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Các hằng số tương đương của một số chuẩn trong không gian các đường cong Bézier bậc ba N mảnh

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TÓM TẮT

Bài báo này nghiên cứu chuẩn của đường cong Bézier N khúc bậc ba và các hằng số tương đương trong không gian $B_{N,3}$ của các đường cong Bézier N khúc bậc ba. Chúng tôi đề xuất các chuẩn $\|\cdot\|_p^{B_3}$ trên không gian B_3 các đường cong Bézier bậc ba và chuẩn $\|\cdot\|_p^{B_{N,3}}$ trên không gian $B_{N,3}$ của các đường cong Bézier N khúc bậc ba. Trong bài báo này chúng tôi bàn về các hằng số tương đương của chuẩn $\|\cdot\|_p^{B_{N,3}}$ và chuẩn L_p trên không gian $B_{N,3}$ các đường cong Bézier N khúc bậc ba. Mỗi đường cong Bézier bậc m có thể được xem như một đường cong Bézier bậc $m+1$. Do đó, chúng tôi cũng nghiên cứu các hằng số tương đương của chuẩn $\|\cdot\|_p^{B_{N,m}}$ và chuẩn $\|\cdot\|_p^{B_{N,m+1}}$ trên không gian các đường cong Bézier N khúc bậc m.

Từ khóa: *Đường cong Bézier bậc ba, hằng số tương đương, chuẩn, khoảng cách.*

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Equivalence constants for some norms on the space of N-piece cubic Bézier curves

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ABSTRACT

This paper is concerned with norms of N-piece cubic Bézier curves and the equivalence relations between some norms on the space $B_{N,3}$ of N-piece cubic Bézier curves. We introduce a norm $\|\cdot\|_p^{B_3}$ on the space B_3 of cubic Bézier curves and a norm $\|\cdot\|_p^{B_{N,3}}$ on the space $B_{N,3}$ of N-piece cubic Bézier curves. This article we deal with the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the L_p norm on the space $B_{N,3}$ of N-piece cubic Bézier curves. An N-piece Bézier curve of degree m can be considered as an N-piece Bézier curve of degree $m+1$. We also study the equivalence constants for the norm $\|\cdot\|_p^{B_{N,m}}$ and the norm $\|\cdot\|_p^{B_{N,m+1}}$ on the space $B_{N,m}$ of N-piece Bézier curves of degree m.
Keywords: *Cubic Bézier curves, equivalence constants, norm, distance.*

1. Introduction

In 1962, the French engineer Pierre Bézier publicized the Bézier curve which is defined based on Bernstein polynomials. However, the mathematician Paul de Casteljau built Bézier curve by using de Casteljau's algorithm.

Pierre Bézier applied Bézier curves for designing the bodywork of Renault cars. Its importance is due to the fact that, Bézier curves are used in many fields of applications, not only mathematics. For instance, Bézier curves are used in computer graphics, computer-aided design system, robotic, industry, walking, communication, path-planning and aerospace (see[1–8]). Bézier curves are also used to find plane shape optimization which appears in many fields such as environment design, aerospace, structural mechanics, networks, automotive, hydraulic, oceanology and wind engineering (see[9–14]).

Bézier curves are mentioned in many books and articles for instance.[15–17] A continuous curve can be approximated by a Bézier curve. However, when the curve is long and complex, the degree of the Bézier curve is high. As a result, the computation is more difficult. Then, the most common use of Bézier curves is as N-piece cubic Bézier curves. We will focus uniform N-piece cubic Bézier curves.

In this article, we define a norm $\|\cdot\|_p^{B_m}$ on the space B_m of Bézier curves of degree m and a norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$ of uniform N-piece Bézier curves of degree m. These norms are computed through control points. This article studies the equivalence relations between the norm $\|\cdot\|_p^{B_{N,3}}$ and the L_p norm on the space $B_{N,3}$ of N-piece cubic Bézier curves.

Theorem 1. *For $p \in [1,\infty]$. Let $\beta \in B_{N,3}$, we get*

$$\|\beta\|_{L_p} \le \|\beta\|_p^{B_{N,3}} \le 2^{10}\,\|\beta\|_{L_p}.$$

An N-piece Bézier curve of degree m can be considered as an N-piece Bézier curve of degree $m+1$. We also study the equivalence constants for the norm $\|\cdot\|_p^{B_{N,m}}$ and the norm $\|\cdot\|_p^{B_{N,m+1}}$ on the space $B_{N,m}$ of N-piece Bézier curves of degree m.

Theorem 2. *For $p \in [1,\infty]$. Let $\beta \in B_{N,m}$, we get*

$$\frac{1}{2(m+1)}\|\beta\|_p^{B_{N,m}} \le \|\beta\|_p^{B_{N,m+1}} \le 2\|\beta\|_p^{B_{N,m}}.$$

2. Preliminaries

For the convenience of reading, we present some definitions and notations that will be used through the article.

Definition 3. ([16] chapter 6, p. 141) Let m be a positive integer and $P_0,\dots,P_m$ be $m+1$ points in $\mathbb{R}^n$. The *Bézier curve of degree m* with control points $P_0,\dots,P_m$ is defined by

$$B([P_0,\dots,P_m],t) := \sum_{i=0}^{m} P_i b_{i,m}(t), \quad t \in [0,1], \qquad (1)$$

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where $b_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}$ is the Bernstein polynomial.

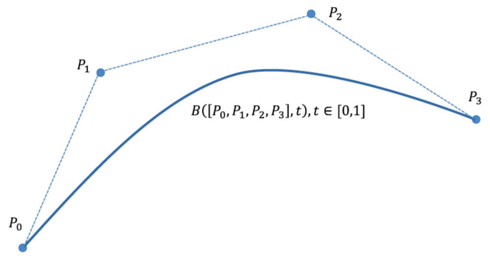

Figure 1. A cubic Bézier curve.

The points P_i are called control points for the Bézier curve of degree m. The polygon formed by connecting the control points with lines, starting with P_0 and finishing with P_m, is called the control polygon. The convex hull of the control polygon contains the Bézier curve.

A uniform N-piece Bézier curve of degree m is a piecewise Bézier curve which has N-pieces, each piece is a Bézier curve of degree m and the point at $t = \frac{j}{N}$, $j = 1, \dots, N-1$, is the connecting point of the pieces. We often drop "uniform". Let us consider the definition of the N-piece Bézier curve of degree m.

