Norms and equivalence constants on the space of piecewise bicubic Bézier surface

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Received: dd/mm/yyyy; Revised: dd/mm/yyyy;
Accepted: dd/mm/yyyy; Published: dd/mm/yyyy

ABSTRACT

In this paper, we present a class of norms of $N \times M$-piece bicubic Bézier surfaces and the equivalence relations between some norms on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surface. A bicubic Bézier surfaces is a 4×4 grid of 16 control points and bounded by cubic Bézier curves. We introduce a class of norms $\|\cdot\|_p^{B_{3,3}}$ on the space $B_{3,3}$ of bicubic Bézier surfaces and a class of norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surfaces. These norms are calculated through control points. Thus, the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p norm are norms on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surface. We will study the equivalence constants for the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p norm on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surface. These equivalence constants do not depend on the number of pieces in piecewise bicubic Bézier surface. From this result, we can consider the convergence of a sequence of piecewise bicubic Bézier surfaces. This keeps an important roll for applying piecewise bicubic Bézier surfaces to find optimal shapes.

Keywords: *Bézier surface, bicubic Bézier surface, equivalence constants, norm, distance.*

1. INTRODUCTION

In 1959, the French physicist and mathematician Paul de Casteljau developed De Casteljau's algorithm while working at the French car company Citroën. This algorithm is a recursive method to evaluate Bézier curves or Bézier surfaces. De Casteljau's method was patented in France. But the company Citroën kept the copyright, preventing publication until the 1980s. The French engineer Pierre Bézier discovered Bézier surfaces independently and used them to design Renault cars. Bézier surfaces were widely publicised in 1962 by Pierre Bézier.

Bézier surfaces are a direct extension of Bézier curves to higher dimensions. Bézier surfaces construction has many benefits. Initially, each Bézier surface is presented by a few control points, then it need very little memory. Besides, these surfaces is intuitive, smooth and compactand beautiful. Bezier surfaces can be infinitely scalable without compromising quality, and efficient render via piece subdivision or evaluation. Points on the Bézier surface are easily calculated using De Casteljau's algorithm. This allows for efficient division into polygonal meshes. It's easy to compute and design, so the designer without mathematical background can be use them. The Bézier surface is flexible. They can be used to create, smooth, and, with proper adjustments, create complex shapes, often by blending multiple patches to achieve desired forms. More, we can easily modify, change, move, turn Bézier surfaces just by changing, moving, turning their control points. (see [1] - [6])

Bézier surfaces appears from practical needs, not only mathematics, then its have many applications. Bézier surfaces are parametric, smooth patches defined by a grid of control points, widely used in computer-aided design

(CAD), computer graphics, and engineering for modeling complex, curved shapes. They enable precise control over surface curvature, such as in automotive body design, font rendering, and animations. (see [4] – [22])

Bézier surface are presented in many books and articles for instance [1] – [3]. A continuous surface can be approximated by a Bézier surface. However, when the surface is large and complex, the degree of the Bézier surface is high. As a result, the computation is more difficult. Then, the most common use of Bézier surface is as $N \times M$-piece bicubic Bézier surfaces. We will focus on $N \times M$-piece bicubic Bézier surfaces.

In this article, we define a norm $\|\cdot\|_p^{B_{3,3}}$ on the space $B_{3,3}$ of bicubic Bézier surface and a norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surfaces. These norms are computed through control points. This article studies the equivalence relations between the norm $B_{3,3}^{N,M}$ and the L_p norm on the space $B_{3,3}^{N,M}$ of $N \times M$-piece cubic Bézier curves.

Theorem 1.1. *For $p \in [1, \infty]$. Let $\Gamma \in B_{3,3}^{N,M}$, we get*

$$\|\Gamma\|_{L_p} \leq \|\Gamma\|_p^{B_{3,3}^{N,M}} \leq 2^9 \|\Gamma\|_{L_p}.$$

2. PRELIMINARIES

For the convenience of reading, we present some definitions and notations that will be used through the article.

Definition 2.1. Let k, l be two positive integers and $P_{i,j}$ be $(k+1)(l+1)$ points in $\mathbb{R}^n$. The Bézier surface of degree (k, l) associated to the points $P_{i,j}$ for $i = 0, \cdots, k, j = 0, \cdots, l$ is defined as follows

$\Gamma: [0,1] \times [0,1] \to \mathbb{R}^n$

$$(u, v) \mapsto \sum_{i=0}^{k} \sum_{j=0}^{l} b_{i,k}(u) b_{j,l}(v) P_{i,j},$$

where $b_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-1}$ is the Bernstein polynomial.

The points $P_{i,j}$, $i = 0, \cdots, k, j = 0, \cdots, l$ are called control points of the Bézier surface of degree (k, l). The Bézier surface does not in general pass through the control points except for the corners of the control point grid. The Bézier surface is contained within the convex hull of the control points.

In practice, the bicubic Bézier surfaces (where $k = l = 3$) are most common. The bicubic Bézier surfaces are Bézier surfaces of degree (3,3).

$\Gamma: [0,1] \times [0,1] \to \mathbb{R}^n$

$$(u, v) \mapsto \sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j}$$

where $P_{i,j}$, $i = 0, \cdots, 3, j = 0, \cdots, 3$ are control points.

