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Chặn dưới tường minh cho hàm năng lượng tự do trong miền nhiệt độ thấp của mô hình Blume–Capel

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu mô hình Blume–Capel một cách hệ thống trong khuôn khổ lý thuyết Pirogov–Sinai. Trọng tâm của bài báo là việc thiết lập các chặn dưới tường minh cho tham số nhiệt độ, tại đó hàm năng lượng tự do tồn tại và có tính giải tích. Các kết quả đạt được dựa trên những cải tiến gần đây về miền hội tụ của khai triển cụm –một công cụ quan trọng trong lý thuyết Pirogov-Sinai– qua đó cho phép chúng tôi cải thiện các chặn dưới đối với tham số nhiệt độ.

Từ khóa: *Mô hình Blume-Capel, lý thuyết Pirogov-Sinai, khai triển cụm.*

Explicit Low-Temperature Bounds for the Free Energy of the Blume–Capel Model

ABSTRACT

We study the Blume–Capel model within the framework of Pirogov–Sinai theory. The main objective of this work is to establish explicit lower bounds on the temperature parameter for which the free energy is well defined and analytic. Our analysis relies on recent advances in the convergence domain of the cluster expansion, a key technical tool in Pirogov–Sinai theory. These improvements allow us to obtain sharper temperature bounds than those previously available, thereby extending the range of validity of the low-temperature phase analysis for the Blume–Capel model.

Keyworks: *Blume–Capel model, Pirogov–Sinai theory, cluster expansion.*

1. INTRODUCTION

In statistical physics, the study of phase transitions and the microscopic structure of matter plays a fundamental role in understanding the behavior of complex physical systems. Among the theoretical models developed to explain these phenomena, the Blume–Capel (BC) model stands out as a natural extension of the classical Ising model, allowing spins to take three possible values $\{-1, 0, +1\}$. As discussed in[1], this three-state structure enables the model to capture features such as the appearance of intermediate phases and first-order phase transitions, which cannot be fully described by the two-state Ising model.

From an applied perspective, the Blume–Capel model possesses not only profound theoretical significance but also wide-ranging practical applications. It has been used to analyze magnetic systems with anisotropy, to model solid–liquid phase transitions in condensed matter physics, and even to describe biological systems characterized by multistate behavior. However, constructing a rigorous mathematical framework for the existence and characterization of stable phases in this model remains a nontrivial challenge, particularly in the low-temperature regime where conventional perturbative methods become insufficient.

In this context, the Pirogov–Sinai (PS) theory has emerged as a central tool in mathematical statistical mechanics, providing a quantitative approach to studying the existence and stability of phases. Based on two key techniques—the cluster expansion presented in[2–4], and the contour model—the theory allows a detailed analysis of Gibbs measures and the conditions under which distinct phases coexist or separate. The pioneering works of Pirogov and Sinai in[5], along with the significant extensions by Zahradník[6], established a rigorous framework for studying multiphase lattice systems such as the Blume–Capel model.

More recently, refined developments in cluster expansion methods, notably those presented in[3,4], have broadened the applicability of Pirogov–Sinai theory to more complex systems while improving estimates on convergence and phase stability. These advances open promising perspectives for constructing the phase diagram of the Blume–Capel model and for deepening our theoretical understanding of first-order phase transitions in modern statistical mechanics.

2. BLUME-CAPEL MODEL AND MAIN RESULTS

Let us recall the distance definition as follow:

$$d_\infty(i,j) \;:=\; \|i-j\|_\infty \;=\; \max_{1\le k\le d}|i_k-j_k| \quad (1)$$

for $i,j\in\mathbb{Z}^d$, and

$$d_\infty(S,S') \;:=\; \inf\{d_\infty(\kappa,\ell) : \kappa\in S, \ell\in S'\}$$

for $S,S'\subset\mathbb{Z}^d$.

Let $\Lambda\subset\mathbb{Z}^d$ be a finite set. A configuration $\boldsymbol{\sigma}_\Lambda\boldsymbol{\eta}^{\#}_{\Lambda^c}\in\Omega^{\#}$ consists of two parts: an interior configuration $\boldsymbol{\sigma}_\Lambda\in\{-1,0,1\}^\Lambda =: \Omega_\Lambda$, and a boundary configuration $\boldsymbol{\eta}^{\#}_{\Lambda^c}$, where for $\#\in\{-1,0,1\}$ we set

$$\boldsymbol{\eta}^{\#}_{\Lambda^c} := \#_{\Lambda^c},$$

that is, all spins outside Λ are fixed to the constant value $\#$. The configuration $\boldsymbol{\eta}^{\#}_{\Lambda^c}$ is referred to as the $\#$-boundary condition for the system in Λ. Hamiltonian is defined as

$$\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda\boldsymbol{\eta}^{\#}_{\Lambda^c}) := -\sum_{\substack{B\subset\mathbb{Z}^d\\ B\cap\Lambda\neq\varnothing}}\Phi_B(\boldsymbol{\sigma}_\Lambda\boldsymbol{\eta}^{\#}_{\Lambda^c}) \quad (2)$$

where $\Phi_B(\boldsymbol{\sigma})$ is defined as

$$\Phi_B(\boldsymbol{\sigma}) := \begin{cases} \beta\sigma_i\sigma_j & B\in\mathcal{E}_{\mathbb{Z}^d}\\ -h\sigma_i-\lambda\sigma_i^2 & B=\{i\}\\ 0 & \text{otherwise}\end{cases}, \quad (3)$$

with

$$\mathcal{E}_{\mathbb{Z}^d} = \big\{\{i,j\}\subset\mathbb{Z}^d : d_\infty(i,j)=1\big\}.$$

More precisely, Hamiltonian can be written as

$$\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda\boldsymbol{\eta}^{\#}_{\Lambda^c}) = -\sum_{\substack{\{i,j\}\subset\mathbb{Z}^d\\ i,j\in\Lambda\\ \|i-j\|_\infty=1}}\sigma_i\sigma_j - \sum_{\substack{\{i,j\}\subset\mathbb{Z}^d\\ i\in\Lambda,\, j\notin\Lambda\\ \|i-j\|_\infty=1}}\sigma_i\eta^{\#}_j - \lambda\sum_{i\in\Lambda}\sigma_i^2 - h\sum_{i\in\Lambda}\sigma_i. \quad (4)$$