Definition 4. [16] Let m, N be positive integers and $P_0, \dots, P_{Nm}$ be $Nm+1$ points in $\mathbb{R}^n$. The *N-piece Bézier curve of degree m* with control points $P_0, \dots, P_{Nm}$ is formed by

$$\beta : [0,1] \to \mathbb{R}^n$$
$$t \mapsto \beta(t) = B([P_{jm}, \dots, P_{jm+m}], Nt - j)$$
$$\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right].$$

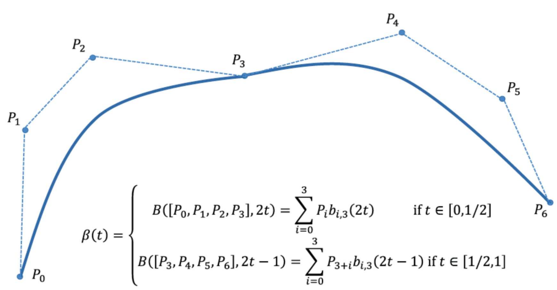

Figure 2. A two-piece cubic Bézier curve.

Notation 5.

- The vector space of Bézier curves of degree m is denoted by the symbol B_m.
- The vector space of N-piece Bézier curves of degree m is denoted by the symbol $B_{N,m}$.
- The set of continuous parametrizations on $[0,1]$ is denoted by the symbol $C^0([0,1], \mathbb{R}^n)$.

We define some norms and distances through control points on the space of Bézier curves of degree m and on the space of N-piece Bézier curves of degree m.

Definition 6. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_m} : B_m \to \mathbb{R}$ is defined by: For any $\beta(t) = \sum_{i=0}^{m} P_i b_{i,m}(t) \in B_m$,

$$\|\beta\|_p^{B_m} := \begin{cases} \left(\sum_{i=0}^{m} \|P_i\|_p^p\right)^{1/p} & \text{if } p \in [1, \infty[\\ \max_{i=0,\dots,m} \{\|P_i\|_\infty\} & \text{if } p = \infty, \end{cases}$$

where $\|\cdot\|_p$ is the p-norm on $\mathbb{R}^n$.

From the properties of the p-norm on $\mathbb{R}^n$ and the Minkowski inequality, it is easily seen that $\|\cdot\|_p^{B_m}$ is a norm on the vector space B_m. Indeed, it is a norm on the space $(\mathbb{R}^n)^{m+1}$ of control polygons. We then have an induced distance on B_m by $d_p^{B_m}(\beta, \gamma) := \|\beta - \gamma\|_p^{B_m}$.

Definition 7. Let $p \in [1, \infty]$. The function $\|\cdot\|_p^{B_{N,m}} : B_{N,m} \to \mathbb{R}$ is defined by: For any $\beta(t) = \beta^{(j)}(Nt - j) = \sum_{i=0}^{m} P_{jm+i} b_{i,m}(Nt - j)$ if $t \in \left[\frac{j}{N}, \frac{j+1}{N}\right]$, $j = 0, \dots, N-1$,

$$\|\beta\|_p^{B_{N,m}} := \begin{cases} \frac{1}{N^{1/p}} \left(\sum_{j=0}^{N-1} \left(\|\beta^{(j)}\|_p^{B_m}\right)^p\right)^{1/p} \\ \qquad \text{if } p \in [1, \infty[\\ \max_{j=0,\dots,N-1} \left\{\|\beta^{(j)}\|_\infty^{B_m}\right\} \text{ if } p = \infty. \end{cases}$$

Using the Minkowski inequality and the properties of the norm $\|\cdot\|_p^{B_m}$ on B_m, it is easy to see that $\|\cdot\|_p^{B_{N,m}}$ is a norm on the vector space $B_{N,m}$. Then we have again an induced distance on $B_{N,m}$ defined by $d_p^{B_{N,m}}(\beta, \gamma) := \|\beta - \gamma\|_p^{B_{N,m}}$.

The norms $\|\cdot\|_p^{B_m}$ and $\|\cdot\|_p^{B_{N,m}}$ can be computed more efficiently than, for instance, the L_p- norm.

3. THE EQUIVALENCE RELATIONS BETWEEN THE NORM $\|\cdot\|_p^{B_{N,3}}$ AND THE L_p NORM ON THE SPACE $B_{N,3}$

Since the space $B_{N,3}$ is a subspace of the space $C^0([0,1], \mathbb{R}^n)$, the norm $\|\cdot\|_{L_p}$ is also a norm on the space $B_{N,3}$ of N-piece cubic Bézier curves. Moreover, N-piece cubic Bézier curves are common and convenient to approximate continuous curves. So, we concentrate on the equivalence relations between the norm $\|\cdot\|_p^{B_{N,3}}$ and the L_p-norm on the space $B_{N,3}$ of N-piece cubic Bézier curves.

For $p \in [1, \infty]$, we first show that the norm $\|\cdot\|_{L_p}$ is weaker than the norm $\|\cdot\|_p^{B_m}$ on the space B_m of Bézier curves of degree m and then show that $\|\cdot\|_{L_p}$ is also weaker than the norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$ of N-piece Bézier curves of degree m.

Lemma 8. *Let $p \in [1, \infty]$. Then for any $\beta \in B_m$,*

$$\|\beta\|_{L_p} \le \|\beta\|_p^{B_m}.$$

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Proof. Let $\beta \in B_m$ and assume that

$$\beta(t) = \sum_{i=0}^{m} P_i b_{i,m}(t), \ \ t \in [0,1].$$

Case $p = 1$. We get

$$\begin{aligned}\|\beta\|_{L_1} &= \int_0^1 \Big\| \sum_{i=0}^{m} P_i b_{i,m}(t) \Big\|_1 dt \\ &\leq \sum_{i=0}^{m} \|P_i\|_1 b_{i,m}(t) \leq \sum_{i=0}^{m} \|P_i\|_1 = \|\beta\|_1^{B_m}.\end{aligned}$$

Case $p \in]1,\infty[$. Using Holder's inequality, we get

$$\begin{aligned}&\sum_{i=0}^{m} \|P_i\|_p b_{i,m}(t) \\ \leq &\Big(\sum_{i=0}^{m} \|P_i\|_p^p\Big)^{1/p} \Big(\sum_{i=0}^{m} (b_{i,m}(t))^{p/(p-1)}\Big)^{(p-1)/p} \\ \leq &\Big(\sum_{i=0}^{m} \|P_i\|_p^p\Big)^{1/p} \Big(\sum_{i=0}^{m} b_{i,m}(t)\Big)^{(p-1)/p} \\ = &\Big(\sum_{i=0}^{m} \|P_i\|_p^p\Big)^{1/p}.\end{aligned}$$

