TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Bài toán ước lượng lỗi H_∞ cho một lớp hệ có trễ sử dụng một phép biến đổi trạng thái mới

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu bài toán ước lượng lỗi H_∞ cho một lớp hệ có trễ với đầu vào không xác định. Đầu tiên, một phép biến đổi trạng thái mới được đề xuất để biến đổi hệ thống có trễ đang xét thành một hệ thống mới trong đó số hạng trễ liên quan đến véc tơ trạng thái được chuyển đến đầu ra và đầu vào của hệ thống. Sau đó, một bộ quan sát mới được thiết kế dựa trên thông tin đầu vào và đầu ra để giải quyết bài toán ước lượng lỗi H_∞. Sử dụng Bổ đề Kalman-Yakubovich-Popov tổng quát, các điều kiện đủ về sự tồn tại bộ quan sát ước lượng lỗi H_∞ được thiết lập và việc giải các ma trận quan sát đạt được quy về việc giải một tập các bất đẳng thức ma trận tuyến tính. Một ví dụ số được đưa ra để minh họa tính hiệu quả của phương pháp được đề xuất.

Từ khóa: *Hệ trễ thời gian, phép biến đổi trạng thái, bổ đề Kalman-Yakubovich-Popov tổng quát, ước lượng lỗi.*

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H_∞ fault estimation problem for a class of time-delay systems using a new state transformation

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ABSTRACT

In this paper, we study the problem of H_∞ fault estimation for a class of time-delay systems with unknown inputs. First, a new state transformation is derived to transform the system into new coordinates where the delay term associated with the state vector is injected into the system output and input. Then, an observer-based H_∞ fault estimator with input and output injections is proposed for fault estimation. Based on the Generalized Kalman-Yakubovich-Popov Lemma, sufficient conditions on the existence of the H_∞ fault estimator are derived and a solution to the observer gain matrices is obtained by solving a set of linear matrix inequalities. A numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: *Time-delay systems, state transformations, generalised Kalman-Yakubovich-Popov Lemma, fault estimation.*

1. INTRODUCTION AND MOTIVATIONS

Fault detection and isolation is widely required in various kinds of practical processes for safety purposes and hence has been extensively studied in the past three decades by many authors (see,[1–5] and the references therein).

As well-known, the time-delays are inherent in many real physical systems, such as chemical processes, long transmission lines in pneumatic systems, power and water distribution networks, air pollution systems, econometric systems, hydraulic and rolling mill systems. Since the delayed state is frequently the cause for instability and poor performance of systems,[6] much attention has been paid to the fault detection design problems of the linear state delayed systems and many significant results have been achieved (see, for example,[7–17]). An adaptive observer based fault estimator was used to estimate the abrupt constant fault.[8] An iterative learning observer was designed for fault estimation and accommodation for nonlinear time-delay systems.[12] A new performance index was introduced to design the reference model, and the fault detection filter design problem was formulated as the H_∞ model-matching problem.[13] However, the above-mentioned studies are characterized in the full frequency domain. While, in many practical systems, the faults usually emerge in the low frequency domain, for example, in aerospace-related fields (see, for instance,[18,19] and in industrial processes[20,21]).

The generalised Kalman-Yakubovich- Popov (GKYP) Lemma was established and, based on this, the equivalence between finite frequency domain inequality for a transfer function and a linear matrix inequality (LMI) was built.[22] The authors have proposed a GKYP-Lemma based approach to the fault detection and fault estimation in a finite frequency domain.[23] The robust fault detection problem in low frequency domain for linear time-delay systems was investigated.[24] The H_∞ filtering problem of discrete-time state-delayed systems with finite frequency specifications was considered.[25] Some results on

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H_∞ model reduction for continuous-time linear systems over finite frequency ranges were obtained.[26]

