TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Một phương pháp mới để thiết kế quan sát trạng thái cho hệ Glucose-Insulin có trễ

Đào Thị Hải Yến[1], **Đinh Công Hướng**[2,*]

[1] *Khoa Khoa học Tự nhiên, Trường Đại học Phú Yên, Phú Yên, Việt Nam*
[2] *Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Bình Định, Việt Nam*

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TÓM TẮT

Bài báo này trình bày một phương pháp mới để thiết kế một bộ quan sát trạng thái cho một lớp hệ Glucose-Insulin phi tuyến với hai độ trễ thời gian. Dựa vào tính dương của nghiệm, chúng tôi đã đề xuất một phép biến đổi tọa độ mới để đưa mô hình đang xem xét về một hệ quan sát được. Trong hệ tọa độ mới này, các độ trễ của hệ đã cho xuất hiện trong véc tơ đầu vào và véc tơ đầu ra mà không xuất hiện trong véc tơ trạng thái. Hệ quả là chúng ta dễ dàng thiết kế được bộ quan sát để ước lượng thông tin của biến trạng thái. Các kết quả minh họa số được trình bày trong bài báo cho thấy tính hiệu quả của phương pháp đề xuất.

Từ khóa: *Quan sát trạng thái, phép biến đổi trạng thái, hệ trễ thời gian.*

*Tác giả liên hệ chính:
Email: dinhconghuong@qnu.edu.vn

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A new method for designing obversvers of nonlinear time-delay Glucose-Insulin system

Dao Thi Hai Yen[1], **Dinh Cong Huong**[2,*]

[1] *Faculty of Natural Sciences, Phu Yen University, Phu Yen, Vietnam*
[2] *Faculty of Mathematics and Statistics, Quy Nhon University, Binh Dinh, Vietnam*

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ABSTRACT

This paper presents a novel method for designing a state observer of a class of nonlinear Glucose-Insulin (GI) systems with two time delays. Based on the positivity of its solutions, we suggested a new state transformation to transform the model into a new observable form. In this new form, the time delays in the system description appear in the input and out put vectors, but not in the state vector. As a result, a state observer can be easily designed. Simulation results are given to illustrate the effectiveness of the suggested method.

Keywords: *State observers, state transformations, time-delay systems.*

1. INTRODUCTION

Diabetes is a world-wide epidemic. In the treatment of diabetes, it is essential to monitor glucose and insulin levels in diabetic patients so that appropriate treatment such as insulin injections can be implemented to maintain satisfactory blood glucose levels. Blood glucose levels can be readily measured by using a glucose-oxidase-based amperometric sensor. The sensor utilizes glucose in interstitial fluid under the skin to indirectly reflect the blood sugar level. Whereas, insulin measurements are slower, harder to obtain and less accurate than glucose measurements. Thus, model-based state observers have been proposed in order to estimate insulin levels.[1−2] The contribution of this paper is in the design of a novel state observer to estimate insulin levels in diabetic patients.

In this paper, we consider a general nonlinear time-delay GI model of the following form

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-\tau)) + Bu(t) \\ &\quad + g(y(t), y(t-\tau_g)), t \geq 0, \end{aligned} \tag{1}$$

$$\begin{aligned} x(\theta) &= \phi(\theta),\ \theta \in [-\tau_{\max}, 0], \\ &\quad \tau_{\max} = \max\{\tau, \tau_g\}, \end{aligned} \tag{2}$$

$$y(t) = Cx(t) = x_1(t), \tag{3}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$ is the state vector, $x_1(t)$ and $x_2(t)$ are the blood glucose and insulin levels, respectively, $x_3(t)$ and $x_4(t)$ are the insulin mass in the accessible and not-accessible subcutaneous depot, respectively. The control input $u(t)$ is the subcutaneous insulin delivery rate while the output variable is defined as the measured glucose levels, $x_1(t)$. In (1)-(3),

$$f(x(t), x(t-\tau)) = \begin{bmatrix} -a_1 x_1(t) x_2(t-\tau) \\ -a_2 x_2(t) + a_3 x_4(t) \\ -a_4 x_3(t) \\ a_5 x_3(t) - a_6 x_4(t) \end{bmatrix},$$

$B = \begin{bmatrix} 0 \\ 0 \\ a_7 \\ 0 \end{bmatrix}$, $C = [1 \quad 0 \quad 0 \quad 0]$, a_i $(i = 1, 2, \ldots, 7)$ are positive parameters, $\phi(\theta) =$