Hence

$$\begin{aligned}\|\beta\|_{L_p} &= \Big(\int_0^1 \Big\| \sum_{i=0}^{m} P_i b_{i,m}(t) \Big\|_p^p dt\Big)^{1/p} \\ &\leq \Big(\int_0^1 \Big(\sum_{i=0}^{m} \|P_i\|_p b_{i,m}(t)\Big)^p dt\Big)^{1/p} \\ &\leq \Big(\int_0^1 \sum_{i=0}^{m} \|P_i\|_p^p dt\Big)^{1/p} = \Big(\sum_{i=0}^{m} \|P_i\|_p^p\Big)^{1/p} \\ &= \|\beta\|_p^{B_m}.\end{aligned}$$

Case $p = \infty$. We get

$$\begin{aligned}\|\beta\|_{L_\infty} &= \Big\| \sum_{i=0}^{m} P_i b_{i,m}(t) \Big\|_{L_\infty} \leq \Big(\sum_{i=0}^{m} b_{i,m}(t)\Big) \max_{i=0,\ldots,m} \|P_i\|_\infty \\ &= \max_{i=0,\ldots,m} \|P_i\|_\infty = \|\beta\|_\infty^{B_m}.\end{aligned}$$

Combining the above cases, we obtain the proof of this lemma. □

Form this estimation, we have the following proposition.

Proposition 9. *Let $p \in [1,\infty]$. Then for any $\beta \in B_{N,m}$,*

$$\|\beta\|_{L_p} \leq \|\beta\|_p^{B_{N,m}}.$$

Proof. Let $\beta \in B_{N,m}$ be an N-piece Bézier curve of degree m with control points $P_{jm+i} \in \mathbb{R}^n$, $i = 0,\ldots,m$, $j = 0,\ldots,N-1$. So

$$\begin{aligned}\beta(t) &= \beta^{(j)}(Nt-j) = \sum_{i=0}^{m} P_{jm+i} b_{i,m}(Nt-j) \\ &\text{if } t \in \Big[\frac{j}{N}, \frac{j+1}{N}\Big], j = 0,\ldots,N-1.\end{aligned}$$

Case $p \in [1,\infty)$. We get

$$\begin{aligned}\|\beta\|_{L_p} &= \Big(\int_0^1 \|\beta(t)\|_p^p dt\Big)^{1/p} \\ &= \Big(\sum_{j=0}^{N-1} \int_{\frac{j}{N}}^{\frac{j+1}{N}} \|\beta^{(j)}(Nt-j)\|_p^p dt\Big)^{1/p} \\ &= \frac{1}{N^{1/p}} \Big(\sum_{j=0}^{N-1} \int_0^1 \|\beta^{(j)}(t)\|_p^p dt\Big)^{1/p}.\end{aligned}$$

Using Lemma 8, we obtain

$$\|\beta\|_{L_p} \leq \frac{1}{N^{1/p}} \Big(\sum_{j=0}^{N-1} \Big(\|\beta^{(j)}\|_p^{B_m}\Big)^p\Big)^{1/p} = \|\beta\|_p^{B_{N,m}}.$$

Case $p = \infty$. We have

$$\begin{aligned}\|\beta\|_{L_\infty} &= \max_{t\in[0,1]} \|\beta(t)\|_\infty \\ &= \max_{j=0,\ldots,N-1} \max_{t\in[\frac{j}{N},\frac{j+1}{N}]} \|\beta^{(j)}(Nt-j)\|_\infty \\ &= \max_{j=0,\ldots,N-1} \max_{t\in[0,1]} \|\beta^{(j)}(t)\|_\infty.\end{aligned}$$

Using Lemma 8, we get

$$\|\beta\|_{L_\infty} \leq \max_{j=0,\ldots,N-1} \|\beta\|_\infty^{B_m} = \|\beta\|_\infty^{B_{N,m}}.$$

Thus, the proof of this proposition is complete. □

Thus, for $p \in [1,\infty]$, we have with respect to the norm $\|\cdot\|_{L_p}$ is weaker than the norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$ of N-piece Bézier curves of degree m. Since $B_{N,3}$ is a subspace of $B_{N,m}$, this result is useful for the space $B_{N,3}$ of N-piece *cubic* Bézier curves.

In practice, the most common use of Bézier curves is as N-piece cubic Bézier curves. So, we concentrate on the space $B_{N,3}$ of N-piece *cubic* Bézier curves. We will find a constant A such that $\|\cdot\|_p^{B_{N,3}} \leq A\|\cdot\|_{L_p}$ on the space $B_{N,3}$ of N-piece *cubic* Bézier curves.

Lemma 10. *Let $p \in [1,\infty[$ and P_0,P_1,P_2,P_3 be four points on $\mathbb{R}^n$, we get*

$$\int_0^1 \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_p^p dt \geq \frac{1}{2^{10p}} \Big(\sum_{i=0}^{3} \|P_i\|_p^p\Big).$$

Proof. Put

$$M = \max\Big\{\|P_0\|_p, \frac{1}{3}\|P_1\|_p, \frac{1}{3}\|P_2\|_p, \|P_3\|_p\Big\}.$$

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- Case 1: $M = \|P_0\|_p$.
  We consider the interval $\left[0, \frac{1}{16}\right]$. For any $t \in \left[0, \frac{1}{16}\right]$, we have

$$\|P_0(1-t)^3\|_p \geq \left(1-\frac{1}{16}\right)^3 \|P_0\|_p = \frac{15^3}{16^3}\|P_0\|_p.$$

$$\|P_1 3t(1-t)^2\|_p \leq 3\frac{1}{16}\left(1-\frac{1}{16}\right)^2\|P_1\|_p$$

$$\leq 3\|P_0\|_p \frac{3.15^2}{16^3} = \frac{9.15^2}{16^3}\|P_0\|_p$$

$$\|P_2 3t^2(1-t)\|_p \leq 3\left(\frac{1}{16}\right)^2\left(1-\frac{1}{16}\right)\|P_2\|_p$$

$$\leq 3\|P_0\|_p \frac{3.15}{16^3} = \frac{9.15}{16^3}\|P_0\|_p$$

$$\|P_3 t^3\|_p \leq \frac{1}{16^3}\|P_3\|_p \leq \frac{1}{16^3}\|P_0\|_p.$$

So, for any $t \in \left[0, \frac{1}{16}\right]$, we have

$$\Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p \geq \|P_0(1-t)^3\|_p$$
$$-\|P_1 3t(1-t)^2\|_p - \|P_2 3t^2(1-t)\|_p - \|P_3 t^3\|_p$$
$$\geq \frac{15^3}{16^3}\|P_0\|_p - \frac{9.15^2}{16^3}\|P_0\|_p$$
$$-\frac{9.15}{16^3}\|P_0\|_p - \frac{1}{16^3}\|P_0\|_p$$
$$= \frac{1214}{4096}\|P_0\|_p.$$