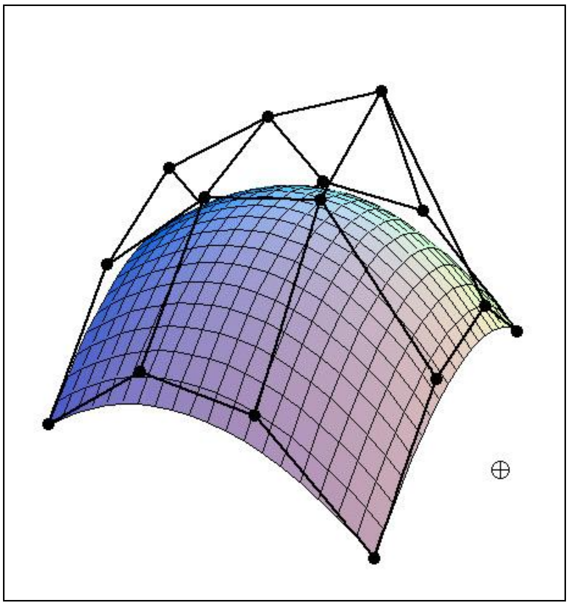

Figure 1. A bicubic Bézier surface.

Definition 2.2. Let N, M be positive integers and $P_{i,j}$ be $(3N+1)(3M+1)$ points in $\mathbb{R}^n$. The piecewise bicubic Bézier surface associated to the points $P_{i,j}$ for $i = 0, \cdots, 3N+1, j = 0, \cdots, 3M+1$ is a function $\Gamma: [0,1] \times [0,1] \to \mathbb{R}^n$ defined by: For any $(u, v) \in \left[\frac{r}{N}, \frac{r+1}{N+1}\right] \times \left[\frac{s}{M}, \frac{s+1}{M}\right]$, $r = 0, \cdots, N-1, s = 0, \cdots, M-1$,

$$\Gamma(u, v) := \Gamma_{r,s}(Nu - r, Mv - s)$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,3}(Nu - r) b_{j,3}(Mv - s) P_{3r+i, 3s+j}.$$

It is clear that $\Gamma_{r,s}$ is a bicubic Bézier surface and Γ is formed from $N \times M$ pieces of bicubic Béziers.

Notation 2.3.

- The vector space of bicubic Bézier surfaces is denoted by the symbol $B_{3,3}$.
- The vector space of $N \times M$-piece

bicubic Bézier surfaces is denoted by the symbol $B_{3,3}^{N,M}$.

We will define some norms and distances through control points on the space of bicubic Bézier surfaces $B_{3,3}$ and on the space of $N \times M$-piece bicubic Bézier surfaces $B_{3,3}^{N,M}$.

Definition 2.4. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_{3,3}}: B_{3,3} \to \mathbb{R}$ is defined by: For any $\Gamma(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)P_{i,j}$,

$$\|\Gamma\|_p^{B_{3,3}} := \begin{cases} \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{1/p} & \text{if } p \in [1, p[\\ \max_{\substack{i=0,\cdots,3 \\ j=0,\cdots,3}} \{\|P_{i,j}\|_\infty\} & \text{if } p = \infty, \end{cases}$$

where $\|\cdot\|_p$ is the p-norm on $\mathbb{R}^n$.

From the properties of the p-norm on $\mathbb{R}^n$ and the Minkowski inequality, it is easily seen that $\|\cdot\|_p^{B_{3,3}}$ is a norm on the vector space $B_{3,3}$. Indeed, it is a norm on the space $(\mathbb{R}^n)^8$ of control polygons. We then have an induced distance on $B_{3,3}$ by $d_p^{B_{3,3}}(\Gamma, \Delta) := \|\Gamma - \Delta\|_p^{B_{3,3}}$ for any $\Gamma, \Delta \in B_{3,3}$.

Definition 2.5. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_{3,3}^{N,M}} : B_{3,3}^{N,M} \to \mathbb{R}$ is defined by: For any

$$\Gamma(u,v) = \Gamma_{r,s}(Nu - r, Mv - s) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(Nu - r)b_{j,3}(Mv - s)P_{3r+i,3s+j}$$

for $(u,v) \in \left[\frac{r}{N}, \frac{r+1}{N+1}\right] \times \left[\frac{s}{M}, \frac{s+1}{M}\right], r = 0, \cdots, N - 1, s = 0, \cdots, M - 1$, then

$$\|\Gamma\|_p^{B_{3,3}^{N,M}} := \begin{cases} \frac{1}{N^{\frac{1}{p}}M^{\frac{1}{p}}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\left(\|\Gamma_{r,s}\|_p^{B_{3,3}}\right)^p\right)^{\frac{1}{p}} & if\ p \in [1, \infty[\\ \max_{\substack{r=0,\cdots,N-1 \\ s=0,\cdots,M-1}} \left\{\|\Gamma_{r,s}\|_\infty^{B_{3,3}}\right\} & if\ p = \infty. \end{cases}$$

Using the Minkowski inequality and the properties of the norm $\|\cdot\|_p^{B_{3,3}}$ on $B_{3,3}$, it is easy to see that $\|\cdot\|_p^{B_{3,3}^{N,M}}$ is a norm on the vector space $B_{3,3}^{N,M}$. Then we have again an induced distance on $B_{3,3}^{N,M}$ defined by $d_p^{B_{3,3}^{N,M}}(\Gamma, \Delta) := \|\Gamma - \Delta\|_p^{B_{3,3}^{N,M}}$ for any $\Gamma, \Delta \in B_{3,3}^{N,M}$.

The norms $\|\cdot\|_p^{B_{3,3}}$ and $\|\cdot\|_p^{B_{3,3}^{N,M}}$ can be computed more efficiently than, for instance, the L_p-norm. In the next section, we will find equivalence constants for the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p-norm on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surfaces.

