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Một đặc trưng cho bậc của đa tạp Fano

Đặng Tuấn Hiệp[1], **Nguyễn Thị Mai Vân**[2,*]

[1] *Khoa Toán - Tin học, Trường Đại học Đà Lạt, Việt Nam*

[2] *Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam*

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TÓM TẮT

Sử dụng một kết quả gần đây về số giao trên đa tạp Grassmann, chúng tôi đưa ra một đặc trưng cho bậc của đa tạp Fano của các không gian con tuyến tính trên giao đầy đủ trong một không gian xạ ảnh phức dưới dạng hệ số của một đa thức đối xứng.

Từ khóa: *Đa tạp Fano, công thức bậc.*

[*] *Tác giả liên hệ chính.*
Email: ntmvan18@gmail.com

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A characterization for the degree of Fano varieties

Dang Tuan Hiep[1], **Nguyen Thi Mai Van**[2,*]

[1] *Faculty of Mathematics and Computer Science, Da Lat University, Vietnam*
[2] *Faculty of Mathematics and Statistics, Quy Nhon University, Vietnam*

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ABSTRACT

Using a recent result on intersection numbers over Grassmannians, we propose a characterization for the degree of Fano varieties of linear subspaces on complete intersections in a complex projective space in terms of the coefficient of a symmetric polynomial.

Keywords: *Fano variety, degree formula.*

1. INTRODUCTION

Let X be a general complete intersection of type $\underline{d} = (d_1, \ldots, d_r)$ in the projective space $\mathbb{P}^n$ over the complex field $\mathbb{C}$, provided that $n, d_1, \ldots, d_r$ are natural numbers with $n \geq 4, d_i \geq 2$ for all i. Recall that the Fano variety $F_k(X)$ parametrizing linear subspaces of dimension k contained in X is a smooth subvariety of the Grassmannian $G(k+1, n+1)$ of linear subspaces of dimension k in $\mathbb{P}^n$, provided that

$$(k+1)(n-k) \geq \sum_{i=1}^{r} \binom{d_i + k}{k}$$

and X is not a quadric, in which last case $n \geq 2k + r$ is required[1−2]. The degree of $F_k(X)$ under the Plücker embedding were formulated by Debarre-Manivel (1998)[2] and Hiep (2016)[6].

In this paper, we show that the degree of $F_k(X)$ can be expressed in terms of the coefficient of a symmetric polynomial. For convenience, we set

$$\delta(n, \underline{d}, k) = (k+1)(n-k) - \sum_{i=1}^{r} \binom{d_i + k}{k},$$

which is the expected dimension of $F_k(X)$. Our main result is the following:

Theorem 1. *With the notations as above, if $\delta(n, \underline{d}, k) \geq 0$, then the degree of $F_k(X)$ under the Plücker embedding is given by*

$$\deg(F_k(X)) = \frac{c(n, \underline{d}, k)}{(k+1)!},$$

where $c(n, \underline{d}, k)$ is the coefficient of $x_0^n \cdots x_k^n$ in the polynomial

$$P_{(n,\underline{d},k)}(x_0, \ldots, x_n) \prod_{i \neq j} (x_i - x_j),$$

where

$$P_{(n,\underline{d},k)}(x_0, \ldots, x_n) = \prod_{i=1}^{r} \prod_{a_0 + \cdots + a_k = d_i, a_i \in \mathbb{N}}$$

$$(a_0 x_0 + \cdots + a_k x_k)(x_0 + \cdots + x_k)^{\delta(n,\underline{d},k)}.$$

The statement of Theorem 1 seems to be similar to that of Debarre-Manivel (1998)[2]. However, we here consider the coefficient of the monomial $x_0^n \cdots x_k^n$ in the product of the polynomial $P_{(n,\underline{d},k)}(x_0, \ldots, x_n)$ by the discriminant

$$\Delta = \prod_{i \neq j} (x_i - x_j)$$

instead of that of the monomial $x_0^n x_1^{n-1} \cdots x_k^{n-k}$ in the product of the same polynomial by the Vandermonde determinant

$$V = \prod_{i<j} (x_i - x_j).$$

*Corresponding author.
Email: ntmvan18@gmail.com

In particular, our approach is completely different from that of Debarre-Manivel (1998)[2]. We apply an integral formula for Grassmannians which has been recently explored by Hiep (2019)[8] (see Theorem 3 below).