Then

$$\int_0^1 \Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p^p dt \geq \int_0^{\frac{1}{16}} \Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p^p dt$$
$$\geq \int_0^{\frac{1}{16}} \left(\frac{1214}{4096}\right)^p \|P_0\|_p^p dt$$
$$\geq \frac{1}{16}\left(\frac{1214}{4096}\right)^p \frac{1}{2+2.3^p}\Big(\sum_{i=0}^{3}\|P_0\|_p^p\Big)$$
$$\geq \frac{1}{2^{10p}}\Big(\sum_{i=0}^{3}\|P_0\|_p^p\Big).$$

- Case 2: $M = \frac{1}{3}\|P_1\|_p$.
  We consider the interval $\left[\frac{7}{32}, \frac{9}{32}\right]$. For any $t \in \left[\frac{7}{32}, \frac{9}{32}\right]$, we have

$$\|P_1 3t(1-t)^2\|_p \geq 3.\frac{7}{32}\left(1-\frac{7}{32}\right)^2\|P_1\|_p$$
$$= \frac{3.7.25^2}{32^3}\|P_1\|_p,$$

$$\|P_0(1-t)^3\|_p \leq \left(1-\frac{7}{32}\right)^3\|P_0\|_p$$
$$\leq \frac{25^3}{32^3}\frac{1}{3}\|P_1\|_p = \frac{25^3}{3.32^3}\|P_1\|_p,$$

$$\|P_2 3t^2(1-t)\|_p \leq 3.\left(\frac{9}{32}\right)^2\left(1-\frac{9}{32}\right)\|P_2\|_p$$
$$\leq \frac{3.9^2.23}{32^3}\|P_1\|_p,$$

$$\|P_3 t^3\|_p \leq \frac{9^3}{32^3}\|P_3\|_p$$
$$\leq \frac{1}{3}\|P_1\|_p\frac{9^3}{32^3} = \frac{9^3}{3.32^3}\|P_1\|_p.$$

So, for any $t \in \left[\frac{7}{32}, \frac{9}{32}\right]$, we have

$$\Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p$$
$$\geq \|P_1 3t(1-t)^2\|_p - \|P_0(1-t)^3\|_p$$
$$-\|P_2 3t^2(1-t)\|_p - \|P_3 t^3\|_p$$
$$\geq \frac{3.7.25^2}{32^3}\|P_1\|_p - \frac{25^3}{3.32^3}\|P_1\|_p$$
$$-\frac{3.9^2.23}{32^3}\|P_1\|_p - \frac{9^3}{3.32^3}\|P_1\|_p$$
$$= \frac{6254}{98304}\|P_1\|_p.$$

Then

$$\int_0^1 \Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p^p dt \geq \int_{\frac{7}{32}}^{\frac{9}{32}} \Big\|\sum_{i=0}^{3} P_i b_{i,3}(t)\Big\|_p^p dt$$
$$\geq \int_{\frac{7}{32}}^{\frac{9}{32}} \left(\frac{6254}{98304}\right)^p \|P_1\|_p^p dt$$
$$\geq \frac{1}{16}\left(\frac{6254}{98304}\right)^p \frac{1}{2+2.3^{-p}}\Big(\sum_{i=0}^{3}\|P_i\|_p^p\Big)$$
$$\geq \frac{1}{2^{10p}}\Big(\sum_{i=0}^{3}\|P_i\|_p^p\Big).$$

- Case 3: $M = \frac{1}{3}\|P_2\|_p$. When we substitute $s = 1-t$, this case become Case 2.
- Case 4: $M = \|P_3\|_p$. When we substitute $s = 1-t$, this case become Case 1.

From the above four cases, we have

$$\int_0^1 \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_p^p dt \geq \frac{1}{2^{10p}} \Big(\sum_{i=0}^{3} \|P_i\|_p^p \Big).$$

□

Lemma 11. *Let P_0, P_1, P_2, P_3 be four points on $\mathbb{R}^n$, we get*

$$\max_{t\in[0,1]} \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_\infty \geq \frac{13}{96} \max_{i=0,\ldots,3} \|P_i\|_\infty.$$

Proof. Put

$$M = \max \Big\{ \|P_0\|_\infty, \frac{1}{3}\|P_1\|_\infty, \frac{1}{3}\|P_2\|_\infty, \|P_3\|_\infty \Big\}.$$

- Case 1: $M = \|P_0\|_\infty$.
  Since $\sum_{i=0}^{3} P_i b_{i,3}(0) = P_0$, hence

$$\max_{t\in[0,1]} \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_\infty \geq \|P_0\|_\infty = \max_{i=0,\ldots,3} \|P_i\|_\infty.$$

- Case 2: $M = \frac{1}{3}\|P_1\|_\infty$.
  At $t = \frac{1}{4}$, we get

$$\begin{aligned} \Big\| \sum_{i=0}^{3} P_i b_{i,3}\Big(\frac{1}{4}\Big) \Big\|_\infty &\geq \frac{27}{64}\|P_1\|_\infty - \frac{27}{64}\|P_0\|_\infty \\ &\quad - \frac{9}{64}\|P_2\|_\infty - \frac{1}{64}\|P_3\|_\infty \\ &\geq \frac{27}{64}\|P_1\|_\infty - \frac{9}{64}\|P_1\|_\infty \\ &\quad - \frac{9}{64}\|P_1\|_\infty - \frac{1}{192}\|P_1\|_\infty \\ &= \frac{13}{96}\|P_1\|_\infty. \end{aligned}$$

  Thus

$$\max_{t\in[0,1]} \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_\infty \geq \frac{13}{96} \max_{i=0,\ldots,3} \|P_i\|_\infty.$$

- Case 3: $M = \frac{1}{3}\|P_2\|_\infty$. This case is the same to Case 2.
- Case 4: $M = \|P_3\|_\infty$. This case is the same to Case 1.