The *partition function* with $\#$ boundary condition in Λ is

$$Z^{\#}_\Phi(\Lambda) \;:=\; \sum_{\boldsymbol{\sigma}_\Lambda\in\Omega_\Lambda}\exp\Big(-\beta\mathrm{H}_{\Lambda;\Phi}(\boldsymbol{\sigma}_\Lambda\boldsymbol{\eta}^{\#}_{\Lambda^c})\Big), \quad (5)$$

where $\beta\in[0,+\infty)$ is the inverse temperature. The (finite-volume) free energy function (pressure function) with $\#$ boundary condition is

$$P^{\#}_\Lambda(\beta,h,\lambda) \;:=\; \frac{1}{\beta\,|\Lambda|}\log Z^{\#}_\Phi(\Lambda). \quad (6)$$

The thermodynamic free energy function $p^{\#}(\beta,h,\lambda)$ is obtained through the thermodynamic limit

$$p^{\#}(\beta,h,\lambda) \;=\; \lim_{\Lambda\uparrow\mathbb{Z}^d}P^{\#}_\Lambda(\beta,h,\lambda) \quad (7)$$

in Fisher sense, where $\# := \{-1,0,1\}$.

The main purpose of this paper is to provide an explicit lower bound on the inverse temperature β for which the pressure function $p(\beta,h)$ exists and is analytic. In this work, we employ a novel convergence criterion for the cluster expansion, presented in Subsection 4.2, which is substantially stronger than the classical Kotecký–Preiss condition commonly used in this context. As a consequence, we expect an improvement over existing results on related problems, such as those obtained in[1]. Our results are summarized in the following theorem.

Theorem 2.1. *For each dimension d, let $\beta_0>0$ be the solution of equation* (28)*. Then, for all $\beta>\beta_0$, the following holds: for every contour $\gamma\in\Gamma^{\#}$,*

$$(\lambda,h)\in\mathcal{U}^{\#}_\beta \quad\Longrightarrow\quad \tilde{w}^{\#}(\gamma)=w^{\#}(\gamma).$$

In particular, whenever $(\lambda,h)\in\mathcal{U}^{\#}_\beta$, the pressure of the model is given by

$$\psi(\lambda,h) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{\beta|\Lambda|}\log Z^{\#}_\Phi(\Lambda) = \widehat{\psi}^{\#}(\beta,\lambda,h). \quad (8)$$

Here,

$$\widehat{\psi}^{\#}(\beta,\lambda,h) := \lim_{n\to\infty}\widehat{\psi}^{\#}_n(\beta,\lambda,h), \quad (9)$$

where $\widehat{\psi}^{\#}_n(\beta,\lambda,h)$ is defined in (22)*.*

The set $\mathcal{U}_\beta^{\#}$ is defined by

$$\mathcal{U}_\beta^{\#} = \Big\{(\lambda, h) \in U : \widehat{\psi}^{\#}(\lambda, h) = \max_{\#'} \widehat{\psi}^{\#'}(\lambda, h)\Big\}, \quad (10)$$

where the domain U is given in (20). Theorem 2.1 is a direct consequence of Proposition 3.12, presented in Subsection 3.2.

3. PIROGOV-SINAI THEORY FOR BLUME-CAPEL MODEL

In the next step, we express the Hamiltonian as a sum over contours on the lattice $\mathbb{Z}^d$, following the strategy developed for the Ising model (see reference[1]), and then apply cluster expansion techniques to study the analyticity of the free energy. However, for the Blume–Capel model this procedure is substantially more involved than in the Ising case, since the local state space consists of three values, namely $\{-1, 0, 1\}$. To overcome this difficulty, we rely on Pirogov–Sinai theory to construct an appropriate notion of contours adapted to the Blume–Capel model.

3.1. Definition of Contours

For each $\# \in \{-1, 0, 1\}$, we add and subtract the term $\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\eta}^{\#})$ to Hamiltonian. We then can rewrite the Hamiltonian as following form

$$\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}^{\#}_{\Lambda^c}) = \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\eta}^{\#}) + \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}^{\#}_{\Lambda^c} | \boldsymbol{\eta}^{\#}) \quad (11)$$

where the term $\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}^{\#}_{\Lambda^c} | \boldsymbol{\eta}^{\#})$ —well-known as the *relative Hamiltonian*— is defined as

$$\mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}_{\Lambda^c} | \boldsymbol{\eta}^{\#}) := \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}^{\#}_{\Lambda^c}) - \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\eta}^{\#}).$$

We then can rewrite the partition function as the follow forms:

$$Z_\Phi^{\#}(\Lambda) = \mathrm{e}^{-\beta \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\eta}^{\#})} \Xi_\Phi^{\#}(\Lambda) \quad (12)$$

with

$$\Xi_\Phi^{\#}(\Lambda) := \sum_{\boldsymbol{\sigma}_\Lambda \in \Omega_\Lambda} \mathrm{e}^{-\beta \mathrm{H}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}_{\Lambda^c} | \boldsymbol{\eta}^{\#})}.$$

To analyze the expression $\Xi_\Phi^{\#}(\Lambda)$, we rewrite the Hamiltonian in terms of contours associated with the configuration, using the framework of Pirogov–Sinai theory. To this end, we introduce several definitions of contours in the sense of Pirogov–Sinai theory, which will be used to express the Hamiltonian in contour form.

Definition 3.1 (Correct and incorrect sites)**.** Let $\boldsymbol{\sigma} \in \{-1, 0, +1\}^{\mathbb{Z}^d}$ be a configuration. A site $i \in \mathbb{Z}^d$ is called $\#$*–correct*, with $\# \in \{+, -, 0\}$, if

$$\sigma_j = \eta_j^{\#} \qquad \text{for all } j \in i + B(1),$$

where

$$B(1) = \{j \in \mathbb{Z}^d : \|j\|_\infty \leq 1\}.$$

A site is called *incorrect* if it is not $\#$–correct for any $\# \in \{-1, 0, 1\}$.

