $N \cong M$.

Proof.

Let $E \in IM_n\left(R\right)$ with $r\left(E\right) = k$. Then there exist an idempotent matrix $F \in IM_k\left(R\right)$ and matrices $M \in M_{n\times k}\left(R\right)$, $N \in M_{k\times n}\left(R\right)$ such that $E = MFN$. Set $G = FNEMF \in M_k\left(R\right)$. Since E is idempotent, we have $G^2 = G$, and hence G is idempotent. We have

$$\left(MF\right)\left(FNE\right) = MFNE = E^2 = E,$$

which implies $E \cong G$. Moreover, since r is a Sylvester matrix rank function, $k = r\left(E\right) = r\left(E^3\right) = r\left(MGN\right) \le r\left(G\right) \le k$, and thus $r\left(G\right) = k$. Therefore, G is a r-full idempotent matrix. □

Remark 3.12. The converse of Proposition 3.11 is, in general, false. Indeed, consider the semiring $\mathbf{R}_{\max}$ (in [4]). Note that since $\mathbf{R}_{\max}$ is an entire quasi-selective semiring, the Gondran-Minoux column rank is a Sylvester matrix rank function. Let $E \in IM_n\left(\mathbf{R}_{\max}\right)$ and $GMc\left(E\right) = k$. By [10, Corollary 1.3], $f\left(E\right) = GMc\left(E\right) = k$, and hence there exist matrices $G \in M_{n\times k}\left(\mathbf{R}_{\max}\right)$ and $H \in M_{k\times n}\left(\mathbf{R}_{\max}\right)$ such that $E = GH$. Set $F = HEG$. Then $F^2 = HE^3G = HEG = F$, and thus $F \in IM_k\left(\mathbf{R}_{\max}\right)$. Moreover, $EGH = E^2 = E$, which implies $E \cong F$, and hence $GMc\left(E\right) = GMc\left(F\right) = k$. Therefore, F is a GMc-full idempotent matrix.

Now consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_4\left(\mathbf{R}_{\max}\right).$$

By [4, Example 8.7], we have $GMc\left(A\right) = 3 < 4 = f\left(A\right)$. If there exist an idempotent matrix $F \in IM_3(\mathbf{R}_{\max})$ and matrices $B \in M_{4\times3}(\mathbf{R}_{\max})$, $C \in M_{3\times4}(\mathbf{R}_{\max})$ such that $A = BFC$, then we have $f(A) \le f(F) \le 3$, which contradicts the fact that $f(A) = 4$.

Theorem 3.13. Let r be a Sylvester matrix rank function on a semiring R. Then the following statements hold:

i) For any matrix A, we have $r(A) \le \min\{r(E) | A = BEC, E \in IM(R)\} \le f(A)$.

ii) $r = f$ if and only if r satisfies the condition: For any matrix $A \in M_{m\times n}(R)$, if $r(A) = k$, then there exist an idempotent matrix $E \in IM_k(R)$ and matrices $B \in M_{m\times k}(R)$, $C \in M_{k\times n}(R)$ such that $A = BEC$.

Proof.

i) For any matrix $A \in M_{m\times n}(R)$, and for any idempotent matrix E such that $A = BEC$ for some matrices B, C. Since r is a Sylvester matrix rank function, we have $r(A) \le r(E)$, and hence $r(A) \le \min\{r(E) | A = BEC, E \in IM(R)\}$.

If $f(A) = k$, then there exist matrices $P \in M_{m\times k}(R)$ and $Q \in M_{k\times n}(R)$ such that $A = PQ$, and therefore $A = PI_kQ$. In this case, $\min\{r(E) | A = BEC, E \in IM(R)\} \le r(I_k) = f(A)$.

ii) Assume that $r = f$. For any matrix $A \in M_{m\times n}(R)$, suppose that $r(A) = f(A) = k$. Then there exist matrices $B \in M_{m\times k}(R)$ and $C \in M_{k\times n}(R)$ such that $A = BC$, and hence $A = BI_kC$, with $I_k \in IM_k(R)$.