3. THE EQUIVALENCE RELATIONS BETWEEN THE NORM $\|\cdot\|_p^{B_{3,3}^{N,M}}$ AND THE L_p-NORM ON THE SPACE $B_{3,3}^{N,M}$

Firstly, we use the properties of absolute values and Holder's inequality to estimate the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p-norm on the space $B_{3,3}^{N,M}$ of $N \times M$-piece bicubic Bézier surfaces.

Lemma 3.1. *Let* $p \in [1, \infty]$. *Then for any* $\Gamma \in B_{3,3}$,

$$\|\Gamma\|_{L_P} \leq \|\Gamma\|_P^{B_{3,3}}.$$

Proof. Let $\Gamma \in B_{3,3}$ and assume that

$$\Gamma(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)P_{i,j}$$

for $(u,v) \in [0,1] \times [0,1]$.

- Case $p = 1$.

$$\|\Gamma\|_{L_p} = \int_0^1\int_0^1 \left\|\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)P_{i,j}\right\|_1 dudv \leq \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)\|P_{i,j}\|_1 \leq \sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_1 = \|\Gamma\|_1^{B_{3,3}}.$$

- Case $p \in]1, \infty[$. Using Holder's inequality, we get

$$\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)\|P_{i,j}\|_p \leq \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}} \times \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\left(b_{i,3}(u)b_{j,3}(v)\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\leq \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}}$$

$$\times\left(\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)\right)^{\frac{p-1}{p}}$$

$$= \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}}.$$

Hence

$$\|\Gamma\|_{L_p}$$

$$= \left(\int_0^1\int_0^1 \left\|\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)P_{i,j}\right\|_p^p dudv\right)^{1/p}$$

$$\leq \left(\int_0^1\int_0^1 \left(\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)\|P_{i,j}\|_p\right)^p dudv\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^1\int_0^1 \sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\, dudv\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}} = \|\Gamma\|_P^{B_{3,3}}.$$

- Case $p = \infty$. We get

$$\|\Gamma\|_{L_\infty} = \left\|\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)P_{i,j}\right\|_{L_\infty}$$

$$\leq \left(\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u)b_{j,3}(v)\right)\max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}}\|P_{i,j}\|_\infty$$

$$= \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}}\|P_{i,j}\|_\infty = \|\Gamma\|_P^{B_{3,3}}.$$

Combining the above cases, we obtain the proof of this lemma. ■

Consider Case $p \in [1, \infty[$, Case $p = \infty$ and use the above lemma, we get the following proposition.

Proposition 3.2. *Let $p \in [1, \infty]$. Then for any $\Gamma \in B_{3,3}^{N,M}$,*

$$\|\Gamma\|_{L_p} \leq \|\Gamma\|_p^{B_{3,3}^{N,M}}.$$

Proof. Let $\Gamma \in B_{3,3}^{N,M}$ be an $N \times M$-piece bicubic Bézier surfaces and assume that

$$\Gamma(u,v) = \Gamma_{r,s}(Nu - r, Mv - s)$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(Nu - r)b_{j,3}(Mv - s)P_{3r+i,3s+j}$$

if $(u,v) \in \left[\frac{r}{N}, \frac{r+1}{N+1}\right] \times \left[\frac{s}{M}, \frac{s+1}{M}\right], r = 0,\cdots,N-1, s = 0,\cdots,M-1.$

- Case $p \in [1, \infty)$. We get

$$\|\Gamma\|_{L_p} = \left(\int_0^1\int_0^1 \|\Gamma(\mathrm{u},\mathrm{v})\|_p^p\, dudv\right)^{1/p}$$

$$= \left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\int_{\frac{r}{N}}^{\frac{r+1}{N}}\int_{\frac{s}{M}}^{\frac{s+1}{M}} \|\Gamma_{r,s}(Nu - r, Mv - s)\|_p^p\, dudv\right)^{1/p}$$

$$= \frac{1}{N^{1/p}M^{1/p}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\int_0^1\int_0^1 \|\Gamma_{r,s}(u,v)\|_p^p\, dudv\right)^{1/p}.$$

Using Lemma 3.1, we obtain

$$\|\Gamma\|_{L_p} \leq \frac{1}{N^{\frac{1}{p}}M^{\frac{1}{p}}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\|\Gamma\|_p^{B_{3,3}}\right)^{\frac{1}{p}} = \|\Gamma\|_p^{B_{3,3}^{N,M}}.$$

- Case $p = \infty$. We have

$$\|\Gamma\|_{L_\infty} = \max_{(u,v)\in[0,1]\times[0,1]}\|\Gamma(\mathrm{u},\mathrm{v})\|_\infty$$

$$= \max_{\substack{r=0,\cdots,N-1\\ s=0,\cdots,M-1}}\max_{(u,v)\in\left[\frac{r}{N},\frac{r+1}{N}\right]\times\left[\frac{s}{M},\frac{s+1}{M}\right]}\|\Gamma_{r,s}(\mathrm{Nu} - \mathrm{r}, \mathrm{Mv} - \mathrm{s})\|_\infty$$

$$= \max_{\substack{r=0,\cdots,N-1\\ s=0,\cdots,M-1}}\max_{(u,v)\in[0,1]\times[0,1]}\|\Gamma_{r,s}(\mathrm{u},\mathrm{v})\|_\infty.$$

Using Lemma 3.1, we get

$$\|\Gamma\|_{L_\infty} \leq \max_{\substack{r=0,\cdots,N-1\\ s=0,\cdots,M-1}}\|\Gamma_{r,s}\|_\infty^{B_{3,3}} = \|\Gamma\|_\infty^{B_{3,3}^{N,M}}.$$

Thus, the proof of this proposition is complete.