*Corresponding author:
Email: dinhconghuong@qnu.edu.vn

$\begin{bmatrix} \phi_1(\theta) & \phi_2(\theta) & \phi_3(\theta) & \phi_4(\theta) \end{bmatrix}^T$ is a continuous initial function, $\phi_i(\theta)$ $(i = 1, 2, 3, 4)$ are positive functions, $\phi_i(t) > 0$, and

$$g(y(t), y(t-\tau_g)) = \begin{bmatrix} g_1(y(t), y(t-\tau_g)) \\ g_2(y(t), y(t-\tau_g)) \\ g_3(y(t), y(t-\tau_g)) \\ g_4(y(t), y(t-\tau_g)) \end{bmatrix}$$

is a function depending on the output $y(t)$ and $y(t-\tau_g)$ with $g_1(y(t), y(t-\tau_g)) > 0$ for all $t \geq 0$. In our model, $\tau > 0$ and $\tau_g > 0$ are known constant time delays. As defined in[3], τ_g is the apparent delay with which the pancreas varies secondary insulin release in response to varying plasma glucose concentrations, while τ is the delay with which insulin acts in stimulating glucose uptake by peripheral tissues.[4]

Note that, when $\tau = 0$, and by letting $a_1 = K_{xgi}$, $a_2 = K_{xi}$, $a_3 = \frac{1}{V_I tmax,I}$, $a_4 = a_5 = a_6 = \frac{1}{tmax,I}$, $a_7 = 1$ and

$$g(y(t), y(t-\tau_g)) = \begin{bmatrix} \frac{T_{gh}}{V_G} \\ \frac{T_{iGmax}}{V_I} \frac{\left(\frac{x_1(t-\tau_g)}{G^*}\right)^\gamma}{1+\left(\frac{x_1(t-\tau_g)}{G^*}\right)^\gamma} \\ 0 \\ 0 \end{bmatrix}$$

for all $t \geq 0$, our model (1)-(3) is reduced to the same GI model as considered in.[3] However, to our knowledge, a direct observer design procedure for (1)-(3) has not yet been reported in the literature as there are some difficulties in dealing with the nonlinear delayed term $x_1(t)x_2(t-\tau)$ in the model.

Recently, the authors of the work[5] proposed an observer design for a nonlinear minimal dynamic model of glucose disappearance and insulin kinetics. They transformed the model into a nonlinear observer normal form and then estimated the state variables that are not directly available from the system, i.e. the remote compartment insulin utilization, the plasma insulin deviation and the infusion rate. However, the results of the work[5] only dealt with nonlinear term $x_1(t)x_2(t)$, that is, $\tau = 0$ in the model. So far, the results of[5] has not been extended to the time-delay model of the form (1)-(3). This motivates the present study.

2. STATE TRANSFORMATION

In this section, we present a novel procedure for designing a state observer of the nonlinear time-delay model (1)-(3). In our design procedure, we propose a two-stage process to transform (1)-(3) into a new observable form where the nonlinear term $x_1(t)x_2(t-\tau)$ is injected into the output and input of the system. To achieve this, we first utilize the concept of diffeomorphism on the output[5] by defining a new output $\bar{y}(t) = -\ln(y(t))$ for the system (1)-(3). To ensure such a diffeomorphism can take place, we show that $x_1(t) > 0$ for all $t \geq 0$ for the model (1)-(3) (i.e., in order for $\ln(y(t))$ to exist, it is necessary that $y(t) > 0$, $\forall t \geq 0$). In the second stage of the process, we introduce a novel state transformation to transform the system into a novel observable form where a state observer can be easily designed.

First, let us prove that $x_1(t) > 0$ for all $t \geq 0$. Indeed, if there exists a $t_0 > 0$ such that $x_1(t_0) = 0$, then according to the continuity of the solution of a differential equation, $\dot{x}_1(t_0) \leq 0$, which is a contradiction since we have

$$\begin{aligned} \dot{x}_1(t_0) &= -a_1 x_1(t_0) x_2(t_0 - \tau) \\ &\quad + g_1(y(t_0), y(t_0 - \tau_g)) \\ &= g_1(0, y(t_0 - \tau_g)) > 0. \end{aligned} \tag{4}$$

Hence, we can conclude that $x_1(t) > 0$ for all $t \geq 0$. With this fact, we can now utilize the concept of diffeomorphism on the output.[5] For this, let us divide both sides of the first equation of (1) by $-x_1(t)$ and let a new output be defined as $\bar{y}(t) = \xi(t) = -\ln(y(t))$. Then (1)-(3) is equivalent to the following