The rest of the paper is organized as follows: Section 2 presents preliminary results. Section 3 presents the proof of the main theorem.

2. PRELIMINARY RESULTS

In this section, we review the basic notions and results which are known.

2.1. Grassmannians and their Schubert classes.

Let $G(k,n)$ be the *Grassmannian* of k-dimensional linear subspaces in a vector space V of dimension n. The *tautological subbundle* S on $G(k,n)$ is the vector bundle of rank k whose fiber at $W \in G(k,n)$ is the vector subspace $W \subset V$ itself. The *tautological quotient bundle* Q on $G(k,n)$ is the vector bundle of rank $n-k$ whose fiber at $W \in G(k,n)$ is the quotient vector space V/W. The *tangent bundle* T on $G(k,n)$ is isomorphic to $\mathrm{Hom}(S,Q) \cong S^\vee \otimes Q$.

Let $\mathcal{V}$ be a flag in V, that is, a strictly increasing sequence of linear subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$

where $\dim V_i = i$.

For any sequence $a = (a_1, \ldots, a_k)$ of integers with

$$n-k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0,$$

we define the *Schubert cycle* by $\Sigma_a(\mathcal{V})$, where

$$\Sigma_a(\mathcal{V}) = \{W \in G(k,n) :$$

$$\dim(V_{n-k+i-a_i} \cap W) \geq i, i = 1, \ldots, k\}.$$

One can show that this is an irreducible subvariety of $G(k,n)$ of codimension

$$|a| = \sum_{i=1}^{k} a_k,$$

and its cycle class $[\Sigma_a(\mathcal{V})]$ does not depend on the choice of flag. We then define the *Schubert class* to be the cycle class $\sigma_a := [\Sigma_a(\mathcal{V})]$.

To shorten the notation, we write Σ_a in place of $\Sigma_a(\mathcal{V})$, write $\Sigma_{a_1,\ldots,a_s}, \sigma_{a_1,\ldots,a_s}$ whenever $a = (a_1, \ldots, a_s, 0, \ldots, 0)$ and $\Sigma_{p^i}, \sigma_{p^i}$ whenever $a = (p, \ldots, p, 0, \ldots, 0)$ with i the first components equal to p. Then the cycle classes $\sigma_i, i = 1, \ldots, n-k$ and $\sigma_{1^i}, i = 1, \ldots, k$ are called *special Schubert classes*.

The special Schubert classes are intimately connected with the tautological bundles on $G(k,n)$, and both $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-k}\}$ and $\{\sigma_1, \sigma_{1^2}, \ldots, \sigma_{1^k}\}$ are minimal generating sets for the Chow ring of $G(k,n)$. More precisely, we have the following statements.

Proposition 1 (Manivel (2001)[10] and Eisenbud-Harris (2016)[3])**.** *The Chern classes of S and Q are as follows:*

$$c_i(S) = (-1)^i \sigma_{1^i}, \quad i = 1, \ldots, k$$

and

$$c_i(Q) = \sigma_i, \quad i = 1, \ldots, n-k.$$

By Corollary 3.5 in Eisenbud-Harris (2016)[3], the Schubert classes form a free $\mathbb{Z}$-basis for $A(G(k,n))$. The multiplication is determined by the following formulas.

Proposition 2 (Duality formula)**.** *(See Corollary 3.2 and Proposition 3.4 in Eisenbud-Harris (2016)[3]) If $|a| + |b| = k(n-k)$, we have*

$$\sigma_a \cdot \sigma_b = \begin{cases} \sigma_{(n-k)^k} & \text{if } a_i + b_{k-i} = n-k \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, both $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{n-k}$ are equal to the class of a point in the Chow ring of $G(k,n)$.

Proposition 3 (Pieri formula)**.** *(See Proposition 3.7 in Eisenbud-Harris (2016)[3]) For any Schubert class $\sigma_a \in A^*(G(k,n))$ and any integer i with $0 \leq i \leq n-k$, we have*

$$\sigma_a \cdot \sigma_i = \sum_c \sigma_c,$$

where the sum is over all c with $n-k \geq c_1 \geq a_1 \geq c_2 \geq \cdots \geq c_k \geq a_k \geq 0$, and $|c| = |a| + i$.