■

To find the remaining equivalence constant, we have to prove some lemmas as follows.

Lemma 3.3. *Let* $p \in [1,\infty[$ *and* P_0, P_1, P_2, P_3 *be four points on* $\mathbb{R}^n$*, we get*

$$\int_0^1 \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt \geq \frac{1}{2^{10p}}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right).$$

Proof. Put

$$A = \max\left\{\|P_0\|_p, \frac{1}{3}\|P_1\|_p, \frac{1}{3}\|P_2\|_p, \|P_3\|_p,\right\}.$$

- Case 1: $A = \|P_0\|_p$. We consider the interval $\left[1, \frac{1}{16}\right]$. For any $t \in \left[1, \frac{1}{16}\right]$, we have

$$\|P_0(1-t)^3\|_p \geq \left(1-\frac{1}{16}\right)^3 \|P_0\|_p = \frac{15^3}{16^3}\|P_0\|_p$$

$$\|P_1 3t(1-t)^2\|_p \leq 3.\frac{1}{16}\left(1-\frac{1}{16}\right)^2 \|P_1\|_p \leq \frac{9.15^2}{16^3}\|P_0\|_p$$

$$\|P_2 3t^2(1-t)\|_p \leq 3.\frac{1}{16^2}\left(1-\frac{1}{16}\right)\|P_2\|_p \leq \frac{9.15}{16^3}\|P_0\|_p$$

$$\|P_3 t^3\|_p \leq \frac{1}{16^3}\|P_3\|_p \leq \frac{1}{16^3}\|P_0\|_p.$$

So, for any $t \in \left[1, \frac{1}{16}\right]$, we have

$$\left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p$$

$$\geq \|P_0(1-t)^3\|_p - \|P_1 3t(1-t)^2\|_p - \|P_2 3t^2(1-t)\|_p - \|P_3 t^3\|_p$$

$$\geq \frac{15^3}{16^3}\|P_0\|_p - \frac{9.15^2}{16^3}\|P_0\|_p - \frac{9.15}{16^3}\|P_0\|_p - \frac{1}{16^3}\|P_0\|_p$$

$$= \frac{1214}{4096}\|P_0\|_p.$$

Then

$$\int_0^1 \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt \geq \int_0^{\frac{1}{16}} \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt$$

$$\geq \int_0^{\frac{1}{16}} \left(\frac{1214}{4096}\right)^p \|P_0\|_p^p dt$$

$$\geq \frac{1}{16}\left(\frac{1214}{4096}\right)^p \frac{1}{2+2.3^p}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right)$$

$$\geq \frac{1}{2^{10p}}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right).$$

- Case 2: $A = \frac{1}{3}\|P_1\|_p$. We consider the interval $\left[\frac{7}{32}, \frac{9}{32}\right]$. For any $t \in \left[\frac{7}{32}, \frac{9}{32}\right]$, we have

$$\|P_1 3t(1-t)^2\|_p \geq 3\frac{7}{32}\left(1-\frac{7}{32}\right)^2 \|P_1\|_p = \frac{21 \cdot 25^2}{32^3}\|P_1\|_p$$

$$\|P_0(1-t)^3\|_p \leq \left(1-\frac{7}{32}\right)^3 \|P_0\|_p \leq \frac{25^3}{3 \cdot 32^3}\|P_1\|_p$$

$$\|P_2 3t^2(1-t)\|_p \leq 3 \cdot \frac{9^2}{32^2}.\left(1-\frac{9}{32}\right)\|P_2\|_p \leq \frac{3 \cdot 9^2 \cdot 23}{32^3}\|P_1\|_p$$

$$\|P_3 t^3\|_p \leq \frac{9^3}{32^3}\|P_3\|_p \leq \frac{9^3}{3 \cdot 32^3}\|P_1\|_p.$$

So, for any $t \in \left[\frac{7}{32}, \frac{9}{32}\right]$, we have

$$\left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p$$

$$\geq \|P_1 3t(1-t)^2\|_p - \|P_0(1-t)^3\|_p - \|P_2 3t^2(1-t)\|_p - \|P_3 t^3\|_p$$

$$\geq \frac{21 \cdot 25^2}{32^3}\|P_1\|_p - \frac{25^3}{3 \cdot 32^3}\|P_1\|_p - \frac{3 \cdot 9^2 \cdot 23}{32^3}\|P_1\|_p - \frac{9^3}{3 \cdot 32^3}\|P_1\|_p$$

$$= \frac{6243}{98304}\|P_1\|_p.$$

Then

$$\int_0^1 \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt \geq \int_{\frac{7}{32}}^{\frac{9}{32}} \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt$$

$$\geq \int_{\frac{7}{32}}^{\frac{9}{32}} \left(\frac{6243}{98304}\right)^p \|P_1\|_p^p dt$$

$$\geq \frac{1}{16}\left(\frac{6243}{98304}\right)^p \frac{1}{2+2.3^p}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right)$$

$$\geq \frac{1}{2^{10p}}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right).$$

- Case 3: $A = \frac{1}{3}\|P_1\|_p$. When we substitute $s = 1 - t$, this case becomes Case 2.
- Case 4 : $A = \|P_3\|_p$. When we substitute $s = 1 - t$, this case becomes Case 1.