Conversely, for any matrix $G \in M_{m\times n}(R)$, $r(G) = l$. Then there exist an idempotent matrix $F \in IM_l(R)$ and matrices $M \in M_{m\times l}(R)$, $N \in M_{l\times n}(R)$ such that $G = MFN$. It follows that $f(G) \le f(F) \le l = r(G)$, and hence $f(G) = r(G)$. Thus, $r = f$. □

Remark 3.14. There exist a semiring R, a Sylvester matrix rank function r, and a matrix A such that $r(A) = k$, but there does not exist any idempotent matrix $E \in IM_k(R)$ for which $A = BEC$ for some ma trices B, C. Indeed, consider the semiring $R = [0,1] \subset \mathbf{R}$ equipped with the operations $x + y = \max\{x, y\}$, $xy = \min\{x, y\}$ for all $x, y \in R$. Then R is an entire quasi-selective semiring. Consequently, the Gondran-Minoux column rank is a Sylvester matrix rank function on R (by Theorem 3.6).

Consider the matrix $A = \begin{pmatrix} 0.5 & 1 \\ 0.1 & 1 \end{pmatrix}$. Since $0.1\begin{pmatrix} 0.5 \\ 0.1 \end{pmatrix} = 0.1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $GMc(A) = 1$. Assume that there exists an idempotent matrix $E = (x) \in IM_1(R)$ and matrices $B = \begin{pmatrix} a \\ b \end{pmatrix} \in M_{2\times1}(R)$, $C = (u \quad v) \in M_{1\times2}(R)$ such that $A = BEC$. Then $axu = 0.5$, $axv = 1$, $bxu = 0.1$ and $bxv = 1$, which implies $ax = bx = 1$, and hence $0.5 = u = 0.1$, a contradiction.

Now let r be a Sylvester matrix rank function on a semiring R satisfying the condition: For any idempotent matrix M, there exists an r-full idempotent matrix N such that $M \cong N$.

Then, $f(A) = \min\{r(E) | A = BEC, E \in IM(R)\}$ for all $A \in M(R)$. Indeed, let $A \in M_{m\times n}(R)$, and set $k = \min\{r(E) | A = BEC, E \in IM(R)\}$. Then there exists an idempotent matrix E such that $r(E) = k$ and $A = BEC$ for some $B, C \in M(R)$. Let F be an r-full idempotent matrix such that $F \cong E$. Then $r(F) = r(E) = k$. Consequently, $f(A) = f(BEC) \le f(E) = f(F) = k$.

Next, we examine some characteristic properties of Sylvester matrix rank functions satisfying the Nullity law on semirings.

Theorem 3.15. Let R be an entire zerosumfree semiring, and let r be a Sylvester matrix rank function on R. Then r satisfies the Nullity law. That is, for any matrices $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$, if $AB = 0$, then $r(A) + r(B) \le n$.

Proof.

For any matrices $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$ such that $AB = 0$, by Lemma 3.3 there exists a permutation matrix $P \in M_n(R)$ such that $AP = \begin{pmatrix} A' & 0 \end{pmatrix}$, $P^{-1}B = \begin{pmatrix} 0 \\ B' \end{pmatrix}$. Here $A' \in M_{m\times k}(R)$ has all columns nonzero, $B' \in M_{(n-k)\times p}(R)$ and $0 \le k \le n$. Since P is invertible, we have

$$r(A) = r(AP) = r(A') \le k,$$

and

$$r(B) = r(P^{-1}B) = r(B') \le n - k.$$

Consequently, $r(A) + r(B) \le k + n - k = n$. □

Remark 3.16. If r is a Sylvester matrix rank function on a semiring R satisfying the Sylvester inequality, that is,

$r(A) + r(B) \le r(AB) + n$ for all $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$.

Then, r satisfies the Nullity law. However, the converse is not true in general, see [1, Example 4.8].

Theorem 3.17. Let r be a Sylvester matrix rank function on semiring R satisfying the condition: For any idempotent matrix E, there exists an r-full idempotent matrix F such that $E \cong F$. Then the following statements are equivalent:

i) r satisfies the Nullity law.

ii) For any matrices $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$, if $AB = 0$, then $r(A) + r(B) \le r(E)$ for all $E \in IM_n(R)$ such that $AE = A$ and $EB = B$.

Proof.

$ii \Rightarrow i$: For any matrices $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$, if $AB = 0$, then since $AI_n = A$ and $I_nB = B$, it follows that $r(A) + r(B) \le r(I_n) = n$.

$i \Rightarrow ii$: Let $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$ satisfy $AB = 0$. For any $E \in IM_n(R)$ such that $AE = A$ and $EB = B$, there exists an r-full idempotent matrix $F \in IM_k(R)$ such that $E \cong F$. Hence, there exist matrices $M \in M_{n\times k}(R)$ and $N \in M_{k\times n}(R)$ such that $E = MN$ and $F = NM$. Then $0 = AB = AEB = (AM)(NB)$, which implies that $r(AM) + r(NB) \le k = r(F) = r(E)$.