Definition 3.2 (Boundary and thickened boundary)**.** The *boundary* of the configuration $\boldsymbol{\sigma}$ is the set

$$B(\boldsymbol{\sigma}) = \{i \in \mathbb{Z}^d : i \text{ is incorrect}\}.$$

The *thickened boundary* is defined by

$$\Gamma(\boldsymbol{\sigma}) = \bigcup_{i \in B(\sigma)} (i + B(1)).$$

Assume that the configuration $\boldsymbol{\sigma}$ coincides with one of the ground states outside a finite region. Then the set $\Gamma(\boldsymbol{\sigma})$ is finite. Moreover, $\Gamma(\boldsymbol{\sigma})$ admits a unique decomposition into maximal connected components, namely,

$$\Gamma(\boldsymbol{\sigma}) = \{\overline{\gamma}_1, \dots, \overline{\gamma}_n\},$$

where the connected components are taken with respect to nearest–neighbor adjacency in the ℓ^∞ metric, that is, two sites are considered connected if their ℓ^∞ distance is equal to one.

Definition 3.3 (Contour)**.** A contour γ of $\boldsymbol{\sigma}$ is defined as a pair of the form

$$\gamma = \big(\overline{\gamma},\ \boldsymbol{\sigma} \restriction_{\overline{\gamma}}\big),$$

where $\overline{\gamma} \subset \mathbb{Z}^d$ denotes the support of the contour γ, and $\boldsymbol{\sigma} \restriction_{\overline{\gamma}}$ is the restriction of the configuration to $\overline{\gamma}$.

This definition ensures that each contour represents a connected region of local energetic defects separating domains dominated by distinct ground states. The size of a contour is given by $|\overline{\gamma}|$, and the Hamiltonian satisfies a Peierls bound proportional to this size.

Definition 3.4 (Exterior of a contour)**.** The complement of the contour support decomposes uniquely into connected components:

$$(\overline{\gamma})^c = A_0 \cup A_1 \cup \cdots \cup A_k.$$

Exactly one of these components is unbounded. This component is called the *exterior* of the contour and is denoted by

$$\operatorname{ext}\gamma := A_0.$$

Definition 3.5 (Labels of components)**.** For each connected component A_j of $(\overline{\gamma})^c$, there exists a unique $\# \in \{-1, 0, 1\}$ such that

$$\sigma_x = \eta^{\#}_x \qquad \text{for all } x \in \partial_{\text{ex}} A_j,$$

where the notion of external boundary ∂^{ex} is defined as follow

$$\partial^{\text{ex}} A = \{i \in A^c : d_\infty(i, A) \leq 1\}. \tag{13}$$

The symbol $\#$ is called the *label* of the component A_j denoted $\operatorname{lab}(A_j)$.

Definition 3.6 (Interior regions)**.** The bounded connected components $A_1, \ldots, A_k$ are called the *interior components* of the contour. For each $\# \in \{-1, 0, 1\}$, the *interior of type* $\#$ is defined by

$$\operatorname{int}_{\#}(\gamma) := \bigcup_{\substack{j \geq 1: \\ \operatorname{lab}(A_j) = \#}} A_j.$$

The exterior label $\operatorname{lab}(\operatorname{ext}\gamma)$ determines the *type* of the contour. The sets $\operatorname{int}_{\#}(\gamma)$ describe finite droplets of competing phases enclosed by the contour. Denote

$$\operatorname{int}(\gamma) = \bigcup_{\# \in \{-1,0,1\}} \operatorname{int}_{\#}(\gamma)$$

Definition 3.7. Let $\boldsymbol{\sigma} \in \Omega^{\#}$. A contours $\gamma \in \Gamma(\boldsymbol{\sigma})$ is external if there exist no contour $\gamma' \in \Gamma(\boldsymbol{\sigma})$ such that $\overline{\gamma}' \subset \operatorname{int}\gamma$.

Let $\Gamma'(\boldsymbol{\sigma}) \subset \Gamma(\boldsymbol{\sigma})$ denote the collection of external contours of $\boldsymbol{\sigma}$, which are pairwise compatible by construction. Let then

$$\operatorname{ext} := \bigcap_{\gamma' \in \Gamma'(\boldsymbol{\sigma})} \operatorname{ext}\gamma' \quad , \quad \Lambda^{\text{ext}} := \Lambda \cap \operatorname{ext}.$$

Note that all the external contours of $\Gamma'(\boldsymbol{\sigma})$ are of type $\#$.

We can now express the Hamiltonian in terms of the external contours of the configuration. More precisely,

$$\mathrm{H}^{\#}_{\Lambda,\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}_{\Lambda^c} | \boldsymbol{\eta}^{\#}) = \mathrm{H}_{\text{ext};\beta}(\boldsymbol{\eta}^{\#}) + \tag{14}$$
$$\sum_{\gamma' \in \Gamma'} \left(\sum_{B \in \overline{\gamma}'} \phi_B(\boldsymbol{\sigma}_{\overline{\gamma}'}) + \sum_{\#'} \mathrm{H}_{\text{int}\gamma',\Phi}(\boldsymbol{\sigma}_\Lambda \boldsymbol{\eta}_{\Lambda^c} | \boldsymbol{\eta}^{\#}) \right).$$

This representation immediately leads to a contour expansion of the partition function. Namely, the partition function with $\#$ boundary condition can be written as

$$\Xi^{\#}_{\Phi}(\Lambda) \tag{15}$$
$$= \sum_{\substack{\Gamma' \text{ compatible} \\ \text{external}}} \prod_{\gamma' \in \Gamma'} w^{\#}(\gamma) \prod_{\#'} \Xi^{\#'}_{\Phi}(\operatorname{int}_{\#'}\gamma')$$

where the contour weight, $w^{\#}(\cdot)$, is defined as

$$w^{\#}(\gamma') := \mathrm{e}^{-\beta\|\gamma'\|} \prod_{\#'} \frac{Z^{\#'}_{\Phi}(\operatorname{int}_{\#'}\gamma')}{Z^{\#}_{\Phi}(\operatorname{int}_{\#'}\gamma')}, \tag{16}$$

with the surface energy associated with the contour γ' defined as

$$\|\gamma'\| = \sum_{B \subset \overline{\gamma}'} \left[\phi_B(\omega_{\overline{\gamma}'}) - \phi_B(\boldsymbol{\eta}^{\#}_{\overline{\gamma}'}) \right].$$

Several nontrivial intermediate steps are involved in the derivation from (14) to (15). For a complete exposition of these computations, we refer the reader to [1].