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + A_d \tilde{x}(t-\tau) \\ &\quad + Bu(t) + \bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)), \end{aligned}$$

$$t \geq 0, \tag{5}$$
$$\tilde{x}(\theta) = \tilde{\phi}(\theta), \; \theta \in [-\tau_{\max}, 0], \tag{6}$$
$$\bar{y}(t) = C\tilde{x}(t) = \xi(t), \tag{7}$$

where

$$\tilde{x}(t) = \begin{bmatrix} \xi(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix},$$
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_4 & 0 \\ 0 & 0 & a_5 & -a_6 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\phi}(\theta) = \begin{bmatrix} -\ln(\phi_1(\theta)) \\ \phi_2(\theta) \\ \phi_3(\theta) \\ \phi_4(\theta) \end{bmatrix},$$
$$\mu(\bar{y}(t), \bar{y}(t-\tau_g)) = \begin{bmatrix} -g_1(\bar{y}(t), \bar{y}(t-\tau_g))e^{\bar{y}(t)} \\ g_2(\bar{y}(t), \bar{y}(t-\tau_g)) \\ g_3(\bar{y}(t), \bar{y}(t-\tau_g)) \\ g_4(\bar{y}(t), \bar{y}(t-\tau_g)) \end{bmatrix}.$$

As we know, in many practical applications, the states of the considered systems are not easily obtained due to technical or economic reasons. In this case, the estimation of actual states and output feedback control law are very necessary. Therefore, the problem of designing state observers for dynamical systems has attracted considerable attention in the literature (see, for example,[6−8]). On the other hand, since time delay is often encountered in many practical control systems[9], the problem of designing a state observer to estimate the state vector of a time-delay system is an important research topic and it has received considerable research attention in the literature. In particular, state observers have important applications in realisation of state-feedback control, system supervision, fault diagnosis of dynamic processes, and general control and diagnosis issues from available information.[10−12,13]

From (5)-(7), we can proceed to design a state observer to estimate the unknown state vector, $\tilde{x}(t)$. In the literature, there are well-known state observer design methods[6−8,14] for various time-delay systems of the type (5)-(7). These methods aim at designing an asymptotic state observer, $\hat{\tilde{x}}(t)$, such that it converges with any prescribed convergence rate to $\tilde{x}(t)$, i.e., $\hat{\tilde{x}}(t) \to \tilde{x}(t)$. However, based on these methods[6−8,14], it is not possible to design a satisfactory state observer for the system (5)-(7). This is due to the fact that the matrix pair (A, C) is not observable[14] as well as there are some fixed poles in the observer error dynamics.[6−8] These stable fixed poles are very close to zeros and thus resulted in a very slow convergent rate for the designed state observers. Recognizing this difficulty, in this paper, we present a new type of state observer, and referred to it as a "delayed" state observer. In this regard, the designed state observer will be able to estimate a delayed version of the state vector instead of the instantaneous state vector, which is impossible based on existing observer design methods.[6−8,14]

Accordingly, in the following, we consider the general form of system (5)-(7), where $\tilde{x} = \begin{bmatrix} \xi(t) & \tilde{x}_2(t) & \ldots & \tilde{x}_n(t) \end{bmatrix}^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$ and $C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$. We will present a new state transformation which transforms the considered system into an observable form where the time-delay term $A_d\tilde{x}(t-\tau)$ will be injected into the input and output of the transformed system. This will then allow a delayed state observer to be easily designed.

For $m, n \in \mathbb{N}$, $n > 1$ and an arbitrary matrix $M \in \mathbb{R}^{1\times n}$, M^T denotes the transpose of M, $0_{m,n}$ denotes the $m \times n$ zero matrix, $M = \begin{bmatrix} [M]_L & [M]_R \end{bmatrix}$, where $[M]_L \in \mathbb{R}$ and

$[M]_R \in \mathbb{R}^{1\times(n-1)}$ are sub-matrices of M.

We define a new change of coordinates as follows

$$z(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} M_1 & N_1 \\ \vdots & \vdots \\ M_n & N_n \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{x}(t-\tau) \end{bmatrix}, \quad (8)$$

where matrices M_i, N_i $(i = 1, 2, \ldots, n)$ are generated by the following algorithm

$$M_1 = C, \quad N_1 = 0, \quad (9)$$
$$M_{i+1} = M_i A - \alpha_i M_1, \quad i = 1, 2, \ldots, n-1, \quad (10)$$
$$N_{i+1} = M_i A_d + N_i A - \beta_i M_1, \quad i = 1, 2, \ldots, n-1, \quad (11)$$

where α_i and β_i are scalars to be determined later.

