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Nghiệm liouvillian hữu tỷ của phương trình vi phân đại số cấp một giống không

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TÓM TẮT

Chúng tôi đưa ra một điều kiện cần và đủ để phương trình vi phân đại số cấp một, hệ số hằng $f(y, y') = 0$ có nghiệm liouvillian hữu tỷ, trong đó $f(X, Y) = 0$ là một đường cong đại số hữu tỷ (giống không) trên trường số phức $\mathbb{C}$. Bài viết được xây dựng trên ba ý chính: mỗi đường cong đại số hữu tỷ luôn có phép tham số riêng, một cặp $(X(t), X'(t))$ luôn là tham số riêng của một đường cong đại số hữu tỷ nào đó, và điều kiện để phương trình vi phân cấp một $y' = f(y)$ có nghiệm liouvillian trên C.

Từ khóa: *Nghiệm liouvillian, phép tham số hóa, đường cong hữu tỷ, phương trình vi phân đại số.*

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Rational liouvillian solution of algebraic ordinary differential equations of order one in genus zero

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ABSTRACT

We study a necessary and sufficient condition for having a rational liouvilian solution of the autonomous algebraic ordinary differential equation $f(y, y') = 0$, where $f(X, Y) = 0$ defines a rational algebraic curve (genus zero) over complex field $\mathbb{C}$. This article based on three ideas: every rational algebraic curve has proper parametrization, pair $(X(t), X'(t))$ is a proper parametrization of a certain algebraic curve, and the condition of differential equation of order one $y' = f(y)$ has liouvillian solution over C.

Keywords: *Liouvillian solution, parametrization, rational algebraic curve, algebraic differential equation.*

1. INTRODUCTION

Let $\mathbb{C}$ be an algebraically closed field of characteristic zero and view $\mathbb{C}$ as a differential field with the trivial derivation $c' = 0$ for all $c \in \mathbb{C}$. Let

$$C(X) = \left\{\frac{P(X)}{Q(X)}, P(X), Q(X) \in \mathbb{C}[X], Q(X) \neq 0\right\}$$

be a field of rational functions in one variable X. We are interested in the non-constant liouvillian solutions of the following differential equation the autonomous algebraic ordinary differential equation of order one given by $f(y, y') = 0$, where $f(X, Y) = 0$ is a rational algebraic curve.

Combining some results of parametrization of algebraic curve of genus zero, and a result of necessary and sufficient condition for the explicit differential equation,[1] $y' = R(y)$ with $R(y) \in \mathbb{C}(y)$ has a non-constant solution y which is liouvillian over $\mathbb{C}$, we show a result of necessary and sufficient condition for the implicit differential equation $f(y, y') = 0$ having a non-constant rational liouvillian over C through parametrization.

2. PRELIMINARIES

2.1. Definitions

A derivation of the field k, denote by $'$ is an additive endomorphism of k that satisfies the Leibniz law $(xy)' = x'y + xy'$ for every x, y in k. A field equipped with a derivation map is called a *differential field.*

Let E be a differential field extension of k and let $'$ denote the derivation on E. We say that E is a *liouvillian extension* of k if $E = k(t_1, t_2, \dots, t_n)$ and there is a tower of differential field $k = k_0 \leq k_1 \leq \dots \leq k_n = E$ such that for each i, $k_i = k_{i-1}(t_i)$ and either $t_i' \in k_{i-1}$ or $t_i'/t_i \in k_{i-1}$ or t_i is algebraic over k_{i-1}. A solution of a differential equation over k is said to be liouvillian over k if the solution belongs to some liouvillian extension of k. Whenever we write $k \leq E$ as differential field, we mean that E is a differential field extension of k and we write $y \in E - k$ to mean that $y \in E$ and y is not in E.

Let $f(X, Y) \in \mathbb{C}(X, Y)$ be an irreducible polynomial in two variables over the algebraically closed field $\mathbb{C}$. The set $L = \{(a, b) \in \mathbb{C}^2 \backslash f(a, b) = 0\}$ defines an *algebraic curve* over $\mathbb{C}$.

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A *rational parametrization* of the algebraic curve $f(X,Y)=0$ is a pair of rational functions $X(t),Y(t)\in \mathbb{C}(t)$ if two following conditions are satisfied:

i) For almost all t_0 the point $(X(t_0),Y(t_0))\in L$.
ii) For almost all point $(x_0,y_0)\in L$ there exists $t_0\in \mathbb{C}$ such that $\left(X(t_0),Y(t_0)\right)=(x_0,y_0)$.