Before presenting an alternative representation of the partition function for large-field polymers, we introduce the notions of compatible and incompatible objects.

Definition 3.8. Two contours γ and γ' are said to be *compatible*, denoted by $\gamma \sim \gamma'$, if

$$d_\infty(\gamma, \gamma') \geq 1.$$

Otherwise, the contours γ and γ' are *incompatible*, and we write $\gamma \nsim \gamma'$. Denote

$$\zeta(\gamma, \gamma') = \begin{cases} 1 & \text{if} \quad \gamma \sim \gamma' \\ 0 & \text{if} \quad \gamma \nsim \gamma' \end{cases}.$$

From expression (15), the same contour analysis may be applied recursively to each factor $Z_{\Phi}^{\#}(\text{int}\gamma')$, since all resulting contours are again of type $\#$. This recursive procedure yields a finite sequence of external contours and terminates when the interior regions become too small to contain further contours, a property commonly referred to as the *thin condition*; see page 342 of reference[1]. In this way, the partition function is written in contour representation.

$$\Xi_{\Phi}^{\#}(\Lambda) \tag{17}$$
$$= \sum_{(\gamma_1,\dots,\gamma_n)\in\Gamma^n} \prod_{1\le i<j\le n} \zeta(\gamma_i,\gamma_j) \prod_{\gamma'\in\Gamma} w^{\#}(\gamma')$$

where function ζ is defined in the definition (3.8), the weight function $w^{\#}$ is defined in (16) and Γ is *the set of of contours of type* $\#$.

In the spirit of Pirogov–Sinai theory, contours are naturally divided into thin and fat according to their energetic weights. Motivated by this classification, we introduce the notion of τ-stability, which characterizes thin contours via an explicit exponential bound and separates them from unstable (fat) contours.

Definition 3.9. A contour $\gamma \in \Gamma$ is τ-stable if

$$w^{\#}(\gamma) \le \mathrm{e}^{-\tau|\bar{\gamma}|}.$$

3.2. Truncated weight and upper bound of partition function

In the next step, we apply the cluster expansion in order to obtain upper bounds on the partition function, which in turn leads to the analysis of the free energy and the phase diagram of the Blume–Capel model. For the cluster expansion to be well defined, it is necessary that all contours satisfy the τ-stability condition. However, directly verifying that every $\gamma \in \Gamma$ is τ-stable is difficult, since this requires controlling ratios of partition functions on possibly intricate interior regions of γ. In addition, these weights depend not only on the geometry of the contour boundary but also on the thermodynamic behavior of the phases inside. For small interiors the surface term dominates and the weights are exponentially small, whereas for large interiors volume contributions may prevail and the weights are no longer guaranteed to be small, violating the assumptions needed for the cluster expansion. This motivates the introduction of *truncated* contour weights: thin contours retain their true weights, whereas fat contours are suppressed. The truncated weights satisfy the required decay properties, and one subsequently shows that, in the stable phase, they coincide with the true physical weights.

Let $\Gamma_n^{\#}$ be the collection of all contour γ such that $|\text{int}\gamma| \le n$. We will use the iterated process to define the truncated weight $\widehat{w}^{\#}(\cdot)$. Denote

$$\widehat{\psi}_0^{\#} = -\mathrm{e}^{\#},$$

and

$$\widehat{w}^{\#}(\gamma) = \mathrm{e}^{-\beta\|\gamma\|} \tag{18}$$

for each $\gamma \in \Gamma_0^{\#}$. Note that, following the Peierls' condition for Blume-Capel model, we have

$$\widehat{w}^{\#}(\gamma) \le \mathrm{e}^{-\beta(\rho-2\|W\|)|\bar{\gamma}|} \tag{19}$$

where $\rho = 5^{-d}$ and $\|W\| = |h| + |\lambda|$. To verify the τ-stable for every contour $\gamma \in \Gamma_0^{\#}$ we need that $(\lambda, h) \in U$ where

$$U := \{(\lambda,h) \in \mathbb{R}^2 : |\lambda| \le \rho/8, |h| \le \rho/8\}. \tag{20}$$

For the sake of convenience in the further computation, we set

$$\rho_0 := \frac{\rho}{2} = \frac{5^{-d}}{2}.$$

Without losing generalization, we assume that $\widehat{w}^{\#}(\gamma)$ is well-defined for every contour $\gamma \in \Gamma_n^{\#}$. Define

$$\widehat{Z}_{n,\Phi}^{\#}(\Lambda) := \mathrm{e}^{-\beta \mathrm{e}^{\#}|\Lambda|}\widehat{\Xi}_{n,\Phi}^{\#}(\Lambda),$$

where the truncated partition function, $\widehat{\Xi}_{n,\Phi}^{\#}(\Lambda)$, is defined as

$$\widehat{\Xi}_{n,\Phi}^{\#}(\Lambda \tag{21}$$
$$= \sum_{(\gamma_1,\dots,\gamma_n)\in(\Gamma_n^{\#})^n} \prod_{1\le i<j\le n} \zeta(\gamma_i,\gamma_j) \prod_{i=1}^{n} \widehat{w}^{\#}(\gamma_i).$$

Set

$$\widehat{\psi}_n^{\#} := \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{\log \widehat{Z}_{n,\Phi}^{\#}(\Lambda)}{\beta|\Lambda|}$$
$$= -\mathrm{e}^{\#} + \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{\log \widehat{\Xi}_{n,\Phi}^{\#}(\Lambda)}{\beta|\Lambda|}. \tag{22}$$

Note that expression (22) is well defined provided that the limit in the second term exists. To verify the existence of this limit, we employ the cluster expansion technique, which allows us to write $\log \widehat{\Xi}^{\#}_{n,\Phi}(\Lambda)$ in the form

$$\log \widehat{\Xi}^{\#}_{n,\Phi}(\Lambda) \tag{23}$$
$$= \sum_{m=1}^{\infty} \sum_{(\gamma_1,\dots,\gamma_m)\in(\Gamma^{\#}_n)^m} \frac{a^T_m(\gamma_1,\dots,\gamma_m)}{m!} \prod_{i=1}^{m} \widehat{w}^{\#}(\gamma_i),$$

where $a^T_m(\gamma_1,\dots,\gamma_m)$ are the Ursell coefficients, defined by

$$a^T_m(\gamma_1,\dots,\gamma_m) := \sum_{G\in\mathcal{C}[m]} \prod_{\{i,j\}\in E(G)} [\zeta(\gamma_i,\gamma_j)-1], \tag{24}$$

with $\zeta(\gamma_i,\gamma_j)$ denoting the compatibility relation between polymers defined in Definition 3.8, and $\mathcal{C}[m]$ the set of all connected graphs with vertex set $\{1,\dots,m\}$.