Theorem 1. *For some scalars γ_i $(i = 2, 3, \ldots, n)$, α_j and β_j $(j = 1, 2, \ldots, n-1)$, if the following equations hold*

$$[N_i A_d]_R = 0, \; i = 2, \ldots, n, \quad (12)$$
$$[M_n A - \sum_{i=2}^{n} \gamma_i M_i]_R = 0, \quad (13)$$
$$[M_n A_d + N_n A - \sum_{i=2}^{n} \gamma_i N_i]_R = 0, \quad (14)$$

then the change of coordinate (8) transforms the system (5)-(7) into the following form

$$\begin{aligned} \dot{z}(t) &= \bar{A}z(t) + \bar{B}u(t) + \bar{B}_1 u(t-\tau) \\ &+ \Gamma \bar{y}(t) + \Gamma_1 \bar{y}(t-\tau) \\ &+ \Gamma_2 \bar{y}(t-2\tau) \\ &+ \Gamma_3 \bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)) \\ &+ \Gamma_4 \bar{\mu}(\bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)), \; t \geq \tau, \end{aligned} \quad (15)$$

$$\bar{y}(t) = \bar{C}z(t), \quad (16)$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & \gamma_2 & \gamma_3 & \ldots & \gamma_n \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} M_1 B \\ M_2 B \\ \vdots \\ M_{n-1} B \\ M_n B \end{bmatrix}, \; \bar{C}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} 0 \\ N_2 B \\ \vdots \\ N_{n-1} B \\ N_n B \end{bmatrix}, \; \Gamma = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \Gamma(n,1) \end{bmatrix},$$

$$\Gamma(n,1) = \left[M_n A - \sum_{j=2}^{n} \gamma_j M_j \right]_L,$$

$$\Gamma_1 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \Gamma_1(n,1) \end{bmatrix}, \; \Gamma_1(n,1) = \left[M_n A_d + N_n A - \sum_{j=2}^{n} \gamma_j N_j \right]_L,$$

$$\Gamma_2 = \begin{bmatrix} 0 \\ \left[N_2 A_d \right]_L \\ \vdots \\ \left[N_{n-1} A_d \right]_L \\ \left[N_n A_d \right]_L \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix}, \; \Gamma_4 = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{n-1} \\ N_n \end{bmatrix}.$$

Proof: For $i = 1, 2, \ldots, n-1$, by taking the derivatives of (8) and using (10)-(12), we obtain

$$\dot{z}_i(t) = \begin{bmatrix} M_i & N_i \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}(t) & \dot{\tilde{x}}(t-\tau) \end{bmatrix}^T$$

$$\begin{aligned} &= M_iA\tilde{x}(t) + (M_iA_d + N_iA)\tilde{x}(t-\tau) \\ &\quad + M_iBu(t) + N_iBu(t-\tau) \\ &\quad + [N_iA_d]_L\bar{y}(t-2\tau) + M_i\bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)) \\ &\quad + N_i\bar{\mu}(\bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)) \\ &= z_{i+1}(t) + M_iBu(t) + N_iBu(t-\tau) \\ &\quad + \alpha_i\bar{y}(t) + \beta_i\bar{y}(t-\tau) + [N_iA_d]_L\bar{y}(t-2\tau) \\ &\quad + M_i\bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)) \\ &\quad + N_i\bar{\mu}(\bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)). \end{aligned} \tag{17}$$

Next, for $i = n$ and from (13)-(14), we have

$$\begin{aligned} \dot{z}_n(t) &= M_nA\tilde{x}(t) + (M_nA_d + N_nA)\tilde{x}(t-\tau) \\ &\quad + M_nBu(t) + N_nBu(t-\tau) \\ &\quad + [N_nA_d]_L\bar{y}(t-2\tau) \\ &\quad + M_n\bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)) \\ &\quad + N_n\bar{\mu}(\bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)) \\ &= \sum_{i=2}^{n}\gamma_i z_i(t) + M_nBu(t) + N_nBu(t-\tau) \\ &\quad + [N_nA_d]_L\bar{y}(t-2\tau) \\ &\quad + [M_nA - \sum_{j=2}^{n}\gamma_jM_j]_L\bar{y}(t) \\ &\quad + [M_nA_d + N_nA - \sum_{j=2}^{n}\gamma_jN_j]_L\bar{y}(t-\tau) \\ &\quad + M_n\bar{\mu}(\bar{y}(t), \bar{y}(t-\tau_g)) \\ &\quad + N_n\bar{\mu}(\bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)). \end{aligned} \tag{18}$$

Finally, note that $N_1 = 0$, therefore (17) and (18) can now be expressed in the form (15)-(16). This completes the proof of Theorem 1.