An algebraic curve $f(X,Y)=0$ is said to be *rational or rational algebraic curve* if it admits a rational parametrization.

A rational parametrization $(X(t),Y(t))$ of the algebraic curve $f(X,Y)=0$ is said to be *proper* if for almost all point $(x_0,y_0)\in L$, there is a unique $t_0\in \mathbb{C}$ such that $\left(X(t_0),Y(t_0)\right)=(x_0,y_0)$.

Rational function $X(t)\in \mathbb{C}(t)$ is called *rational liouvillian solution* of $f(y,y')=0$ over $\mathbb{C}$ if three following conditions are satisfied:

i) $X(t)$ belongs to liouvillian extension of $\mathbb{C}$.
ii) $X(t),X'(t)\in \mathbb{C}(t)$.
iii) $f\left(X(t),X'(t)\right)=0$.

With $X'(t)\in \mathbb{C}(t)$, so $X'(t)=\frac{\partial X}{\partial t}t'\in \mathbb{C}(t)$, and we obtain $t'\in \mathbb{C}(t)$. Write $X(t)=\frac{P(t)}{Q(t)}, P(t),Q(t)\in \mathbb{C}[t]$, we have $X(t)Q(t)-P(t)=0$. Hence t is algebraic over $\mathbb{C}(X(t))$. So we obtain t belongs to some liouvillian extension field of $\mathbb{C}$. Since t is liouvillian, the differential field $C(t)$ is contained in some liouvillian extension field of $\mathbb{C}$. If we say that $X(t)$ is a *non-constant rational liouvillian solution* of differential equation $f(y,y')=0$ over $\mathbb{C}$, this means $X'(t)\neq 0$. So $t'\neq 0$, and t is obviously transcendental over $\mathbb{C}$. We refer the reader two books for basic theory of differential fields and rational algebraic curves.[2,3]

2.2 Some results

For the reader's convenience and for easy reference, we record some basic results concerning liouvillian extensions and rational algebraic curves in the following these lemmas and theorems.[1,3-5]

Lemma 1. *Algebraic curve $f(X,Y)=0$ can be parametrized if and only if it has genus zero.*[3]

Lemma 2. *Every rational curve can be properly parametrized.*[3]

Lemma 3. *Let $(X_1(t),Y_1(t))$ be a proper parametrization of rational algebraic curve $f(X,Y)=0$. Then for any proper parametrization $(X_2(t),Y_2(t))$ of $f(X,Y)=0$ there exists $R(t)=(at+b)\,/\,(ct+d)\in \mathbb{C}(t)$ such that $(X_2(t),Y_2(t))=(X_1(R(t)),Y_1(R(t))$.*[3]

Lemma 4. *Let $(X_1(t),Y_1(t))$, $(X_2(t),Y_2(t))$ be two proper parametrizations of rational algebraic curve $f(X,Y)=0$. The differential equation $t'=\frac{Y_1(t)}{\frac{\partial X_1}{\partial t}}$ has liouvillian solution if and only if the differential equation $t'=\frac{Y_2(t)}{\frac{\partial X_2}{\partial t}}$ has liouvillian solution.*[4]

Theorem 1. *If $\left(X(t),X'(t)\right)$ is a rational parametrization of algebraic curve $f(X,Y)=0$ then it is a proper parametrization.*[5]

Theorem 2. *Let $\mathbb{C}$ be complex field and $c'=0$ for all $\in \mathbb{C}$. If E is a liouvillian extension field of $\mathbb{C}$ then there is an element $z\in E-\mathbb{C}$ such that $z'=1$ or $z'=az$ for some a in $\mathbb{C}-0$.*[1]

3. MAIN RESULT

Theorem 3. *Let $f(X,Y)=0$ be a rational algebraic curve with proper parametrization $(X_1(t),Y_1(t))$. The differential equation $f(y,y')=0$ has a non-constant rational solution $X(t)\in \mathbb{C}(t)$ which is liouvillian over $\mathbb{C}$ if and only if there is an element $z\in \mathbb{C}(t)$ such that $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)}=\frac{\partial z}{\partial t}$ or $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)}=\frac{\frac{\partial z}{\partial t}}{az}$, $a\in \mathbb{C}-0$.*

Proof.