From the above four cases, we have

$$\max_{t\in[0,1]} \Big\| \sum_{i=0}^{3} P_i b_{i,3}(t) \Big\|_\infty \geq \frac{13}{96} \max_{i=0,\ldots,3} \|P_i\|_\infty.$$

□

Combining the above lemmas, we obtain the following proposition.

Proposition 12. *Let $p \in [1,\infty]$. Then for any $\beta \in B_{N,3}$, we have*

$$\|\beta\|_p^{B_{N,3}} \leq 2^{10} \|\beta\|_{L_p}.$$

Proof. Let $\beta \in B_{N,3}$, assume that

$$\beta(t) = \beta^{(j)}(Nt - j) = \sum_{i=0}^{3} P_{j3+i} b_{i,3}(Nt - j)$$

$$\text{if } t \in \Big[\frac{j}{N}, \frac{j+1}{N}\Big], j = 0, \ldots, N-1.$$

Case $p \in [1,\infty)$, we have

$$\begin{aligned} \|\beta\|_{L_p} &= \Big(\int_0^1 \|\beta(t)\|_p^p dt \Big)^{1/p} \\ &= \Big(\sum_{j=0}^{N-1} \int_{\frac{j}{N}}^{\frac{j+1}{N}} \Big\| \sum_{i=0}^{3} P_{j3+i} b_{i,3}(Nt - j) \Big\|_p^p dt \Big)^{1/p} \\ &= \frac{1}{N^{1/p}} \Big(\sum_{j=0}^{N-1} \int_0^1 \Big\| \sum_{i=0}^{3} P_{j3+i} b_{i,3}(t) \Big\|_p^p dt \Big)^{1/p}. \end{aligned}$$

Using Lemma 10, we obtain

$$\begin{aligned} \|\beta\|_{L_p} &\geq \frac{1}{N^{1/p}} \Big(\sum_{j=0}^{N-1} \frac{1}{2^{10p}} \Big(\sum_{i=0}^{3} \|P_{j3+i}\|_p^p \Big) \Big)^{1/p} \\ &\geq \frac{1}{2^{10}} \|\beta\|_p^{B_{N,3}}. \end{aligned}$$

Case $p = \infty$, we have

$$\begin{aligned} \|\beta\|_{L_\infty} &= \max_{t\in[0,1]} \|\beta(t)\|_\infty \\ &= \max_{j=0,\ldots,N-1} \max_{t\in[\frac{j}{N},\frac{j+1}{N}]} \Big\| \sum_{i=0}^{3} P_{j3+i} b_{i,3}(Nt - j) \Big\|_\infty \\ &= \max_{j=0,\ldots,N-1} \max_{t\in[0,1]} \Big\| \sum_{i=0}^{3} P_{j3+i} b_{i,3}(t) \Big\|_\infty. \end{aligned}$$

Using Lemma 11, we get

$$\begin{aligned} \|\beta\|_{L_\infty} &\geq \max_{j=0,\ldots,N-1} \frac{13}{96} \max_{i=0,\ldots,3} \big\|P_{j3+i}\big\|_\infty \\ &= \frac{13}{96} \|\beta\|_\infty^{B_{N,3}} \geq \frac{1}{2^{10}} \|\beta\|_\infty^{B_{N,3}}. \end{aligned}$$

□

From the above propositions, we obtain the equivalence relations between the norms $\|\cdot\|_p^{B_{N,3}}$ and $\|\cdot\|_{L_p}$ on the space $B_{N,3}$ of N-piece cubic Bézier curve as follows.

Theorem 1. *For $p \in [1,\infty]$. Let $\beta \in B_{N,3}$, we get*

$$\|\beta\|_{L_p} \leq \|\beta\|_p^{B_{N,3}} \leq 2^{10} \|\beta\|_{L_p}.$$

Proof. Using Propositions 9 and 12, we get the proof of this theorem. □

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The equivalence constants do not depend on the number of pieces in piecewise cubic Bézier curves. From the above theorem, we get the following corollary:

$$d_{L_p}(\beta,\gamma) \le d_p^{B_{N,3}}(\beta,\gamma) \le 2^{10}\, d_{L_p}(\beta,\gamma)$$

for any $\beta,\gamma \in B_{N,3}$.

4. EQUIVALENCE CONSTANTS FOR THE NORMS $\|\cdot\|_p^{B_{N,m+1}}$ AND $\|\cdot\|_p^{B_{N,m}}$ ON $B_{N,m}$

For any Bézier curve of degree m with $m+1$ control points

$$\beta(t) = \sum_{i=0}^{m} P_i b_{i,m}(t), \ \ t \in [0,1],$$

we can choose $m+2$ points $Q_0,\dots,Q_m,Q_{m+1}$ as follows

$$Q_i = \begin{cases} P_0 & \text{if } i=0, \\ \dfrac{i}{m+1}P_{i-1} + \dfrac{m+1-i}{m+1}P_i & \text{if } i=1,\dots,m, \\ P_m & \text{if } i=m+1 \end{cases} \quad (2)$$

such that

$$\beta(t) = \sum_{i=0}^{m} P_i b_{i,m}(t) = \sum_{i=0}^{m+1} Q_i b_{i,m}(t), \ \ \forall t \in [0,1].$$

For simplicity of notation, we admits $P_{-1} = P_{m+1} = O_{\mathbb{R}^n}$. Hence

$$Q_i = \frac{i}{m+1}P_{i-1} + \frac{m+1-i}{m+1}P_i, \ \ \forall i = 0,\dots,m+1.$$

This means that a Bézier curve of degree m is also a Bézier curve of degree $m+1$ and then an N-piece Bézier curve of degree m can be considered as an N-piece Bézier curve of degree m. So, $\|\cdot\|_p^{B_{N,m+1}}$ is also a norm on the space $B_{N,m}$. This section deals with the equivalence relations between the norm $\|\cdot\|_p^{B_{N,m+1}}$ and the norm $\|\cdot\|_p^{B_{N,m}}$ on the space $B_{N,m}$.

We first consider on the space B_m of Bézier curves of degree m.