Proposition 4 (Giambelli formula)**.** *(see Section 1.5 in Griffiths-Harris (1978)*[5]*) For any $a = (a_1, \dots, a_k)$ with $n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$, we have*

$$\sigma_a = \det(\sigma_{a_i+j-i})_{1\leq i,j\leq k},$$

where $\sigma_0 = 1$ and $\sigma_m = 0$ whenever $m < 0$ or $m > n - k$.

Pieri's formula shows how to determine the product of an arbitrary Schubert class and a special Schubert class. Giambelli's formula shows how to express an arbitrary Schubert class in terms of special ones. Therefore, both formulas give us an effective way to determine the product of two arbitrary Schubert classes.

2.2. Splitting Principle.

The splitting principle is a useful technique for reducing questions concerning vector bundles to questions concerning line bundles.

Let E be a vector bundle of rank r on a variety X. The splitting principle says that we can regard the Chern classes of E as the elementary symmetric polynomials in r variables α_i for all $i = 1, \dots, r$, which are called the *Chern roots* of E. More precisely, we have

$$\begin{aligned} c_0(E) &= 1, \\ c_1(E) &= \sum_{1\leq i\leq r} \alpha_i, \\ c_2(E) &= \sum_{1\leq i<j\leq r} \alpha_i\alpha_j, \\ &\vdots \\ c_r(E) &= \alpha_1\alpha_2\dots\alpha_r. \end{aligned}$$

By the splitting principle and the Chern roots, one has the following statements.

Proposition 5. *(See Remark 3.2.3 and Example 3.2.6 in Fulton (1997)*[4]*) Let E and F be two vector bundles with Chern roots $(\alpha_i)_i$ and $(\beta_j)_j$, respectively. Then we have the following statemants:*

(i) $E^\vee$ has the Chern roots $(-\alpha_i)_i$. Hence $c_k(E^\vee) = (-1)^k c_k(E)$ for all k.

(ii) $E \otimes F$ has the Chern roots

$$(\alpha_i + \beta_j)_{i,j}.$$

(iii) $\mathrm{Sym}^d E$ has the Chern roots

$$(\alpha_{i_1} + \cdots + \alpha_{i_d})_{i_1\leq\cdots\leq i_d}.$$

(iv) $\wedge^d E$ has the Chern roots

$$(\alpha_{i_1} + \cdots + \alpha_{i_d})_{i_1<\cdots<i_d}.$$

Here we denote by $\mathrm{Sym}^d E$ the d-th symmetric power of E, and $\wedge^d E$ the d-th extorior power of E.

Example 1. Let E be a vector bundle of rank 2 on a variety X of dimension 4. We want to compute the Chern classes of $\mathrm{Sym}^3 E$ in terms of the Chern classes of E. If α_1 and α_2 are the Chern roots of E, then $\mathrm{Sym}^3 E$ has the Chern roots $3\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+2\alpha_2, 3\alpha_2$. Thus we have

$$\begin{aligned} c_1(\mathrm{Sym}^3 E) &= 3\alpha_1 + 2\alpha_1 + \alpha_2 + \alpha_1 + 2\alpha_2 + 3\alpha_2 \\ &= 6(\alpha_1 + \alpha_2) \\ &= 6c_1(E). \end{aligned}$$

Similarly, we have

$$\begin{aligned} c_2(\mathrm{Sym}^3 E) &= 11c_1(E)^2 + 10c_2(E), \\ c_3(\mathrm{Sym}^3 E) &= 6c_1(E)^3 + 30c_1(E)c_2(E), \\ c_4(\mathrm{Sym}^3 E) &= 18c_1(E)^2c_2(E) + 9c_2(E)^2. \end{aligned}$$

2.3. Fano varieties: the hypersurface case.

Let X be a general hypersurface of degree d in the projective space $\mathbb{P}^n$ over the complex field $\mathbb{C}$. The *Fano variety* $F_k(X)$ is defined to be the set of k-dimensional subspaces of $\mathbb{P}^n$ which are contained in X. This is a subvariety of the Grassmannian $G(k+1, n+1)$. For convenience, we set

$$\delta = (k+1)(n-k) - \binom{d+k}{k}.$$

Suppose that $d \neq 2$ (or $n \geq 2k + r$) and $\delta \geq 0$. Langer (1996)[9] showed that $F_k(X)$ is smooth of expected dimension δ. By the language of Schubert calculus, Debarre-Manivel (1998)[2] showed that the degree of $F_k(X)$ is equal to a certain coefficient of an explicit polynomial, gives as the product of linear forms. Hiep (2016)[6] proposed and proved an explicit formula for computing the degree of $F_k(X)$ via equivariant intersection theory.