Remark 1. *In the Appendix, we provide an algorithm (Algorithm 1) which allows us to solve for the unknowns γ_i $(i = 2, 3, \dots, n)$, α_j and β_j $(j = 1, 2, \dots, n-1)$ as defined in Theorem 1.*

Remark 2. *Once, a transformed system as described by (15)-(16) has been obtained, we can easily apply any Luenberger-typed state observers design method (see, for example,[8]) to design a state observer to estimate $z(t)$ since the matrix pair $(\bar{A}, \bar{C})$ is now observable. After a satisfactory state observer $z(t)$ has been designed, we can use the method of backward state transformations reported in.[12,13]*

3. APPLICATION TO THE GI MODEL

3.1. State transformation

In this section, we will apply the results obtained in the previous section to the GI model (5)-(7). By following the steps (Step 1-Step 4) of Algorithm 1, we obtain

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ N_1 = 0_{1,4},$$

$$M_2 = 0_{1,4},\ N_2 = \begin{bmatrix} 0 & a_1 & 0 & 0 \end{bmatrix},$$

$$M_3 = 0_{1,4},\ N_3 = \begin{bmatrix} 0 & -a_1a_2 & 0 & a_1a_3 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} \frac{a_2a_4a_6}{2} & 0 & 0 & 0 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 0 & a_1a_2^2 & a_1a_3a_5 & -a_1a_3(a_2+a_4) \end{bmatrix}.$$

Hence, we obtain the following state transformations

$$\begin{aligned} z_1(t) &= \xi(t), \\ z_2(t) &= a_1x_2(t-\tau), \\ z_3(t) &= -a_1a_2x_2(t-\tau) + a_1a_3x_4(t-\tau), \\ z_4(t) &= \frac{a_2a_4a_6}{2}\xi(t) + a_1a_2^2x_2(t-\tau) \\ &\quad + a_1a_3a_5x_3(t-\tau) \\ &\quad - a_1a_3(a_2+a_6)x_4(t-\tau) \end{aligned}$$

and a transformed system of the forms (15)-(16) is obtained, where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1a_3a_5 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ \gamma_1 = -\frac{a_2a_4a_6}{2},$$

$$\gamma_2 = -(a_2a_6 + a_2a_4 + a_4a_6),$$

$$\gamma_3 = -(a_2 + a_4 + a_6),$$

$$\bar{B}_1 = 0_{4,1},\ \Gamma = \begin{bmatrix} 0 \\ 0 \\ -\frac{a_2a_4a_6}{2} \\ \frac{a_2a_4a_6(a_2+a_4+a_6)}{2} \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_2a_4a_6}{2} & 0 & 0 & 0 \end{bmatrix},\ \Gamma_1 = 0_{4,1},$$

$$\Gamma_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & -a_1a_2 & 0 & a_1a_3 \\ 0 & a_1a_2^2 & a_1a_3a_5 & -a_1a_3(a_2+a_6) \end{bmatrix}.$$

3.2. Observer design

Since the matrix pair $(\bar{A}, \bar{C})$ is observable, it is easy to design a state observer to estimate any linear function of the state vector $z(t)$. Let

$$h(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} = Fz(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} z(t)$$

be a vector that is required to be estimated. To reconstruct the state function, $h(t)$, we consider a functional observer of order 3 as follows:

$$\hat{h}(t) = \omega(t) + E\bar{y}(t), \quad (19)$$

$$\dot{\omega}(t) = N\omega(t) + J\bar{y}(t) + Hu(t-\tau) + L\bar{\mu}(\bar{y}(t), \bar{y}(t-\tau), \bar{y}(t-\tau-\tau_g)), \quad t \geq \tau_{\max}, \quad (20)$$

where $\omega(t) \in \mathbb{R}^3$, $\hat{h}(t) \in \mathbb{R}^3$ is the estimate of $h(t)$, E, N, J, H and L are observer parameters to be determined. Let us define the following error vectors $\epsilon(t)$ and $e(t)$ as

$$\epsilon(t) = \omega(t) - Lz(t), \quad (21)$$

$$e(t) = \hat{h}(t) - Fz(t). \quad (22)$$

Based on[8], $\hat{h}(t)$ converges asymptotically to $Fz(t)$ if the following conditions are satisfied

$$N \text{ is Hurwitz}, \quad (23)$$

$$NL + J\bar{C} - L\bar{A} = 0, \quad (24)$$

$$H - L\bar{B} = 0, \quad (25)$$

$$F - E\bar{C} - L = 0. \quad (26)$$

Accordingly, for the given matrices $\bar{A}$, $\bar{C}$, and $\bar{B}$ as above, we can easily solve (23)-(26) to obtain the following matrices: $N = \bar{A}_{22} + L_1\bar{A}_{12}$, L_1 is chosen such that N is Hurwitz, $E = -L_1$, $J = -NL_1$, $H = L\bar{B}$, where $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$, $\bar{A}_{12} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\bar{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$ and $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that the matrix pair $(\bar{A}_{22}, \bar{A}_{12})$ is observable and thus L_1 can be easily found to ensure that N is stable with any prescribed eigenvalues.