Lemma 13. *For $p \in [1,\infty]$. Let $\beta \in B_m$, we get*

$$\|\beta\|_p^{B_{m+1}} \le 2\|\beta\|_p^{B_m}.$$

Proof. Let $\beta \in B_m$, assume that

$$\begin{aligned}\beta(t) =& B([P_0,\dots,P_m],t) = \sum_{i=0}^{m} P_i b_{i,m}(t), \\ =& B([Q_0,\dots,Q_{m+1}],t) = \sum_{i=0}^{m+1} Q_i b_{i,m+1}(t), \\ & \forall t \in [0,1],\end{aligned}$$

where $P_i \in \mathbb{R}^n$, $Q_i \in \mathbb{R}^n$ and

$$Q_i = \frac{i}{m+1}P_{i-1} + \frac{m+1-i}{m+1}P_i, \ \ \forall i = 0,\dots,m+1.$$

Case $p \in [1,\infty)$, we have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} =& \left(\sum_{i=0}^{m+1} \|Q_i\|_p^p\right)^{1/p} \\ =& \left(\sum_{i=0}^{m+1} \left\|\frac{i}{m+1}P_{i-1} + \frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p}.\end{aligned}$$

From Minkowski's inequality, we obtain

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} \le& \left(\sum_{i=0}^{m+1} \left\|\frac{i}{m+1}P_{i-1}\right\|_p^p\right)^{1/p} \\ & + \left(\sum_{i=0}^{m+1} \left\|\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p} \\ \le& \left(\sum_{i=0}^{m} \|P_i\|_p^p\right)^{1/p} + \left(\sum_{i=0}^{m} \|P_i\|_p^p\right)^{1/p} \\ =& 2\|\beta\|_p^{B_m}.\end{aligned}$$

Case $p=\infty$, we have

$$\begin{aligned}\|\beta\|_\infty^{B_{m+1}} =& \max_{i=0,\dots,m+1} \|Q_i\|_\infty \\ =& \max_{i=0,\dots,m+1} \left\|\frac{i}{m+1}P_{i-1} + \frac{m+1-i}{m+1}P_i\right\|_\infty \\ \le& \max_{i=0,\dots,m+1} \left\|\frac{i}{m+1}P_{i-1}\right\|_\infty \\ & + \max_{i=0,\dots,m+1} \left\|\frac{m+1-i}{m+1}P_i\right\|_\infty \\ \le& 2\max_{i=0,\dots,m} \|P_i\|_\infty = 2\|\beta\|_\infty^{B_m}.\end{aligned}$$

□

Then we estimate an upper bound on the space $B_{N,m}$ of N-piece Bézier curves of degree m.

Proposition 14. *For $p \in [1,\infty]$. Let $\beta \in B_{N,m}$, we get*

$$\|\beta\|_p^{B_{N,m+1}} \le 2\|\beta\|_p^{B_{N,m}}.$$

Proof. For any $\beta \in B_{N,m}$,

$$\begin{aligned}\beta(t) =& \beta^{(j)}(Nt-j) = \sum_{i=0}^{m} P_{jm+i} b_{i,m}(Nt-j) \\ & \text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 1,\dots,N-1.\end{aligned}$$

Since a Bézier curve of degree m is also a Bézier curve of degree $m+1$, an N-piece Bézier curve of degree m is also an N-piece Bézier curve of degree $m+1$. We consider two cases:

Case $p \in [1,\infty]$. Applying Lemma 13, we obtain

$$\begin{aligned}\|\beta\|_p^{B_{N,m+1}} =& \frac{1}{N^{1/p}}\left(\sum_{j=0}^{N-1}\left(\|\beta^{(j)}\|_p^{B_{m+1}}\right)^p\right)^{1/p} \\ \le& \frac{1}{N^{1/p}}\left(\sum_{j=0}^{N-1}\left(2\|\beta^{(j)}\|_p^{B_m}\right)^p\right)^{1/p} \\ =& 2\|\beta\|_p^{B_{N,m}}.\end{aligned}$$

Case $p=\infty$. From Lemma 13, we obtain

$$\begin{aligned}\|\beta\|_\infty^{B_{N,m+1}} &= \max_{j=0,\dots,N-1} \|\beta^{(j)}\|_\infty^{B_{m+1}} \\ &\le \max_{j=0,\dots,N-1} 2\|\beta^{(j)}\|_\infty^{B_m} = 2\|\beta\|_\infty^{B_{N,m}}.\end{aligned}$$

□

Consider $p \in [1,\infty[$. We use the parity of m to evaluate a lower bound for the norm $\|\cdot\|_p^{B_{m+1}}$ with respect to the norm $\|\cdot\|_p^{B_m}$ on the space B_m. From this estimation, we find equivalence constants on the space $B_{N,m}$ of N-piece Bézier curves of degree m.

Lemma 15. *Let $p \in [1,\infty[$. The inequality*

$$\frac{1}{2(m+1)}\|\beta\|_p^{B_m} \le \|\beta\|_p^{B_{m+1}}$$

holds for all $\beta \in B_m$.

Proof. Let $\beta \in B_m$, assume that

$$\begin{aligned}\beta(t) &= B([P_0,\dots,P_m],t) = \sum_{i=0}^{m} P_i b_{i,m}(t), \\ &= B([Q_0,\dots,Q_{m+1}],t) = \sum_{i=0}^{m+1} Q_i b_{i,m+1}(t), \\ &\quad \forall t \in [0,1],\end{aligned}$$

where $P_i \in \mathbb{R}^n$, $Q_i \in \mathbb{R}^n$ and

$$Q_i = \frac{i}{m+1}P_{i-1} + \frac{m+1-i}{m+1}P_i,\ \forall i = 0,\dots,m+1.$$

We consider 2 cases.