From the above four cases, we have

$$\int_0^1 \left\|\sum_{i=0}^{3} P_i b_{i,3}(t)\right\|_p^p dt \geq \frac{1}{2^{10p}}\left(\sum_{i=0}^{3}\|P_i\|_p^p\right).$$

■

Lemma 3.4. *Let* $p \in [1, \infty[$. *For* $\Gamma \in B_{3,3}$, *we have*

$$\|\Gamma\|_{L_p} \geq \frac{1}{2^{20}}\|\Gamma\|_p^{B_{3,3}}.$$

Proof. Let $\Gamma \in B_{3,3}$ be a bicubic Bézier surface and assume that

$$\Gamma(\mathrm{u},\mathrm{v}) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j}$$

for $(u, v) \in [0,1] \times [0,1]$. Using Lemma 3.3, we get

$$\|\Gamma\|_{L_p}$$

$$= \left(\int_0^1\int_0^1 \left\|\sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j}\right\|_p^p du dv\right)^{\frac{1}{p}}$$

$$\geq \left(\int_0^1 \frac{1}{2^{10p}}\sum_{i=0}^{3}\left\|\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j}\right\|_p^p dv\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{2^{10p}}\sum_{i=0}^{3}\int_0^1\left\|\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j}\right\|_p^p dv\right)^{\frac{1}{p}}$$

$$\geq \left(\frac{1}{2^{10p}}\sum_{i=0}^{3}\frac{1}{2^{10p}}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{2^{20p}}\sum_{i=0}^{3}\sum_{j=0}^{3}\|P_{i,j}\|_p^p\right)^{\frac{1}{p}} = \frac{1}{2^{20}}\|\Gamma\|_p^{B_{3,3}}.$$

■

For Case $p = \infty$, we also estimate on each cubic Bézier curve and then consider on each bicubic Bézier surface.

Lemma 3.5. *Let* P_0, P_1, P_2, P_3 *be four points on* $\mathbb{R}^n$, *we get*

$$\max_{t\in[0,1]}\left\|\sum_{i=0}^{3} P_i\, b_{i,3}(t)\right\|_\infty \geq \frac{13}{96}\max_{i=0,\cdots,3}\|P_i\|_\infty.$$

Proof. Put

$$A = \max\left\{\|P_0\|_\infty, \frac{1}{3}\|P_1\|_\infty, \frac{1}{3}\|P_2\|_\infty, \|P_3\|_\infty\right\}.$$

- Case 1: $A = \|P_0\|_\infty$. Since $\sum_{i=0}^{3} P_i\, b_{i,3}(0) = P_0$, hence

$$\max_{t\in[0,1]}\left\|\sum_{i=0}^{3} P_i\, b_{i,3}(t)\right\|_\infty \geq \|P_0\|_\infty = \max_{i=0,\cdots,3}\|P_i\|_\infty.$$

- Case 2: $A = \frac{1}{3}\|P_1\|_\infty$. At $t = \frac{1}{4}$, we get

$$\left\|\sum_{i=0}^{3} P_i\, b_{i,3}\left(\frac{1}{4}\right)\right\|_\infty$$

$$\geq \frac{27}{64}\|P_1\|_\infty - \frac{27}{64}\|P_0\|_\infty - \frac{9}{64}\|P_2\|_\infty - \frac{1}{64}\|P_3\|_\infty$$

$$\geq \frac{27}{64}\|P_1\|_\infty - \frac{9}{64}\|P_1\|_\infty - \frac{9}{64}\|P_1\|_\infty - \frac{1}{192}\|P_1\|_\infty$$

$$= \frac{13}{96}\|P_1\|_\infty.$$

Thus

$$\max_{t\in[0,1]}\left\|\sum_{i=0}^{3} P_i\, b_{i,3}(t)\right\|_\infty \geq \frac{13}{96}\max_{i=0,\cdots,3}\|P_i\|_\infty.$$

- Case 3: $A = \frac{1}{3}\|P_2\|_\infty$. This case is the same to Case 2.
- Case 4: $A = \|P_2\|_\infty$. This case is the same to Case 2.

From the above four cases, we have

$$\max_{t\in[0,1]} \left\| \sum_{i=0}^{3} P_i\, b_{i,3}(t) \right\|_\infty \geq \frac{13}{96} \max_{i=0,\cdots,3} \|P_i\|_\infty.$$

■

Lemma 3.6. *For* $P_{i,j} \in \mathbb{R}^n, i = 0,\cdots,3, j = 0,\cdots,3$. *We have*

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty \geq \frac{1}{2^9} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty.$$

Proof. Put

$$A = \Big\{ \|P_{0,0}\|_\infty, \frac{1}{64}\|P_{0,1}\|_\infty, \frac{1}{64}\|P_{0,2}\|_\infty, \|P_{0,3}\|_\infty$$
$$\frac{1}{64}\|P_{1,0}\|_\infty, \frac{1}{64^2}\|P_{1,1}\|_\infty, \frac{1}{64^2}\|P_{1,2}\|_\infty, \frac{1}{64}\|P_{1,3}\|_\infty$$
$$\frac{1}{64}\|P_{1,0}\|_\infty, \frac{1}{64^2}\|P_{1,1}\|_\infty, \frac{1}{64^2}\|P_{1,2}\|_\infty, \frac{1}{64}\|P_{1,3}\|_\infty$$
$$\|P_{3,0}\|_\infty, \frac{1}{64}\|P_{3,1}\|_\infty, \frac{1}{64}\|P_{3,2}\|_\infty, \|P_{3,3}\|_\infty \Big\}.$$

- Case 1:

$$A = \max\left\{ \|P_{0,0}\|_\infty, \frac{1}{64}\|P_{0,1}\|_\infty, \frac{1}{64}\|P_{0,2}\|_\infty, \|P_{0,3}\|_\infty \right\}.$$