Upon $\hat{h}(t)$ has been obtained, then based the method of backward state transformations (Case 2) reported in[13], we obtain

$$\hat{x}_2(t-\tau) = \frac{1}{a_1}\hat{h}_1(t), \quad (27)$$

$$\hat{x}_3(t-\tau) = \frac{1}{a_1a_3a_5}[a_2a_6\hat{h}_1(t) + (a_2+a_6)\hat{h}_2(t) + \hat{h}_3(t) - \frac{a_2a_4a_6}{2}\bar{y}(t)], \quad (28)$$

$$\hat{x}_4(t-\tau) = \frac{1}{a_1a_3}[a_2\hat{h}_1(t) + \hat{h}_2(t)]. \quad (29)$$

3.3. Simulation results

In order to obtain simulation results, we consider the nonlinear time-delay GI model (1)-(3) with a set of parameters, the initial conditions and the input $u(t)$ are as follows: $a_1 = 3.11\times10^{-5}$, $a_2 = 1.211\times10^{-2}$, $a_3 = \frac{1}{0.25\times55}$, $a_4 = a_5 = a_6 = \frac{1}{55}$, $a_7 = 1$, $\tau = 3\,\text{min}$, $g(y(t), y(t-\tau_g)) = \begin{bmatrix} \frac{3}{187} & \frac{1.573}{0.25}\frac{\left(\frac{x_1(t-\tau_g)}{9}\right)^{3.205}}{1+\left(\frac{x_1(t-\tau_g)}{9}\right)^{3.205}} & 0 & 0 \end{bmatrix}^T$, $\tau_g = 4\,\text{min}$, $x_1(\theta) = 10.66$, $x_2(\theta) = 49.29$, $x_3(\theta) = 0$, $x_4(\theta) = 0$ for all $\theta \in [-4, 0]$, $\omega_1(\zeta) = 20e^{-0.07t}$, $\omega_2(\zeta) = 2e^{-t}$, $\omega_3(\zeta) = 5e^{-t}$ for all $t \geq 0$, $\zeta \in [-8, 0]$ and

$$u(t) = \begin{cases} \sin t + 50, & 0 \leq t \leq 100, \\ \sin t + 3, & 100 < t \leq 180. \end{cases}$$

Let us now apply the reduced-order state observer (19)-(20) for this example. The eigenvalues of matrix N are chosen as, say, $\lambda_1 = -0.05$, $\lambda_2 = -0.07$, $\lambda_3 = -0.08$, hence we obtain $L = \begin{bmatrix} -0.1515 & 1 & 0 & 0 \\ -0.0050 & 0 & 1 & 0 \\ 0.0001 & 0 & 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0.0180 \\ -0.0008 \\ 0 \end{bmatrix}$, $E =$

$$\begin{bmatrix} 0.1515 \\ 0.0050 \\ -0.0001 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 \\ 0 \\ 0.4112 \times 10^{-5} \end{bmatrix}.$$

Figure 1 shows the responses of $x_2(t)$ and its delayed-estimation, i.e., $\hat{x}_2(t-3)$, while, Figure 2 shows the responses of $x_2(t-3)$ and its estimation, i.e., $\hat{x}_2(t-3)$. It is clear from Figure 2 that the designed observer able to track the delayed version of the state vector, as expected.