- Case 1: m is an odd number. We have

$$\|\beta\|_p^{B_{m+1}} = \left(\sum_{i=0}^{m+1}\|Q_i\|_p^p\right)^{1/p}$$
$$\ge\left(\sum_{i=0}^{(m+1)/2}\left\|\frac{i}{m+1}P_{i-1}+\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p}.$$

Using the Minkowski inequality, we obtain

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} &\ge \left(\sum_{i=0}^{(m+1)/2}\left\|\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p} - \left(\sum_{i=0}^{(m+1)/2}\left\|\frac{i}{m+1}P_{i-1}\right\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=0}^{\frac{m+1}{2}-1}\frac{m+1-i}{m+1}\|P_i\|_p^p\right)^{1/p} - \left(\sum_{i=0}^{\frac{m+1}{2}-1}\frac{i+1}{m+1}\|P_i\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=0}^{\frac{m+1}{2}-1}\frac{m+3}{2(m+1)}\|P_i\|_p^p\right)^{1/p} - \left(\sum_{i=0}^{\frac{m+1}{2}-1}\frac{m+1}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &\ge \frac{1}{m+1}\left(\sum_{i=0}^{\frac{m+1}{2}-1}\|P_i\|_p^p\right)^{1/p}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} &= \left(\sum_{i=0}^{m+1}\|Q_i\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=\frac{m+1}{2}+1}^{m+1}\left\|\frac{i}{m+1}P_{i-1}+\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p}.\end{aligned}$$

From the Minkowski inequality, we get

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} &\ge \left(\sum_{i=\frac{m+1}{2}+1}^{m+1}\left\|\frac{i}{m+1}P_{i-1}\right\|_p^p\right)^{1/p} - \left(\sum_{i=\frac{m+1}{2}+1}^{m+1}\left\|\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=\frac{m+1}{2}}^{m}\frac{i+1}{m+1}\|P_i\|_p^p\right)^{1/p} - \left(\sum_{i=\frac{m+1}{2}}^{m}\frac{m+1-i}{m+1}\|P_i\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=\frac{m+1}{2}}^{m}\frac{m+3}{2(m+1)}\|P_i\|_p^p\right)^{1/p} - \left(\sum_{i=\frac{m+1}{2}}^{m}\frac{m+1}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &= \frac{1}{m+1}\left(\sum_{i=\frac{m+1}{2}}^{m}\|P_i\|_p^p\right)^{1/p}.\end{aligned}$$

Then

$$\begin{aligned}2\|\beta\|_p^{B_{m+1}} &\ge \frac{1}{m+1}\left(\sum_{i=0}^{\frac{m+1}{2}-1}\|P_i\|_p^p\right)^{1/p} + \frac{1}{m+1}\left(\sum_{i=\frac{m+1}{2}}^{m}\|P_i\|_p^p\right)^{1/p} \\ &\ge \frac{1}{m+1}\left(\sum_{i=0}^{m}\|P_i\|_p^p\right)^{1/p} \\ &= \frac{1}{m+1}\|\beta\|_p^{B_m}.\end{aligned}$$

- Case 2: m is an even number. We have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} &= \left(\sum_{i=0}^{m+1}\|Q_i\|_p^p\right)^{1/p} \\ &\ge \left(\sum_{i=0}^{m/2}\left\|\frac{i}{m+1}P_{i-1}+\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p}.\end{aligned}$$

Applying Minkowski's inequality, we have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} \geq& \left(\sum_{i=0}^{m/2}\left\|\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=0}^{m/2}\left\|\frac{i}{m+1}P_{i-1}\right\|_p^p\right)^{1/p} \\ =&\left(\sum_{i=0}^{m/2}\frac{m+2}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=0}^{\frac{m}{2}-1}\frac{m}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ \geq&\left(\sum_{i=0}^{m/2}\frac{m+2}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=0}^{m/2}\frac{m}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ =&\frac{1}{m+1}\left(\sum_{i=0}^{m/2}\|P_i\|_p^p\right)^{1/p}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} =& \left(\sum_{i=0}^{m+1}\|Q_i\|_p^p\right)^{1/p} \\ \geq& \left(\sum_{i=\frac{m}{2}+1}^{m+1}\left\|\frac{i}{m+1}P_{i-1}+\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p}.\end{aligned}$$

From the Minkowski inequality, we have

$$\begin{aligned}\|\beta\|_p^{B_{m+1}} \geq& \left(\sum_{i=\frac{m}{2}+1}^{m+1}\left\|\frac{i}{m+1}P_{i-1}\right\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=\frac{m}{2}+1}^{m+1}\left\|\frac{m+1-i}{m+1}P_i\right\|_p^p\right)^{1/p} \\ \geq&\left(\sum_{i=m/2}^{m}\frac{m+2}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=\frac{m}{2}+1}^{m}\frac{m}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ \geq&\left(\sum_{i=\frac{m}{2}+1}^{m}\frac{m+2}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ &- \left(\sum_{i=\frac{m}{2}+1}^{m}\frac{m}{2(m+1)}\|P_i\|_p^p\right)^{1/p} \\ =&\frac{1}{m+1}\left(\sum_{i=\frac{m}{2}+1}^{m}\|P_i\|_p^p\right)^{1/p}.\end{aligned}$$

Then

$$\begin{aligned}2\|\beta\|_p^{B_{m+1}} \geq& \frac{1}{m+1}\left(\sum_{i=0}^{m/2}\|P_i\|_p^p\right)^{1/p} \\ &+\frac{1}{m+1}\left(\sum_{i=\frac{m}{2}+1}^{m}\|P_i\|_p^p\right)^{1/p} \\ \geq& \frac{1}{m+1}\left(\sum_{i=0}^{m}\|P_i\|_p^p\right)^{1/p} = \frac{1}{m+1}\beta\|_p^{B_m}.\end{aligned}$$

Combining the above results, we obtain

$$\|\beta\|_1^{B_{D+1}} \geq \frac{1}{2(m+1)}\|\beta\|_1^{B_m}.$$

□

We now estimate on the space B_m of Bézier curve of degree m.

Lemma 16. *Let $\beta \in B_m$, we get*

$$\frac{1}{m+1}\|\beta\|_\infty^{B_m} \leq \|\beta\|_\infty^{B_{m+1}}.$$

Proof. Let $\beta \in B_m$, assume that

$$\begin{aligned}\beta(t) =&B([P_0,\dots,P_m],t) = \sum_{i=0}^{m}P_ib_{i,m}(t), \\ =&B([Q_0,\dots,Q_{m+1}],t) = \sum_{i=0}^{m+1}Q_ib_{i,m+1}(t), \\ &\forall t \in [0,1],\end{aligned}$$

where $P_i \in \mathbb{R}^n$, $Q_i \in \mathbb{R}^n$ and

$$Q_i = \frac{i}{m+1}P_{i-1}+\frac{m+1-i}{m+1}P_i,\ \ \forall i = 0,\dots,m+1.$$

We assume that $\|\beta\|_\infty^{B_m} = \|P_{i_0}\|_\infty$.