Expression (23) follows from the cluster–expansion method. Its validity requires checking the convergence conditions of the cluster expansion, notably the Fernandez–Procacci criterion. Further details may be found in[8?].

Definition 3.10. The *truncated weight* is defined by

$$\widehat{w}^{\#}(\gamma) = \mathrm{e}^{-\beta\|\gamma\|} \tag{25}$$
$$\times \prod_{\#'} \left[\chi\left((\widehat{\psi}^{\#'}_n - \widehat{\psi}^{\#}_n)|\mathrm{int}_{\#'}\gamma|^{1/d} \right) \frac{Z^{\#'}_{\Phi}(\mathrm{int}_{\#'}\gamma')}{Z^{\#}_{\Phi}(\mathrm{int}_{\#'}\gamma')} \right],$$

where the function $\chi : \mathbb{R} \to [0,1]$ is a continuously differentiable satisfying the following conditions:

i) $\chi(s) = 1$ if $s \le \rho_0/4$,

ii) $\chi(s) = 1$ if $s \ge \rho_0/2$.

Remark 3.11. Denote

$$a^{\#}_n := \max_{\#'} [\widehat{\psi}^{\#'}_n - \widehat{\psi}^{\#}_n] \ge 0. \tag{26}$$

If $a^{\#}_n|\mathrm{int}\gamma|^{1/d} \le \rho_0/4$ for $\gamma \in \Gamma^{\#}_{n+1}$ then $\widehat{w}^{\#}(\gamma) = w^{\#}(\gamma)$.

We define

$$\mathbf{H}(\beta,d) := 2\max\big\{\tau_1,\ \mathbf{F}(\beta,d),\ \mathbf{G}(\beta,d)\big\}, \tag{27}$$

where τ_1, $\mathbf{F}(\beta,d)$, and $\mathbf{G}(\beta,d)$ are defined in (39), (61), and (66) respectively.

The following proposition is the key statement to prove the Theorem 2.1.

Proposition 3.12. Define

$$\tau \overset{\text{def}}{=} \frac{1}{2}\beta\rho_0 - \frac{3}{3^d\epsilon_0 - 1} - 3,$$

with ϵ_0 defined in (37), and

$$\beta_0 = \min\{\beta : \tau \ge \mathbf{H}(\beta,d)\} \tag{28}$$

If $\beta > \beta_0$ then there exists an increasing sequence $c_n \uparrow c_\infty < c_{\tau_1} < 2$ such that, for all $\#$ and all $n \ge 0$, the following statements hold.

1. **(Bounds on the truncated weights.)** For all $k \le n$ and for each $\gamma \in \Gamma^{\#}_k$ is τ-stable uniformly on U. More precisely,

$$\widehat{w}^{\#}(\gamma) \le e^{-\tau|\overline{\gamma}|} \tag{29}$$

and

$$a^{\#}_n\, |\mathrm{int}\gamma|^{1/d} \le \rho_0/8 \quad \Longrightarrow \quad \widehat{w}^{\#}(\gamma) = w^{\#}(\gamma). \tag{30}$$

2. **(Bounds on the partition functions.)** Assume that $\Lambda \subset \mathbb{Z}^d$ is connected component and $|\Lambda| \le n$. Then

$$Z^{\#}_{\Lambda}(\Lambda) \le e^{\beta\widehat{\psi}_n|\Lambda| + c_n|\partial^{\mathrm{ex}}\Lambda|}, \tag{31}$$

uniformly in $(\lambda, h) \in U$.

The proof of Proposition 3.12 proceeds by induction on the size of the contours and relies fundamentally on cluster expansion techniques. Exploiting the contour representation of the Hamiltonian, the truncated contour weights are defined recursively in terms of partition functions associated with the corresponding interior regions. A thin–fat decomposition of contours is then performed, separating τ-stable (thin) contours from unstable (fat) contours. For $\beta \ge \beta_0$, thin contours satisfy a Peierls-type estimate, whereas the contributions of fat contours are suppressed by truncation.

Upper bounds on both the finite-volume partition functions and the truncated contour weights are obtained via the cluster expansion for the gas of external contours, which yields convergent series representations. These expansions lead to exponential bounds on the contour weights in terms of the relevant parameters. In particular, the cluster expansion plays a crucial role in establishing analyticity of the logarithms of the interior partition functions, together with uniform surface-to-volume bounds, which are subsequently inherited by the contour weights. The same expansion also provides the required estimates for the finite-volume partition functions appearing in Proposition 3.12.

To complete the proof of the two statements in Proposition 3.12, three additional conditions are required, as stated in Subsection 4.3. More precisely, condition (a) follows from the verification of (30), while conditions (b) and (c) are needed to derive the upper bound for the partition function $Z_\Phi(\Lambda)$ given in (31). Since the proof closely follows the arguments presented in Chapter 7 of[1], we omit the details and refer the reader to that reference for a complete exposition.

4. LOWER BOUND ON THE INVERSE TEMPERATURE

In this section, we present the main contribution of the present paper, namely the enlargement of the parameter domain in which the τ-stability condition holds, as formulated in Lemma 4.1 and established in Subsection 4.3. This improvement is achieved by replacing the classical Kotecký–Preiss convergence criterion, which is commonly employed in the literature (see, for instance[1]), with the more recent Fernández–Procacci condition.