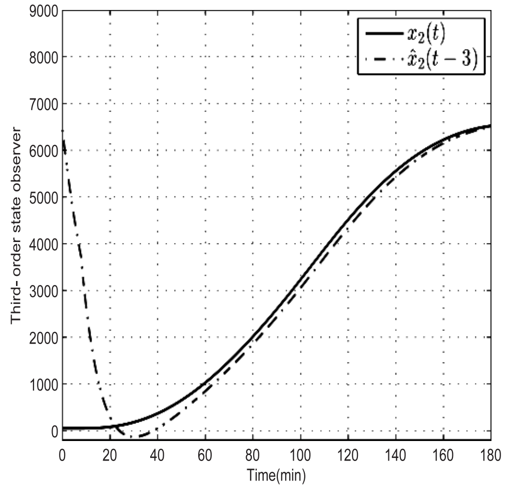

Figure 1. Responses of $x_2(t)$ and $\hat{x}_2(t-3)$

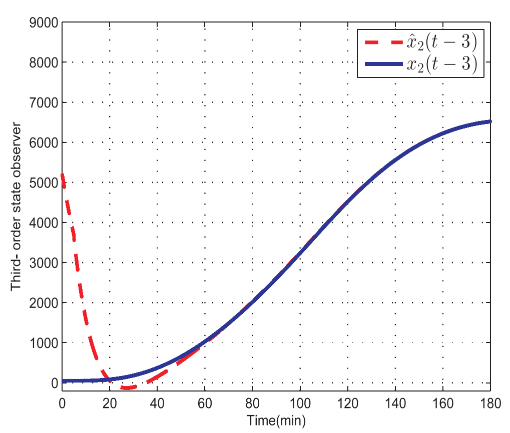

Figure 2. Responses of $\hat{x}_2(t-3)$ and $x_2(t-3)$

4. CONCLUSION

In this paper, we have presented a novel procedure for designing a state observer of a general nonlinear time-delay GI model. The reported result is significant as the two-stage design process transforms a nonlinear time-delay model into a new observable form which allows a third-order delayed state observer to be easily designed. Simulation results have been given to illustrate the effectiveness of our results.

Appendix: An algorithm for solving unknown parameters according to Theorem 1.

In the following development, we will provide a procedure for solving the unknowns γ_i $(i = 2, 3, \ldots, n)$, α_j and β_j $(j = 1, 2, \ldots, n-1)$ as defined in Theorem 1. Let us denote the following recursive matrices

$$X_1^i = [M_1 A^i]_R, \; \bar{X}_1^i = [M_1 A^{i-1} A_d]_R, \quad (30)$$

$$Y_1^i = \left[\sum_{j=1}^{i} M_1 A^{j-1} A_d A^{i-j} A_d \right]_R,$$

$$\bar{Y}_1^i = \left[\sum_{j=1}^{i+1} M_1 A^{j-1} A_d A^{i+1-j} \right]_R, \quad (31)$$

for $i = 1, 2, \ldots, n-1$.

First, we consider (12) and by using (30)-(31), we obtain the following recursive equations

$$\begin{cases} \beta_1 \bar{X}_1^1 = Y_1^1, \\ \alpha_1 Y_1^1 + \beta_1 \bar{X}_1^2 + \beta_2 \bar{X}_1^1 = Y_1^2, \\ \alpha_1 Y_1^2 + \alpha_2 Y_1^1 + \beta_1 \bar{X}_1^3 + \beta_2 \bar{X}_1^2 + \beta_3 \bar{X}_1^1 = Y_1^3, \\ \vdots \\ \alpha_1 Y_1^{n-2} + \alpha_2 Y_1^{n-3} + \ldots + \alpha_{n-2} Y_1^1 + \beta_1 \bar{X}_1^{n-1} \\ + \beta_2 \bar{X}_1^{n-2} + \ldots + \beta_{n-1} \bar{X}_1^1 = Y_1^{n-1}. \end{cases} \quad (32)$$

Equation (32) can be expressed in the following compact form

$$\chi_n X_n = Y_n, \quad (33)$$

where

$$\chi_n = \begin{bmatrix} \chi_n^1 & \chi_n^2 \end{bmatrix},$$

$$X_n = \begin{bmatrix} X_n^1 \\ X_n^2 \end{bmatrix}, \; Y_n = \begin{bmatrix} Y_1^1 & Y_1^2 & \ldots & Y_1^{n-1} \end{bmatrix},$$

with χ_n^1, χ_n^2, X_n^1 and X_n^2 are as defined below

$$\chi_n^1 = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{n-1} \end{bmatrix},$$

$$X_n^1 = \begin{bmatrix} \bar{X}_1^1 & \bar{X}_1^2 & \bar{X}_1^3 & \dots & \bar{X}_1^{n-1} \\ 0_{1,n-1} & \bar{X}_1^1 & \bar{X}_1^2 & \dots & \bar{X}_1^{n-2} \\ 0_{1,n-1} & 0_{1,n-1} & \bar{X}_1^1 & \dots & \bar{X}_1^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{1,n-1} & 0_{1,n-1} & 0_{1,n-1} & \dots & \bar{X}_1^1 \end{bmatrix},$$

$$\chi_n^2 = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n-2} \end{bmatrix},$$

$$X_n^2 = \begin{bmatrix} 0_{1,n-1} & Y_1^1 & Y_1^2 & \dots & Y_1^{n-2} \\ 0_{1,n-1} & 0_{1,n-1} & Y_1^1 & \dots & Y_1^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{1,n-1} & 0_{1,n-1} & 0_{1,n-1} & \dots & Y_1^1. \end{bmatrix}.$$