Then

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty \geq \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(0) b_{j,3}(v) P_{i,j} \right\|_\infty = \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{j,3}(v) P_{i,j} \right\|_\infty.$$

Using Lemma 3.5, we get

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty \geq \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{j,3}(v) P_{i,j} \right\|_\infty \geq \frac{13}{96} \max_{j=0,\cdots,3} \|P_{0,j}\|_\infty \geq \frac{13}{96}\cdot\frac{1}{64} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty \geq \frac{1}{2^9} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty.$$

- Case 2:

$$A = \max\left\{ \|P_{3,0}\|_\infty, \frac{1}{64}\|P_{3,1}\|_\infty, \frac{1}{64}\|P_{3,2}\|_\infty, \|P_{3,3}\|_\infty \right\}.$$

Thus

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty \geq \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(1) b_{j,3}(v) P_{i,j} \right\|_\infty = \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{j,3}(v) P_{1,j} \right\|_\infty.$$

Using Lemma 3.5, we get

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty \geq \max_{v\in[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{j,3}(v) P_{3,j} \right\|_\infty \geq \frac{13}{96} \max_{j=0,\cdots,3} \|P_{3,j}\|_\infty \geq \frac{13}{96}\cdot\frac{1}{64} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty \geq \frac{1}{2^9} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty.$$

- Case 3:

$$A = \max\left\{ \|P_{0,0}\|_\infty, \frac{1}{64}\|P_{1,0}\|_\infty, \frac{1}{64}\|P_{2,0}\|_\infty, \|P_{3,0}\|_\infty \right\}.$$

This case is similar to Case 1 and Case 2.

- Case 4:

$A =$

$\max\left\{\|P_{0,4}\|_\infty, \frac{1}{64}\|P_{1,4}\|_\infty, \frac{1}{64}\|P_{2,4}\|_\infty, \|P_{3,4}\|_\infty\right\}$.

This case is similar to Case 1 and Case 2.

- Case 5: $A = \frac{1}{64^2}\|P_{1,1}\|_\infty$. Consider at $(u,v) = \left(\frac{1}{4}, \frac{1}{4}\right)$, we have

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty$$
$$\geq \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}\left(\frac{1}{4}\right) b_{j,3}\left(\frac{1}{4}\right) P_{i,j} \right\|_\infty$$
$$= \left\| \frac{27^2}{64^2}P_{0,0} + \frac{27^2}{64^2}P_{0,1} + \frac{243}{64^2}P_{0,2} + \frac{27}{64^2}P_{0,3} + \frac{27^2}{64^2}P_{1,0} + \frac{27^2}{64^2}P_{1,1} + \frac{243}{64^2}P_{1,2} + \frac{27}{64^2}P_{1,3} + \frac{243}{64^2}P_{2,0} + \frac{243}{64^2}P_{2,1} + \frac{81}{64^2}P_{2,2} + \frac{9}{64^2}P_{2,3} + \frac{27}{64^2}P_{3,0} + \frac{27}{64^2}P_{3,1} + \frac{81}{9}P_{3,2} + \frac{1}{64^2}P_{3,3} \right\|_\infty.$$

Since $\|P_{i,0}\|_\infty \leq \frac{1}{64^2}\|P_{1,1}\|_\infty, \|P_{i,1}\|_\infty \leq \frac{1}{64}\|P_{1,1}\|_\infty$, $\|P_{i,2}\|_\infty \leq \frac{1}{64}\|P_{1,1}\|_\infty$, $\|P_{i,3}\|_\infty \leq \frac{1}{64^2}\|P_{1,1}\|_\infty$ for $i = 0, i = 3$ and $\|P_{j,0}\|_\infty \leq \frac{1}{64}\|P_{1,1}\|_\infty, \|P_{j,1}\|_\infty \leq \|P_{1,1}\|_\infty, \|P_{j,2}\|_\infty \leq \|P_{1,1}\|_\infty, \|P_{j,3}\|_\infty \leq \frac{1}{64}\|P_{1,1}\|_\infty$ for $j = 1, j = 2$, thus

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty$$
$$\geq \frac{27^2}{64^2}\|P_{1,1}\|_\infty - \frac{27^2}{64^4}\|P_{1,1}\|_\infty - \frac{27^2}{64^3}\|P_{1,1}\|_\infty - \frac{243}{64^3}\|P_{1,1}\|_\infty - \frac{27}{64^4}\|P_{1,1}\|_\infty - \frac{27^2}{64^3}\|P_{1,1}\|_\infty - \frac{243}{64^2}\|P_{1,1}\|_\infty - \frac{27}{64^3}\|P_{1,1}\|_\infty - \frac{243}{64^3}\|P_{1,1}\|_\infty - \frac{243}{64^2}\|P_{1,1}\|_\infty - \frac{81}{64^2}\|P_{1,1}\|_\infty - \frac{9}{64^3}\|P_{1,1}\|_\infty - \frac{27}{64^4}\|P_{1,1}\|_\infty - \frac{27}{64^3}\|P_{1,1}\|_\infty - \frac{81}{64^3}\|P_{1,1}\|_\infty - \frac{1}{64^4}\|P_{1,1}\|_\infty = \frac{529136}{64^4}\|P_{1,1}\|_\infty$$
$$\geq \frac{1}{2^5}\|P_{1,1}\|_\infty \geq \frac{1}{2^{17}} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty.$$