4.1. Application of cluster expansion for Blume-Capel model

At each step of the induction procedure used to construct the truncated contour weights, we employ the same contour representation of the partition function,

$$\Xi(\Lambda) = \sum_{n=1}^{\infty} \sum_{(\gamma_1,\dots,\gamma_n)\in\Gamma} \prod_{1\le i<j\le n} \zeta(\gamma_i,\gamma_j) \prod_{i=1}^{n} w(\gamma_i), \tag{32}$$

where $w(\gamma)$ denotes the truncated contour weight produced at the corresponding induction step, and Γ denotes the associated family of contours $\Gamma_n^{\#}$. The infinite-volume free energy density is defined by

$$g = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{\log \Xi(\Lambda)}{\beta|\Lambda|}, \tag{33}$$

where the logarithm is evaluated via the cluster expansion and takes the form

$$\log \Xi(\Lambda) \tag{34}$$

$$= \sum_{m=1}^{\infty} \sum_{(\gamma_1,\dots,\gamma_m)\in(\Gamma)^m} \frac{a_m^T(\gamma_1,\dots,\gamma_m)}{m!} \prod_{i=1}^{m} w(\gamma_i), \tag{35}$$

where $a_m^T(\gamma_1,\dots,\gamma_m)$ are the Ursell coefficients. At every stage of the induction one shows that, for all $\gamma\in\Gamma_n^{\#}$,

$$w^{\#}(\gamma) \le e^{-\tau|\overline{\gamma}|}. \tag{36}$$

These bounds guarantee the uniform τ-stability and regularity properties required for the subsequent cluster-expansion analysis. Denote

$$\epsilon_0 = \min\{|\overline{\gamma}| : \gamma\in\Gamma\} = 2^d. \tag{37}$$

Lemma 4.1. *Assume that, for all $\gamma\in\Gamma$, the weight $w(\gamma)$ is a continuously differentiable function in (a,b) with respect to a parameter s, and that, uniformly on (a,b),*

$$w(\gamma) \le e^{-\tau|\overline{\gamma}|}. \tag{38}$$

Then, with m_0 defined in (57), there exists

$$\tau_1 := 1 + 3^d \ln\left[1 + \frac{1}{3^d\epsilon_0 - 1}\right] + 2\log(2d) + \frac{d}{d-1} - \ln(m_0), \tag{39}$$

such that the following holds: If $\tau > \tau_1$, then g defined in (33) is given by the following absolutely convergent series,

$$g = \sum_{X:\,\overline{X}\ni 0} \frac{1}{|X|}\Psi(X), \tag{40}$$

where the sum is over clusters X made of contours $\gamma\in\Gamma$ and $\overline{X} \stackrel{\text{def}}{=} \bigcup_{\gamma\in X}\overline{\gamma}$. Moreover,

$$|g| \le \frac{1}{3^d\epsilon_0 - 1},$$

and, for all $\Lambda\subset\mathbb{Z}^d$, g provides the volume contribution to $\log\Xi(\Lambda)$, in the sense that

$$\Xi(\Lambda) = \exp\big(g|\Lambda| + \Delta\big), \tag{41}$$

where Δ is a boundary term satisfying

$$|\Delta| \le \frac{1}{3^d\epsilon_0 - 1}|\partial^{\text{in}}\Lambda|. \tag{42}$$

Indeed, Lemma 4.1 follows directly from cluster expansion theory; more specifically, it is an immediate consequence of Proposition 4.2 stated in the next subsection. Since in this paper we are only concerned with the temperature domain, we omit the proof of Lemma 4.1. The detailed proof can be found in Chapter 7 of reference [1].

4.2. Gruber-Kunz–type condition

In this section, we establish the well-posedness of the cluster expansion by employing the Fernández–Procacci criterion[3], together with the refined compatibility relation introduced in Section 2. This combination allows us to derive an improved version of the classical Gruber–Kunz condition. We begin by recalling the standard Theorem given by Fernandez-Procacci in[3].

Proposition 4.2. If there exists a positive function $a(\cdot) : \Gamma \to \mathbb{R}$ such that

$$\sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\Gamma^n \\ \gamma_0 \nsim \gamma_i,\, \gamma_i \sim \gamma_j,\, 1\leq i<j\leq n}} \prod_{i=1}^{n} \mathrm{e}^{a(\gamma_i)}\, \omega(\gamma_i) \leq \mathrm{e}^{a(\gamma_0)} - 1. \quad (43)$$

Then the infinite-volume free energy density, defined in (33), is absolutely convergent series.

Dealing directly with the Fernández–Procacci condition turns out to be too difficult in the present setting. Instead, following standard practice, we slightly relax this condition and work with a weaker but more tractable criterion, commonly referred to as the Gruber–Kunz-type condition. This approach is still sufficient to obtain explicit bounds on the parameter τ and yields a significantly larger domain of validity than the classical Kotecký–Preiss criterion. For a detailed comparison, see[3]. To derive a Gruber–Kunz–type condition, we introduce the one-neighborhood of a polymer γ, defined by

$$[\overline{\gamma}]_1 := \{x \in \mathbb{Z}^d : d_\infty(x, \overline{\gamma}) \leq 1\}. \quad (44)$$

By construction,

$$\gamma \nsim \gamma' \iff d_\infty(\gamma, \gamma') \leq 1 \iff \gamma \cap [\overline{\gamma}']_1 \neq \varnothing.$$

Using this characterization, the left-hand side of the convergence condition (43) can be bounded from above by

$$1 + \sum_{n\geq 1} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\subset\Gamma \\ [\gamma_0]_1\cap\gamma_i\neq\varnothing,\, \gamma_i\cap\gamma_j=\varnothing,\, i\neq j}} \prod_{i=1}^{n} \omega(\gamma_i)\, \mathrm{e}^{a|[\overline{\gamma}_i]_1|} \leq \mathrm{e}^{a|[\overline{\gamma}_0]_1|}, \quad (45)$$

where we choose the control function $a(\gamma) = a|[\overline{\gamma}]_1|$ with $a > 0$.

Starting with the constraint in the sum $[\overline{\gamma}_0]_1 \cap \gamma_i \neq \varnothing$ and $\gamma_i \cap \gamma_j = \varnothing$, this implies that each polymer γ_i must intersect different points in $[\overline{\gamma}_0]_1$ to avoid overlapping. Consequently, the number of polymers in the family is bounded by $n \leq |[\overline{\gamma}_0]_1|$, and the set of sites touched by $\gamma_1, \dots, \gamma_n$ consists of n distinct points in $[\overline{\gamma}_0]_1$. The number of ways to choose these points is therefore $\binom{|[\overline{\gamma}_0]_1|}{n}$.