From (33), a solution for χ_n exists if and only if

$$\operatorname{rank}\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X_n \end{bmatrix}. \tag{34}$$

Next, to determine the remaining unknowns α_{n-1} and γ_j $(j = 2, 3, \dots, n)$, let us look at the solvability of equations (13)-(14). Substituting (10) - (11) into (13) - (14), using (30) - (31) and after some rearranging, we obtain the following equation expressed in a compact vector-matrix form

$$\zeta_n Z_n = T_n, \tag{35}$$

where

$$\zeta_n = \begin{bmatrix} \alpha_{n-1} & \gamma_2 & \gamma_3 & \gamma_4 & \dots & \gamma_n \end{bmatrix}, \tag{36}$$

$$Z_n = \begin{bmatrix} X_1^1 & \bar{X}_1^1 \\ X_1^1 & \bar{X}_1^1 \\ Z_n^1(3, n-1) & Z_n^2(3, n-1) \\ \vdots & \vdots \\ Z_n^1(n, n-1) & Z_n^2(n, n-1) \end{bmatrix}, \tag{37}$$

$$T_n = \begin{bmatrix} T_n^1 & T_n^2 \end{bmatrix}. \tag{38}$$

In (37)-(38), T_n^1, T_n^2, $Z_n^1(k, n-1)$ and $Z_n^2(k, n-1)$ $(k = 3, 4, \dots, n)$ are defined as follows

$$T_n^1 = X_1^n - \alpha_1 X_1^{n-1} - \dots - \alpha_{n-2} X_1^2, \tag{39}$$

$$\begin{aligned} T_n^2 &= \bar{Y}_1^{n-1} - \alpha_1 \bar{Y}_1^{n-2} - \dots - \alpha_{n-2} \bar{Y}_1^1 \\ &\quad -\beta_1 X_1^{n-1} - \beta_2 X_1^{n-2} - \dots \\ &\quad -\beta_{n-1} X_1^1, \end{aligned} \tag{40}$$

$$Z_n^1(k, n-1) = X_1^{k-1} - \alpha_1 X_1^{k-2} - \dots - \alpha_{k-2} X_1^1, \tag{41}$$

$$\begin{aligned} Z_n^2(k, n-1) &= \bar{Y}_1^{k-2} - \alpha_1 \bar{Y}_1^{k-3} - \dots \\ &\quad -\alpha_{k-3} \bar{Y}_1^1 - \alpha_{k-2} \bar{X}_1^1 \\ &\quad -\beta_1 X_1^{k-2} - \beta_2 X_1^{k-3} \\ &\quad - \dots - \beta_{k-2} X_1^1. \end{aligned} \tag{42}$$

It is clear from (39)-(42), Z_n and T_n are two known constant matrices since β_k $(k = 1, 2, \dots, n-1)$ and α_ℓ $(\ell = 1, 2, \dots, n-2)$ have already been derived from the solution to equation (33). From (35), a solution for ζ_n always exists if and only if

$$\operatorname{rank}\begin{bmatrix} Z_n \\ T_n \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Z_n \end{bmatrix}. \tag{43}$$

Accordingly, we present an effective algorithm to transform a general n-order time-delay system $(n \geq 3)$ with single output into the observable form (15)-(16).

Algorithm 1

Step 1: Obtain matrices X_n and Y_n according to (33). Check if condition (34) is satisfied or not. If so, obtain χ_n where $\chi_n = Y_n X_n^+$, where X_n^+ denotes the Moore-Penrose inverse of X_n.

Step 2: Substitute β_k $(k = 1, 2, \dots, n-1)$ and α_ℓ, α_ℓ $(\ell = 1, 2, \dots, n-2)$ into (37)-(38) and obtain Z_n and T_n. Check if condition (35) is satisfied or not. If so, obtain $\zeta_n = T_n Z_n^+$, where Z_n^+ denotes the Moore-Penrose inverse of Z_n.

Step 3: From (9)-(11), obtain matrices M_i and N_i $(i = 1, 2, \dots, n)$ and hence the state transformation (8). Finally, obtain a transformed system according to (15)-(16).

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