(Necessary) If $y=X(t)$ is a non-constant rational liouvillian solution of differential equation $f(y,y')=0$, then following Theorem 1 this means $(X(t),X'(t))$ is a proper parametrization of algebraic curve . Following

Lemma 4, there exists a non-constant liouvillian solution of diferential equation $t' = \frac{Y_1(t)}{\frac{\partial X_1}{\partial t}}$. So we obtain $\mathbb{C}(t, t')$ is contained in some liouvillian extension field of $\mathbb{C}$. Following Theorem 2, there is an element $z \in \mathbb{C}(t) - \mathbb{C}$ such that $z' = 1$ or $z' = az$. Since $z' = \frac{\partial z}{\partial t} t'$, we obtain $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} = \frac{\partial z}{\partial t}$ *or* $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} = \frac{\frac{\partial z}{\partial t}}{az}$, $a \in \mathbb{C} - 0$.

(**Sufficient**) If there is an element $z \in \mathbb{C}(t)$ such that $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} = \frac{\partial z}{\partial t}$, then we consider differential field $(\mathbb{C}(x), ')$ with $x' = a \in \mathbb{C} - 0$. Set $= z(t) = \frac{x}{a}$, we obtain $z' = 1$, and t is algebraic over $\mathbb{C}(x)$.

Since $\frac{X_1'(t)}{Y_1(t)} = \frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} t' = \frac{\partial z}{\partial t} t' = z' = 1$, then $X_1'(t) = Y_1(t)$. From $f\big(X_1(t), Y_1(t)\big) = 0$, it means $f\left(X_1(t), X_1'(t)\right) = 0$. Since $X_1(t) \in \mathbb{C}(t)$ and t is algebraic over $\mathbb{C}(x)$, we obtain $X_1(t)$ is a non-constant rational liouvillian solution over $\mathbb{C}$.

If there is an element $z \in \mathbb{C}(t)$ such that $\frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} = \frac{\frac{\partial z}{\partial t}}{az}$, then we consider differential field $(\mathbb{C}(x), ')$ with $x' = ax \in \mathbb{C} - 0$. Set $z = z(t) = x$, we obtain $z' = az$, and t is algebraic over $\mathbb{C}(x)$. Since $\frac{X_1'(t)}{Y_1(t)} = \frac{\frac{\partial X_1}{\partial t}}{Y_1(t)} t' = \frac{\frac{\partial z}{\partial t}}{az} t' = \frac{z'}{az} = 1$, one obtain $X_1'(t) = Y_1(t)$. Repeating previous argument, we conclude $y = X_1(t)$ is a non-constant rational liouvillian solution.

4. EXAMPLES

Example 1. Consider the differential equation

$$f(y, y') := -9y^2 + 30y - 12y'^2 + 36y' - 25 + y'^3 = 0 \quad (1)$$

The corresponding algebraic curve

$$f(X, Y) := -9X^2 + 30X - 12Y^2 + 36Y - 25 + Y^3 = 0 \quad (2)$$

has a proper rational parametrization

$$\big(X(t), Y(t)\big) = \left(\frac{(4t-1)(164t^2-58t+5)}{3(6t-1)^3}, \frac{4t^2}{(6t-1)^2}\right).$$

We have $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{34t^2-12t+1}{(t(6t-1))^2}$. Set $z = \frac{1}{3(6t-1)} - \frac{1}{t} \in \mathbb{C}(t)$, then $\frac{\partial z}{\partial t} = \frac{\frac{\partial X}{\partial t}}{Y(t)}$. Following Theorem 3 $X(t)$ is a non-constant rational liouvillian solution of equation (1).

Example 2. Consider the differential equation

$$f(y, y') := -9y^2 + 36yy' - 36y^2 + 216y^3 - 432y'y^2 + 288y'^2 y - 64y'^3 = 0 \quad (3)$$

The corresponding algebraic curve

$$f(X, Y) := -9Y^2 + 36XY - 36X^2 + 216X^3 - 432YX^2 + 288Y^2X - 64Y^3 = 0 \quad (4)$$

has a proper rational parametrization

$$\big(X(t), Y(t)\big) = \left(\frac{9}{8}\frac{t(4t^2-4t+1)}{27t^3-54t^2+36t-8}, \frac{9}{8}\frac{(4t^2-4t+1)}{27t^3-54t^2+36t-8}\right).$$

We have

$\frac{\frac{\partial X}{\partial t}}{Y(t)} = -\frac{2(3t-1)}{(2t-1)(3t-2)}$. Set $z = \frac{2t-1}{(3t-2)^2} \in \mathbb{C}(t)$, then $\frac{\frac{\partial z}{\partial t}}{z} = \frac{\frac{\partial X}{\partial t}}{Y(t)}$. So $X(t)$ is a non-constant rational liouvillian solution of equation (3).