Proceeding as in the derivation of the Gruber–Kunz condition in[3], we obtain the bound

$$1 + \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma, \\ x\in\gamma}} w(\gamma), \mathrm{e}^{a|[\overline{\gamma}]_1|} \leq \mathrm{e}^{a}, \quad (46)$$

which yields a weaker, but well-known, *Gruber–Kunz–type* convergence condition. From now on, we shall work with the Gruber–Kunz–type condition given in (46) to determine an explicit domain for the parameter τ. The resulting bounds are derived and discussed throughout the remainder of this subsection.

On the lattice $\mathbb{Z}^d$, each site of $\overline{\gamma}$ generates at most 3^d sites in its unit neighborhood. Consequently,

$$|[\overline{\gamma}]| \leq 3^d\, |\overline{\gamma}|.$$

This estimate follows from a simple local covering argument and holds uniformly for all admissible contours, independently of their geometry. Although sharper bounds may be obtained for specific contour shapes, the precise value of the geometric constant is irrelevant for the convergence of the contour and cluster expansions. The constant 3^d is therefore adopted for its robustness and universality rather than for optimality, and is standard in Pirogov–Sinai analyses of lattice spin systems such as the Blume–Capel model; see[1] for more details.

We thus obtain the following weaker condition:

$$1 + \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\gamma}} w(\gamma)\, e^{3^d a|\overline{\gamma}|} \;\leq\; e^{a}. \tag{47}$$

Combining this with the bounds on the contour weights in the Blume–Capel model, we arrive at the condition

$$1 + \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\gamma}} e^{-(\tau - a3^d)|\overline{\gamma}|} \;\leq\; e^{a}. \tag{48}$$

However, in order to establish an upper bound on the derivative of the pressure with respect to the parameter s, we must also incorporate the bound on the derivative of the contour weights. This leads to the stronger condition

$$\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\overline{\gamma}}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d a|\overline{\gamma}|} \leq e^{a} - 1. \tag{49}$$

Using the translation invariance of the Blume–Capel model, we can rewrite the left-hand side of (49) as

$$\begin{aligned}
&\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\gamma}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d a|\overline{\gamma}|}\\
&= \sum_{\substack{\gamma\in\Gamma\\ 0\in\overline{\gamma}}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d a|\overline{\gamma}|}\\
&= \sum_{\ell=\epsilon_0}^{\infty} |A_\ell|\, \ell^{d/(d-1)} e^{-(\tau/2-1)\ell} e^{3^d a\ell},
\end{aligned} \tag{50}$$

where

$$A_\ell := \{\gamma\in\Gamma : 0\in\overline{\gamma},\ |\overline{\gamma}| = \ell\}. \tag{51}$$

Moreover, the cardinality of A_ℓ can be bounded by

$$|A_\ell| \leq (2d)^{2\ell-2},$$

see[8] for a proof. Consequently, we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\gamma}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|}\\
&\leq \sum_{\ell=\epsilon_0}^{\infty} (2d)^{2\ell-2}\, \ell^{d/(d-1)} e^{-(\tau/2-1)\ell} e^{3^d a\ell}\\
&\leq \sum_{\ell=\epsilon_0}^{\infty} e^{-(\tau/2-1-3^d a-2\log(2d)-d/(d-1))\ell}\\
&= \frac{z^{\epsilon_0}}{(1-z)},
\end{aligned} \tag{52}$$

with $z := e^{-(\tau/2-1-3^d a-2\log(2d)-d/(d-1))} < 1$.

Although the inequality (52) can be solved numerically, we adopt a simpler sufficient condition in order to avoid unnecessary technical complications. From the estimation in (52) and for $z \leq m < 1$, we have

$$\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\gamma\in\Gamma\\ x\in\gamma}} |\overline{\gamma}|^{d/(d-1)} e^{-(\tau/2-1)|\overline{\gamma}|} \leq e^{a} - 1 \tag{53}$$

This implies

$$z \leq \left((1-m)(e^{a}-1)\right)^{1/\epsilon_0} \leq m. \tag{54}$$

with

$$e^{a} \leq 1 + \frac{m^{\epsilon_0}}{1-m}.$$

Substituting this bound into the definition of τ yields

$$\begin{aligned}
\frac{\tau}{2} &\geq 1 + 3^d a + 2\log(2d) + \frac{d}{d-1}\\
&\quad - \frac{1}{\epsilon_0}\left[\ln(e^{a}-1) + \ln(1-m)\right]
\end{aligned} \tag{55}$$

Optimizing over a then gives the explicit sufficient condition

$$\begin{aligned}
\frac{\tau}{2} &\geq \min\left\{\phi^{(1)}(a) \,\middle|\, a>0, m<1, e^{a}-1 \leq \frac{m^{\epsilon_0}}{1-m}\right\}\\
&= 1 + 3^d \ln\left[1 + \frac{1}{3^d\epsilon_0 - 1}\right] + 2\log(2d)\\
&\quad + \frac{d}{d-1} - \ln(m_0)\\
&:= \tau_1,
\end{aligned} \tag{56}$$

where m_0 comes from the equation

$$\frac{m^{\epsilon_0}}{1-m} = \frac{3^d\epsilon_0}{3^d\epsilon_0 - 1}. \tag{57}$$

All the computations from (47) to (56) lead to the following result.

Lemma 4.3. *If* (56) *holds, then the Gruber–Kunz–type condition stated in* (46) *is also satisfied.*

4.3. Sufficient conditions for the estimation of the Partition Function

Beside the convergence condition for cluster expansion, the proof of Proposition 3.12 requires three additional conditions presented precisely in[1], which are stated below.