- If $0 \leq i_0 < \dfrac{m+1}{2}$.

$$\begin{aligned}\|\beta\|_\infty^{B_{m+1}} =& \max_{i=0,\dots,m+1}\{\|Q_i\|_\infty\} \\ \geq&\left\|\frac{i_0}{m+1}P_{i_0-1}+\frac{m+1-i_0}{m+1}P_{i_0}\right\|_\infty \\ \geq&\frac{m+1-i}{m+1}\|P_{i_0}\Big\|_\infty - \frac{i_0}{m+1}\|P_{i_0-1}\|_\infty \\ =&\frac{m+1-2i_0}{m+1}\|P_{i_0}\|_\infty \\ \geq&\frac{1}{m+1}\|P_{i_0}\|_\infty = \frac{1}{m+1}\|\beta\|_\infty^{B_m}.\end{aligned}$$

- If $i_0 = \dfrac{m+1}{2}$. (This case requires that m is an odd number.)

$$\begin{aligned}
\|\beta\|_\infty^{B_{m+1}} &= \max_{i=0,\dots,m+1}\{\|Q_i\|_\infty\} \\
&\geq \left\|\frac{i_0+1}{m+1}P_{i_0} + \frac{m+1-(i_0+1)}{m+1}P_{i_0+1}\right\|_\infty \\
&\geq \frac{i_0+1}{m+1}\|P_{i_0}\|_\infty - \frac{m-i_0}{m+1}\|P_{i_0+1}\|_\infty \\
&= \frac{2i_0+1-m}{m+1}\|P_{i_0}\|_\infty \\
&\geq \frac{1}{m+1}\|P_{i_0}\|_\infty = \frac{1}{m+1}\|\beta\|_\infty^{B_m}.
\end{aligned}$$

- If $i_0 > \dfrac{m+1}{2}$.

$$\begin{aligned}
\|\beta\|_\infty^{B_{m+1}} &= \max_{i=0,\dots,m+1}\{\|Q_i\|_\infty\} \\
&\geq \left\|\frac{i_0+1}{m+1}P_{i_0} + \frac{m+1-(i_0+1)}{m+1}P_{i_0+1}\right\|_\infty \\
&\geq \frac{i_0+1}{m+1}\|P_{i_0}\|_\infty - \frac{m-i_0}{m+1}\|P_{i_0+1}\|_\infty \\
&= \frac{2i_0+1-m}{m+1}\|P_{i_0}\|_\infty \\
&\geq \frac{1}{m+1}\|P_{i_0}\|_\infty = \frac{1}{m+1}\|\beta\|_\infty^{B_m}.
\end{aligned}$$

Combining the above results, we obtain

$$\|\beta\|_\infty^{B_{m+1}} \geq \frac{1}{m+1}\|\beta\|_\infty^{B_m}.$$

□

We then consider on the space $B_{N,m}$ of N-piece Bézier curves of degree m.

Proposition 17. *Let $p \in [1,\infty]$. The inequality*

$$\frac{1}{2(m+1)}\|\beta\|_p^{B_{N,m}} \leq \|\beta\|_p^{B_{N,m+1}}$$

holds for any $\beta \in B_{N,m}$.

Proof. Let $\beta \in B_{N,m}$, assume that

$$\begin{aligned}
\beta(t) =& \beta^{(j)}(Nt-j) = \sum_{i=0}^{m} P_{jm+i} b_{i,m}(Nt-j) \\
&\text{if } t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], j = 1,\dots,N-1.
\end{aligned}$$

Because a Bézier curve of degree m is also a Bézier curve of degree $m+1$, an N-piece Bézier curve of degree m is also an N-piece Bézier curve of degree $m+1$. We consider two cases: Case $p \in [1,\infty)$. From Lemma 15, we get

$$\begin{aligned}
\|\beta\|_p^{B_{N,m+1}} =& \frac{1}{N^{1/p}}\left(\sum_{j=0}^{N-1}\left(\|\beta^{(j)}\|_p^{B_{m+1}}\right)^p\right)^{1/p} \\
\geq& \frac{1}{N^{1/p}}\left(\sum_{j=0}^{N-1}\left(\frac{1}{2(m+1)}\|\beta^{(j)}\|_p^{B_m}\right)^p\right)^{1/p} \\
=& \frac{1}{2(m+1)}\|\beta\|_p^{B_{N,m}}.
\end{aligned}$$

Case $p = \infty$. From Lemma 16, we get

$$\begin{aligned}
\|\beta\|_\infty^{B_{N,m+1}} &= \max_{j=0,\dots,N-1}\|\beta^{(j)}\|_\infty^{B_{m+1}} \\
&\geq \max_{j=0,\dots,N-1}\frac{1}{m+1}\|\beta^{(j)}\|_\infty^{B_m} \\
&= \frac{1}{m+1}\|\beta\|_\infty^{B_{N,m}}.
\end{aligned}$$

Thus, the proof of this proposition is complete. □

From the above result, we obtain the following theorem.

Theorem 2. *For $p \in [1,\infty]$. Let $\beta \in B_{N,m}$, we get*

$$\frac{1}{2(m+1)}\|\beta\|_p^{B_{N,m}} \leq \|\beta\|_p^{B_{N,m+1}} \leq 2\|\beta\|_p^{B_{N,m}}.$$

Proof. Using Propositions 14 and 17, we get the proof of this theorem. □

Thus, we have the following corollary

$$\begin{aligned}
\frac{1}{2(m+1)}d_p^{B_{N,m}}(\beta,\gamma) \leq& d_p^{B_{N,m+1}}(\beta,\gamma) \\
\leq& 2d_p^{B_{N,m}}(\beta,\gamma),
\end{aligned}$$

for any $\beta,\gamma \in B^{N,m}$.

5. Conclusion

This article introduces a norm $\|\cdot\|_p^{B_{N,3}}$ of piecewise cubic Bézier curves which is defined by control points. This norm is convenient to compute. We also show the equivalence constants for the norm $\|\cdot\|_p^{B_{N,3}}$ and the norm L_p. These equivalence constants do not depend on the number of pieces. Thus, we can use the norm $\|\cdot\|_p^{B_{N,3}}$ to consider the convergence for sequences of piecewise cubic Bézier curves. This result is important for using piecewise cubic Bézier curves to find optimal trajectories.

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