- Case 6: $A = \frac{1}{64^2}\|P_{1,2}\|_\infty$. Consider at $(u,v) = \left(\frac{1}{4}, \frac{3}{4}\right)$, we have

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty$$
$$\geq \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}\left(\frac{1}{4}\right) b_{j,3}\left(\frac{3}{4}\right) P_{i,j} \right\|_\infty$$
$$= \left\| \frac{27}{64^2}P_{0,0} + \frac{243}{64^2}P_{0,1} + \frac{27^2}{64^2}P_{0,2} + \frac{27^2}{64^2}P_{0,3} + \frac{27}{64^2}P_{1,0} + \frac{243}{64^2}P_{1,1} + \frac{27^2}{64^2}P_{1,2} + \frac{27^2}{64^2}P_{1,3} + \frac{9}{64^2}P_{2,0} + \frac{81}{64^2}P_{2,1} + \frac{243}{64^2}P_{2,2} + \frac{243}{64^2}P_{2,3} + \frac{1}{64^2}P_{3,0} + \frac{9}{64^2}P_{3,1} + \frac{27}{9}P_{3,2} + \frac{27}{64^2}P_{3,3} \right\|_\infty.$$

Note that $\|P_{i,0}\|_\infty \leq \frac{1}{64^2}\|P_{1,2}\|_\infty, \|P_{i,1}\|_\infty \leq \frac{1}{64}\|P_{1,2}\|_\infty$, $\|P_{i,2}\|_\infty \leq \frac{1}{64}\|P_{1,2}\|_\infty$, $\|P_{i,3}\|_\infty \leq \frac{1}{64^2}\|P_{1,2}\|_\infty$ for $i = 0, i = 3$ and $\|P_{j,0}\|_\infty \leq \frac{1}{64}\|P_{1,2}\|_\infty, \|P_{j,1}\|_\infty \leq \|P_{1,2}\|_\infty, \|P_{j,2}\|_\infty \leq \|P_{1,2}\|_\infty, \|P_{j,3}\|_\infty \leq \frac{1}{64}\|P_{1,2}\|_\infty$ for $j = 1, j = 2$, then

$$\max_{(u,v)\in[0,1]\times[0,1]} \left\| \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(u) b_{j,3}(v) P_{i,j} \right\|_\infty$$
$$\geq \frac{27^2}{64^2}\|P_{1,2}\|_\infty - \frac{27}{64^4}\|P_{1,2}\|_\infty - \frac{243}{64^3}\|P_{1,2}\|_\infty - \frac{27^2}{64^3}\|P_{1,2}\|_\infty - \frac{27^2}{64^4}\|P_{1,2}\|_\infty - \frac{27}{64^3}\|P_{1,2}\|_\infty - \frac{243}{64^2}\|P_{1,2}\|_\infty - \frac{27^2}{64^3}\|P_{1,2}\|_\infty - \frac{9}{64^3}\|P_{1,2}\|_\infty - \frac{81}{64^2}\|P_{1,2}\|_\infty - \frac{243}{64^2}\|P_{1,2}\|_\infty - \frac{243}{64^3}\|P_{1,2}\|_\infty - \frac{1}{64^4}\|P_{1,2}\|_\infty - \frac{9}{64^3}\|P_{1,2}\|_\infty - \frac{27}{64^3}\|P_{1,2}\|_\infty - \frac{27}{64^4}\|P_{1,2}\|_\infty = \frac{533744}{64^4}\|P_{1,2}\|_\infty$$
$$\geq \frac{1}{2^5}\|P_{1,2}\|_\infty \geq \frac{1}{2^{17}} \max_{\substack{i=0,\cdots,3\\ j=0,\cdots,3}} \|P_{i,j}\|_\infty.$$

- Case 7: $A = \frac{1}{64^2}\|P_{2,1}\|_\infty$. This case is similar to Case 5 and Case 6.

- Case 8: $A = \frac{1}{64^2}\|P_{2,2}\|_\infty$. This case is similar to Case 5 and Case 6.

From the above cases, we get the proof of Lemma. ■

Combining the above lemmas, we obtain the following proposition.

Proposition 3.7. *Let* $p \in [1,\infty]$. *For* $\Gamma \in B_{3,3}^{N,M}$, *we have*

$$\|\Gamma\|_p^{B_{3,3}^{N,M}} \le 2^9 \|\Gamma\|_{L_p}.$$

Proof. For any $\Gamma \in B_{3,3}^{N,M}$ be an $N\times M$-piece bicubic Bézier surfaces and assume that

$$\Gamma(u,v) = \Gamma_{r,s}(Nu-r, Mv-s)$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(Nu-r)b_{j,3}(Mv-s)P_{3r+i,3s+j}$$

if $(u,v) \in \left[\frac{r}{N},\frac{r+1}{N+1}\right]\times\left[\frac{s}{M},\frac{s+1}{M}\right], r = 0,\cdots,N-1, s = 0,\cdots,M-1$.