(a) We seek parameters (β,τ) such that

$$k^{1/d}\,\mathrm{e}^{-\frac{\tau}{2}k^{(d-1)/d}} \;\le\; \frac{\rho_0\beta}{16}, \qquad \forall\, k\in\mathbb{N}. \quad (58)$$

Equivalently, this condition can be rewritten as

$$\frac{\tau}{2} \;\ge\; \max_{k\in\mathbb{N}}\left\{-\,k^{-(d-1)/d}\ln\!\left(k^{-1/d}\frac{\rho_0\beta}{16}\right)\right\}. \quad (59)$$

Let

$$\beta_c \;:=\; \frac{16}{\rho_0}\exp\!\left(-\frac{1}{d-1}\right).$$

The maximization in (59) can be solved explicitly and yields

$$\max_{k\in\mathbb{N}}\left\{-\,k^{-(d-1)/d}\ln\!\left(k^{-1/d}\frac{\rho_0\beta}{16}\right)\right\} = \begin{cases} \dfrac{1}{e(d-1)}\left(\dfrac{16}{\rho_0\beta}\right)^{d-1}, & \text{if } \beta\ge\beta_c,\\[2ex] -\ln\!\left(\dfrac{\rho_0\beta}{16}\right), & \text{if } \beta<\beta_c. \end{cases} \quad (60)$$

Consequently, condition (59) is equivalent to

$$\frac{\tau}{2} \ge \begin{cases} \dfrac{1}{e(d-1)}\left(\dfrac{16}{\rho_0\beta}\right)^{d-1}, & \text{if } \beta\ge\beta_c,\\[2ex] -\ln\!\left(\dfrac{\rho_0\beta}{16}\right), & \text{if } \beta<\beta_c. \end{cases}$$
$$:= \mathbf{F}(\beta,d). \quad (61)$$

(b) We next require (β,τ) such that

$$\mathrm{e}^{-\frac{\tau}{2}\max\{(\rho_0/4a)^{d-1},\,\epsilon_0\}} \;\le\; \frac{a\beta}{2}. \quad (62)$$

Let

$$a_c \;:=\; \frac{\rho_0}{4\,\epsilon_0^{1/(d-1)}}, \qquad a_* \;:=\; \frac{2}{\beta}\,\mathrm{e}^{-1/(d-1)}.$$

We consider the inequality

$$\mathrm{e}^{-\frac{\tau}{2}\max\{(\rho_0/4a)^{d-1},\,\epsilon_0\}} \;\le\; \frac{a\beta}{2}, \qquad a>0. \quad (63)$$

Case (b1): $a_* < a_c$. In this regime the maximum in (63) is attained at the interior point $a=a_*$. The inequality (63) is satisfied provided that

$$\frac{\tau}{2} \;\ge\; \frac{1}{d-1}\left(\frac{8}{\rho_0\beta}\right)^{d-1}. \quad (64)$$

Case (b2): $a_* \ge a_c$. In this case the maximum in (63) is attained at the boundary point $a=a_c$. The condition (63) then reduces to

$$\frac{\tau}{2} \;\ge\; \frac{1}{\epsilon_0}\ln\!\left(\frac{8\,\epsilon_0^{1/(d-1)}}{\rho_0\beta}\right). \quad (65)$$

Hence, the inequality (63) holds for all $a>0$ if and only if

$$\frac{\tau}{2} \ge \begin{cases} \dfrac{1}{d-1}\left(\dfrac{8}{\rho_0\beta}\right)^{d-1}, & \text{if } \dfrac{2}{\beta}\mathrm{e}^{-1/(d-1)} < a_c,\\[2ex] \dfrac{1}{\epsilon_0}\ln\!\left(\dfrac{8\,\epsilon_0^{1/(d-1)}}{\rho_0\beta}\right), & \text{if } \dfrac{2}{\beta}\mathrm{e}^{-1/(d-1)} \ge a_c. \end{cases}$$
$$:= \mathbf{G}(\beta,d). \quad (66)$$

(c) Besides conditions *(a)* and *(b)*, we require an additional condition,

$$\sum_{n=1}(n+1)^{1/d}\mathrm{e}^{-\tau n^{(d-1)/d}/2} < \infty. \quad (67)$$

This condition arises in the derivation of the bound on the partition function (31). In the proof of Proposition 3.12 in[1], a stronger assumption is imposed, namely,

$$\sum_{n=1}(n+1)^{1/d}\mathrm{e}^{-\tau n^{(d-1)/d}/2} < 1. \quad (68)$$

In the present work, we relax this requirement by replacing it with the weaker condition (67). Indeed, condition (67) is automatically satisfied for any $\tau>0$. However, number τ must satisfy greater or equal to τ_1 defined in (39). Denote

$$c_{\tau_1} = \sum_{n=1}(n+1)^{1/d}\mathrm{e}^{-\tau_1 n^{(d-1)/d}/2} \quad (69)$$

This quantity replaces the definition of τ in (7.67) of Proposition 7.47 in[1] by introducing a new constant c_{τ_1}, defined as

$$c_{\tau_1} := \sum_{n=1}^{\infty}(n+1)^{1/d}\,\mathrm{e}^{-\tau_1 n^{(d-1)/d}/2}$$

$$
\begin{aligned}
&\leq 2^{1/d} \sum_{n=1}^{\infty} n^{1/d}\, \mathrm{e}^{-\tau_1 n^{(d-1)/d}/2} \\
&\leq \frac{2^{1/d} d}{d-1} \widehat{\mathbf{\Gamma}}\left(1 + \frac{2}{d-1}\right) \left(\frac{2}{\tau_1}\right)^{1+\frac{2}{d-1}} \\
&< 2, \qquad (70)
\end{aligned}
$$

for all $d \geq 2$, where $\widehat{\mathbf{\Gamma}}$ denotes the Gamma function.

4.4. Conclusion

We now combine the conditions obtained in Subsections 4.2 and 4.3 in order to derive an explicit lower bound on the inverse temperature β that ensures the validity of Proposition 3.12. By combining (56), (61), and (66), we conclude that the τ-stability condition holds provided that

$$\frac{\tau}{2} \geq \max\{\tau_1,\ \mathbf{F}(\beta, d),\ \mathbf{G}(\beta, d)\}\,. \qquad (71)$$

As a consequence, there exists an explicit lower bound β_0 such that, for all $\beta \geq \beta_0$, one may choose

$$\tau = \frac{\rho_0 \beta}{2} - \frac{1}{3^d \epsilon_0 - 1} - 3 = \mathbf{H}(\beta, d)$$

and all required conditions to prove Proposition 3.12 are satisfied simultaneously.

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