Moreover, since $A = AE = AMN$ and $B = EB = MNB$, we obtain $r(A) \le r(AM)$ and $r(B) \le r(NB)$. Therefore, $r(A) + r(B) \le r(E)$. □

The following result provides a generalization of the Nullity law to semirings with an ε - function.

Theorem 3.18. Let R be a semiring with an ε - function satisfying the following condition: For any matrices $U \in M_{m\times n}(R)$ and $V \in M_{n\times p}(R)$, if $\varepsilon(UV) = UV$, then $f(U) + f(V) \le n$.

Then, for any matrices $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$, and $C \in M_{p\times q}(R)$, we have $f(AB) + f(BC) \le f(ABC) + f(B)$.

Proof.

For any matrices $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$, and $C \in M_{p\times q}(R)$, assume that $f(B) = k$ and $f(ABC) = l$.

Then there exist matrices $P \in M_{m\times l}(R)$, $Q \in M_{l\times q}(R)$, $E \in M_{n\times k}(R)$, and $F \in M_{k\times p}(R)$ such that $ABC = PQ$ and $B = EF$. We have $\begin{pmatrix} AE & P \end{pmatrix}\begin{pmatrix} FC \\ \varepsilon(Q) \end{pmatrix} = ABC + P\varepsilon(Q) = PQ + \varepsilon(PQ)$.

Since $\varepsilon\left[PQ + \varepsilon(PQ)\right] = PQ + \varepsilon(PQ)$, it follows that $\varepsilon\left[\begin{pmatrix} AE & P \end{pmatrix}\begin{pmatrix} FC \\ \varepsilon(Q) \end{pmatrix}\right] = \begin{pmatrix} AE & P \end{pmatrix}\begin{pmatrix} FC \\ \varepsilon(Q) \end{pmatrix}$.

Hence,

$$f\begin{pmatrix} AE & P \end{pmatrix} + f\begin{pmatrix} FC \\ \varepsilon(Q) \end{pmatrix} \le k + l = f(ABC) + f(B).$$

Moreover, since

$$f(AB) = f(AEF) \le f(AE) \le f\begin{pmatrix} AE & P \end{pmatrix}$$

and

$$f(BC) = f(EFC) \le f(FC) \le f\begin{pmatrix} FC \\ \varepsilon(Q) \end{pmatrix},$$

we obtain $f(AB) + f(BC) \le f(ABC) + f(B)$. □

The following Corollary follows directly from Theorem 3.18 with the observation that, over a commutative semiring, $f(I_n) = n$ for all positive integer n.

Corollary 3.19. Let R be a semiring with an ε-function satisfying the condition: For any matrices $U \in M_{m\times n}(R)$ and $V \in M_{n\times p}(R)$, if $\varepsilon(UV) = UV$, then $f(U) + f(V) \le n$.

Then, for any matrices $A \in M_{m\times n}(R)$ and $B \in M_{n\times p}(R)$, we have

$f(A) + f(B) \le f(AB) + n$.

4. CONCLUSION

The main results of this paper can be summarized as follows: Theorems 3.4 and 3.6 provide two distinct Sylvester matrix rank functions on entire quasi-selective semirings. Several characteristic properties of Sylvester matrix rank functions are presented in Propositions 3.8, 3.9, and 3.11, as well as in Theorem 3.13. Moreover, further characterizations of Sylvester matrix rank functions satisfying the Nullity law are investigated in Theorems 3.15, 3.17, and 3.18.

Conflict of interest

The author declare no competing interests.

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Về các hàm hạng ma trận sylvester trên nửa vành giao hoán

TÓM TẮT

Trong bài báo này, tác giả cung cấp hai hàm hạng ma trận Sylvester phân biệt trên nửa vành tựa lựa chọn không có ước của không. Tác giả tiến hành khảo sát một số tính chất cơ bản của hàm hạng ma trận Sylvester trên nửa vành giao hoán và chỉ ra các trường hợp mà hàm hạng ma trận Sylvester không còn đúng như trên vành. Tác giả đã tiến hành so sánh các hàm hạng ma trận Sylvester với hạng nhân tử của ma trận và cung cấp một số tính chất đặc trưng của các hàm hạng ma trận Sylvester thỏa mãn luật Nullity trên các nửa vành có đối xứng.

Từ khóa: *Nửa vành, Ma trận, Hàm hạng ma trận Sylvester, luật Nullity.*