- Case $p \in [1,\infty[$. We have

$$\|\Gamma\|_{L_p} = \left(\int_0^1\int_0^1 \|\Gamma(\mathrm{u},\mathrm{v})\|_p^p\, dudv\right)^{1/p}$$

$$= \left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\int_{\frac{r}{N}}^{\frac{r+1}{N}}\int_{\frac{s}{M}}^{\frac{s+1}{M}} \left\|\Gamma_{r,s}(Nu-r, Mv-s)\right\|_p^p dudv\right)^{1/p}$$

$$= \frac{1}{N^{1/p}M^{1/p}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\int_0^1\int_0^1 \left\|\Gamma_{r,s}(u,v)\right\|_p^p dudv\right)^{1/p}.$$

Using Lemma 3.4, we obtain

$$\|\Gamma\|_{L_p} \ge \frac{1}{N^{\frac{1}{p}}M^{\frac{1}{p}}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\frac{1}{2^{20}}\left(\left\|\Gamma_{r,s}\right\|_p^{B_{3,3}}\right)^p\right)^{\frac{1}{p}}$$

$$= \frac{1}{2^{20}}\frac{1}{N^{\frac{1}{p}}M^{\frac{1}{p}}}\left(\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\left(\left\|\Gamma_{r,s}\right\|_p^{B_{3,3}}\right)^p\right)^{\frac{1}{p}}$$

$$= \frac{1}{2^{20}}\|\Gamma\|_p^{B_{3,3}^{N,M}}.$$

- Case $p = \infty$. We have

$$\|\Gamma\|_{L_\infty} = \max_{r,s}\left\|\Gamma_{r,s}\right\|_{L_\infty}.$$

Using the above lemmas, we get

$$\|\Gamma\|_{L_\infty} = \max_{r,s}\left\|\Gamma_{r,s}\right\|_{L_\infty} \ge \frac{1}{2^9}\max_{r,s}\|\Gamma\|_\infty^{B_{3,3}} = \frac{1}{2^9}\|\Gamma\|_\infty^{B_{3,3}^{N,M}}.$$

Thus, from two above cases, we obtain the proof of the proposition. ■

From the above results, we have following theorem about equivalence constants for the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the norm $\|\Gamma\|_{L_p}$ on the space of $N\times M$-piece bicubic Bézier surface.

Theorem 1.1. *For* $p \in [1,\infty]$. *Let* $\Gamma \in B_{3,3}^{N,M}$, *we get*

$$\|\Gamma\|_{L_p} \le \|\Gamma\|_p^{B_{3,3}^{N,M}} \le 2^9 \|\Gamma\|_{L_p}.$$

Proof. Using Propositions 3.2 and 3.7, we get the proof of this theorem. ■

The equivalence constants do not depend on the number of pieces in piecewise bicubic Bézier surface. From the above theorem, we get the following corollary:

$$d_{L_p}(\Gamma,\Delta) \le d_p^{B_{3,3}^{N,M}}(\Gamma,\Delta) \le 2^9 d_{L_p}(\Gamma,\Delta)$$

for any $\Gamma,\Delta \in B_{3,3}^{N,M}$.

4. CONCLUSION

In this article, we propose the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ on the space of $N\times M$-piece bicubic Bézier surfaces. Piecewise bicubic Bézier surfaces are defined through its control points. This norm is also determined through its control points. Then, the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ is convenient to compute. On the space of $N\times M$-piece bicubic Bézier surface, the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p norm are equivalent. The article give equivalence constants for the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$ and the L_p norm on the space of $N\times M$-piece bicubic Bézier surface. The equivalence constants do not depend on the number of pieces in piecewise bicubic Bézier surface. Form the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$, we have the induced distance $d_p^{B_{3,3}^{N,M}}$. Using the norm $\|\cdot\|_p^{B_{3,3}^{N,M}}$, we can consider the convergence for sequences of piecewise bicubic Bézier surfaces. Thus, by using this norm and piecewise bicubic Bézier surfaces, it is convenient to find optimal shapes.

DECLARATION OF CONFLICTS OF INTEREST

The author(s) declare that this research has no conflicts of interest.

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Chuẩn và các hằng số tương đương trong không gian mặt phẳng Bézier song bậc ba từng mảnh

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TÓM TẮT

Trong bài báo này chúng tôi trình bày một lớp chuẩn của các mặt phẳng Bézier song bậc ba $N \times M$ mảnh và sự tương đương của một số chuẩn trên không gian $B_{3,3}^{N,M}$ của các mặt phẳng Bézier song bậc ba $N \times M$ mảnh. Một mặt phẳng Bézier song bậc ba là một lưới 4×4 của 16 điểm điều khiển và được giới hạn bởi các đường cong Bézier bậc ba. Chúng ta định nghĩa một lớp chuẩn $\|\cdot\|_p^{B_{3,3}}$ trên không gian các mặt phẳng Bézier song bậc ba và một lớp chuẩn $\|\cdot\|_p^{B_{3,3}^{N,M}}$ trên không gian các mặt phẳng Bézier $N \times M$ mảnh. Các chuẩn này được xác định bởi các điểm điều khiển. Khi đó chuẩn $\|\cdot\|_p^{B_{3,3}^{N,M}}$ và chuẩn L_p là các chuẩn trên không gian các mặt phẳng Bézier song bậc ba $N \times M$ mảnh. Chúng tôi sẽ nghiên cứu các hằng số tương đương giữa chuẩn $\|\cdot\|_p^{B_{3,3}^{N,M}}$ và chuẩn L_p trên không gian các mặt phẳng Bézier song bậc ba $N \times M$ mảnh. Các hằng số tương đương này không phụ thuộc vào số mảnh của mặt phẳng Bézier song bậc ba từng mảnh. Từ kết quả này, chúng ta có thể xét sự hội tụ của một dãy các mặt phẳng Bézier song bậc ba từng mảnh. Điều này đóng vai trò quan trọng trong việc trong việc ứng dụng mặt phẳng Bézier song bậc ba từng mảnh để tìm hình dạng tối ưu.

Từ khóa: *Mặt phẳng Bézier, mặt phẳng Bézier song bậc ba, hằng số tương đương, chuẩn, khoảng cách.*