Example 3. Differential equation

$$y^2 + y'^2 = 1 \quad (5)$$

has a rational liouvilian solution over $\mathbb{C}$.

Algebraic curve $X^2 + Y^2 = 1$ (6)

has proper parametrization $\big(X(t), Y(t)\big) = \left(\frac{2t}{t^2+1}, \frac{-t^2+1}{t^2+1}\right)$. We have $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{2}{t^2+1}$.

Set $i = \sqrt{-1}$ and $z = \frac{t-i}{t+i} \in \mathbb{C}(t)$, since $\frac{\partial z}{\partial t} = \frac{2i}{(i+t)^2}$, we obtain $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{\frac{\partial z}{\partial t}}{iz} = \frac{2}{t^2+1}$. So $X(t)$ is a non-constant rational liouvillian solution of equation (5).

Example 4. Differential equation

$$f(y, y') := y'^5 - 3y^2y' + y^2y'^3 - y^2 = 0 \quad (7)$$

has no non-constant rational liouvillian solution.

Algebraic curve

$$f(X, Y) := Y^5 - 3X^2Y + X^2Y^3 - X^2 = 0 \quad (8)$$

has a proper parametrization

$\left(X(t), Y(t)\right) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1}\right).$

We have $\frac{\frac{\partial X}{\partial t}}{Y(t)} = 2t^2 - 6 + \frac{6}{t^2+1}$. If there exists $z \in C(t)$ such that $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{\frac{\partial z}{\partial t}}{az}$, one can write $z = \frac{r(t)}{S(t)}$, $r(t), s(t) \in \mathrm{C}[t]$. Since $\frac{\frac{\partial z}{\partial t}}{az} = \frac{1}{a}\left(\frac{\frac{\partial r}{\partial t}}{r(t)} - \frac{\frac{\partial s}{\partial t}}{s(t)}\right)$, $\deg\left(\frac{\partial r}{\partial t}\right) < \deg(r(t))$ and $\deg\left(\frac{\partial s}{\partial t}\right) < \deg(s(t))$. So there is not an element z such that $\frac{\frac{\partial z}{\partial t}}{az} = 2t^2 - 6 + \frac{6}{t^2+1}$.

If $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{\partial z}{\partial t}$, one can write $z = \frac{r(t)}{S(t)}$, with $r(t), s(t) \in \mathrm{C}[t]$ and $\gcd\left(r(t), s(t)\right) = 1$, where gcd is abbreviated word of "greatest common divisor". Then we have $\frac{\partial z}{\partial t} = \frac{\frac{\partial r}{\partial t}s - \frac{\partial s}{\partial t}r}{s^2} = \frac{2t^4-4t^2}{t^2+1}$. So we obtain $s(t)$ divides (t^2+1). Rewrite $s(t) = q(t)(t^2+1)^k, k \geq 1$, with $\gcd\left(t^2+1, q(t)\right) = 1$. Then we have $\frac{\partial r}{\partial t}q(t).(t^2+1)^k - \left(\frac{\partial q}{\partial t}(t^2+1)^k + 2t(t^2+1)^{k-1}q(t)r(t)\right) = (2t^4 - 4t^2)q^2(t)(t^2+1)^{2k-1}$. So $q(t).r(t)$ divides (t^2+1). This is a contradiction. Hence there is not an element z such that $\frac{\frac{\partial X}{\partial t}}{Y(t)} = \frac{\partial z}{\partial t}$. Following Theorem 3, differential equation (7) has no non-constant rational liouvillian solution.

5. CONCLUSION

In this article, we focus on rational liouvillian solution of differential equation $f\left(y, y'\right) = 0$, with form $y = X(t) \in \mathrm{C}(t)$ and $t' \in \mathrm{C}(t)$. This means y, y' belongs to some certain differential field $\mathrm{C}(t)$ through parametrization. In the future, we hope to develop some techniques to find condition for differential equation $f\left(y, y'\right) = 0$ having a liouvillian solution in other forms.

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