Consider the diagonal action of $T = (\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$ given in coordinates by

$$(t_0, \ldots, t_n)(x_0 : \ldots : x_n) = (t_0 x_0 : \ldots : t_n x_n)$$

This induces an action of T on the Grassmannian $G(k+1, n+1)$ with $\binom{n+1}{k+1}$ isolated fixed points L_I corresponding to the coordinate k-subspaces in $\mathbb{P}^n$, which are indexed by the subsets I of size $k+1$ of the set $\{1, \ldots, n+1\}$. Let $\mathcal{I}$ denote the set of all these subsets. Then the degree of $F_k(X)$ can be expressed as a sum of rational polynomials, where the sum ranges over all elements I of $\mathcal{I}$.

Theorem 2. *(See Theorem 1.1 in Hiep (2016)[6]) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d and $k < n$ be a positive integer. Then the degree of the Fano variety $F_k(X)$ can be computed by the following formula*

$$\deg(F_k(X)) = (-1)^\delta \sum_{I \in \mathcal{I}} \frac{S_I Q_I^\delta}{T_I},$$

where

$$S_I = \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i h_i \right),$$

$$Q_I = \sum_{j \notin I} h_j \text{ and } T_I = \prod_{i \in I} \prod_{j \notin I} (h_i - h_j)$$

are polynomials in $\mathbb{C}[h_1, \ldots, h_{n+1}]$.

Remark 1. The right-hand-side of the formula in Theorem 2 is the sum of rational polynomials, and the above theorem claims in other words that it is in fact a constant function, moreover it is an integer. Namely, for any numbers h_i such that $h_i \neq h_j$ for $i \neq j$, the right-hand-side of the formula becomes the same integer.

Example 2. Let $k = 1$ and $X \subset \mathbb{P}^3$ be a general cubic surface. In this case, the Fano variety $F_1(X)$ has the expected dimension $\delta = 0$. The degree of $F_1(X)$ can be computed as follows:

$$\deg(F_1(X)) = \sum_{\{i_1, i_2\} \subset \{1,2,3,4\}} \frac{3h_{i_1}(2h_{i_1} + h_{i_2})(h_{i_1} + 2h_{i_2})3h_{i_2}}{(h_{i_1} - h_{j_1})(h_{i_1} - h_{j_2})(h_{i_2} - h_{j_1})(h_{i_2} - h_{j_2})},$$

where $\{j_1, j_2\}$ is the complement of the subset $\{i_1, i_2\}$ in the set $\{1, 2, 3, 4\}$. After simplifying, we obtain the degree of $F_1(X)$ is 27 as desired.

2.4. Intersection numbers on Grassmannians.

Consider the following intersection number on the Grassmannian $G(k, n)$

$$\int_{G(k,n)} \Phi(\mathcal{S}),$$

where $\Phi(\mathcal{S})$ are respectively characteristic classes of the tautological sub-bundle $\mathcal{S}$.

Using an identity involving symmetric polynomials, Hiep (2019)[8] expressed the intersection number in terms of a coefficient of a certain monomial in the expansion of a symmetric polynomial.

Theorem 3. *(See Corollary 1 in Hiep (2019)[8]) Suppose that $\Phi(\mathcal{S})$ is represented by a symmetric polynomial $P(x_1, \ldots, x_k)$ of degree not greater than $k(n-k)$ in k variables $x_1, \ldots, x_k$ which are the Chern roots of $\mathcal{S}$. Then we have the following formula:*

$$\int_{G(k,n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)} \frac{c(k,n)}{k!},$$

where $c(k, n)$ is the coefficient of $x_1^{n-1} \ldots x_k^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_k) \prod_{j \neq i} (x_i - x_j).$$

2.5. Debarre-Manivel's result.

As mentioned in the introduction, our main result seems to be similar to that of Theorem 4.3 in Debarre-Manivel (1998)[2]. Let us recall the result of Debarre-Manivel for comparison.

Theorem 4. *(See Theorem 4.3 in Debarre-Manivel (1998)*[2]*) With the notations as mentioned in the introduction and* $\delta(n,\underline{d},k) \geq 0$*. Then the degree of* $F_k(X)$ *under the Plücker embedding is given by*

$$\deg(F_k(X)) = e(n,\underline{d},k),$$

where $e(n,\underline{d},k)$ *is the coefficient of* $x_0^n x_1^{n-1} \cdots x_k^{n-k}$ *in the polynomial*

$$P_{(n,\underline{d},k)}(x_0,\ldots,x_n)\prod_{i<j}(x_i - x_j).$$

Example 3. Let us come back Example 2 above. By the statement of Theorem 4, the degree of $F_1(X)$ can be computed as follows:

$$\deg(F_1(X)) = e(3,(3),1),$$

where $e(3,(3),1)$ is the coefficient of $x_0^3x_1^2$ in the expansion of the polynomial

$$3x_0(2x_0 + x_1)(x_0 + 2x_1)3x_1(x_0 - x_1).$$

After expanding, we obtain $e(3,(3),1) = 27$, then the degree is 27 as desired.

3. PROOF OF THEOREM 1

We first prove the following lemma.

Lemma 1. *(see Proposition 6.4 in Eisenbud-Harris (2016)*[3] *and Lemma 3 in Hiep-Tu-Van (2019)*[7]*) Let* $X \subset \mathbb{P}^n$ *be a general complete intersection of type* $(d_1,\ldots,d_r)$*. The variety* $F = F_k(X)$ *is the zero locus of a global section of the vector bundle*

$$\mathcal{F} = \bigoplus_{i=1}^{r} \operatorname{Sym}^{d_i} S^\vee.$$

Proof of Lemma 1. Assume that X is the intersection of r hypersurfaces $X_1,\ldots,X_r$ with $\deg(X_i) = d_i$ for all i. Each $F_k(X_i)$ is the zero locus of a global section s_i of $\operatorname{Sym}^{d_i} S^\vee$. Thus the variety F, which is the intersection of the $F_k(X_i)$, is the zero locus of a global section $s = (s_1,\ldots,s_r)$ of the vector bundle $\mathcal{F}$.

We now prove Theorem 1. By the Gauss-Bonnet formula (see, for example, Section 3.5.3 in Manivel (2001)[10]), the class of $F_k(X)$ is the top Chern class of the vector bundle $\mathcal{F}$. If $\delta(n,\underline{d},k) \geq 0$, then the degree of $F_k(X)$ can be expressed as follows:

$$\deg(F_k(X)) =$$

$$\int_{G(k+1,n+1)} \prod_{i=1}^{r} c_{\text{top}}(\operatorname{Sym}^{d_i} S^\vee) \cdot c_1(S^\vee)^{\delta(n,\underline{d},k)},$$

where $c_{\text{top}}(E)$ is the top Chern class of the vector bundle E. By the splitting principle, the characteristic class

$$\prod_{i=1}^{r} c_{\text{top}}(\operatorname{Sym}^{d_i} S^\vee) \cdot c_1(S^\vee)^{\delta(n,\underline{d},k)}$$

is represented by the symmetric polynomial

$$(-1)^{(k+1)(n-k)} P_{(n,\underline{d},k)}(x_0,\ldots,x_n).$$

Note that $x_0,\ldots,x_k$ are the Chern roots of the tautological sub-bundle S on the Grassmannian $G(k+1,n+1)$. By Theorem 3, Theorem 1 follows.

Example 4. Let us come back Example 2 above. By the statement of Theorem 1, the degree of $F_1(X)$ can be computed as follows:

$$\deg(F_1(X)) = \frac{c(3,(3),1)}{2!},$$

where $c(3,(3),1)$ is the coefficient of $x_0^3x_1^3$ in the expansion of the polynomial

$$3x_0(2x_0 + x_1)(x_0 + 2x_1)3x_1(x_0 - x_1)(x_1 - x_0).$$

After expanding, we obtain $c(3,(3),1) = 54$, then the degree is 27 as desired.

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