In recent years, the H_∞ problem of time-delay systems has attracted a lot of attention and many results have been reported in the literature (for instance,[27–29]). Based on the state transformation,[30] the authors[31] considered the problem of H_∞ fault estimation for a class of linear time-delay systems. Nevertheless, the existence conditions of the state transformations reported[30] are quite restrictive.[32,33] Thus, when the existence conditions[30] are not fulfilled, the methods reported[31] can not be applied. To support this statement, let us consider the following motivated fourth-order example,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2) \\ &\quad +Bu(t) + B_dd(t) + B_ff(t), \ t \geq 0, \end{aligned} \tag{1}$$

$$y(t) = Cx(t) + D_dd(t) + D_ff(t), \tag{2}$$

$$\begin{aligned} x(t) &= \phi(t), \ t \in [-\tau, 0), \\ &\quad \tau = \max\{\tau_1, \tau_2\}, \end{aligned} \tag{3}$$

where $x(t)$ is the state vector, $u(t)$ is the control input vector, $y(t)$ is the output measurement vector, $\tau_1 > 0$ and $\tau_2 > 0$ are known constant time delays, $\phi(t)$ is a continuous initial function, $d(t)$ is the disturbance, $f(t)$ is the potential fault and

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & -3 & -4 \\ -5 & -5 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & -1 & -1 & -2 \\ -2 & -2 & -3 & -3 \\ -4 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -4 & -4 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}, \ B_f = \begin{bmatrix} 1.1 \\ 1.2 \\ 1.3 \\ 1.4 \end{bmatrix},$$

$$D_d = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, D_f = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, it is not hard to show that for the above example, the condition for the existence of a state transformation[30] (i.e., column unimodular) is not satisfied, and hence, the observer-based H_∞ fault estimator with input and output injections for fault estimation with known frequency range[31] can not be applied.

Motivated by the above discussions, in this paper, we consider the following class of time-delay systems with unknown inputs in finite frequency domain

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{s=1}^{q} A_sx(t-\tau_s) + Bu(t) \\ &\quad +B_dd(t) + B_ff(t), \ t \geq 0, \end{aligned} \tag{4}$$

$$y(t) = Cx(t) + D_dd(t) + D_ff(t), \tag{5}$$

$$x(t) = \phi(t), \ t \in [-\tau, 0), \ \tau = \max_{1 \leq s \leq q} \tau_s, \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the output measurement vector, $\tau_s > 0$ $(s = 1, 2, \ldots, q)$ are known constant time delays, $\phi(t)$ is a continuous initial function, $d(t) \in \mathbb{R}^{n_d}$ is the disturbance, $f(t) \in \mathbb{R}^{n_f}$ is the potential fault, $d(t)$ and $f(t)$ are assumed to be L_2-norm bounded, matrices A, A_s $(s = 1, 2, \ldots, q)$, B, C, B_d, B_f, D_d and D_f are known constant and of appropriate dimensions.

The remainder of this paper is organized as follows. Section 2 proposes a novel state transformation for system (4)-(6). The problem of H_∞ fault estimation is studied in Section 3. Finally, a numerical example to demonstrate the obtained results is presented in Section 4.

Notations: For integer numbers $m, n, p, \ n > p$

and matrices $G \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{1\times n}$, we have the following notations:

- $n_z = n + p - 1$.
- M^T, M^*, $M^\perp$, M^+ denote the transpose, complex conjugate transpose, orthogonal complement and the Moore-Penrose-inverse, respectively.
- I_n denotes the identity matrix of size n, $0_{m,n}$ denotes the $m \times n$ zero matrix.
- $H = \begin{bmatrix} [H]_L & [H]_R \end{bmatrix}$, where $[H]_L \in \mathbb{R}^{1\times p}$, $[H]_R \in \mathbb{R}^{1\times(n-p)}$.
- $||\cdot||$ denotes the Euclidean vector norm of $(\cdot)$.
- L_2 is the space of square integrable functions over $[0,\infty)$ with $||\cdot||_{L_2} = \left(\int_0^\infty ||\cdot||^2 dt\right)^{\frac{1}{2}}$.
- $\sigma_{\max}(G)$ denotes maximum singular value of the transfer function matrix. H_∞ norm of a transfer function $G(s)$ over a finite frequency range $|w| \leqslant \bar{w}$ is defined as $||G(s)||_\infty^{[-\bar{w},\bar{w}]} = \sup \sigma_{\max}(G(iw))$, $\forall |w| \leqslant \bar{w}$.

2. STATE TRANSFORMATION

Let us assume that matrix C takes the following form

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_p \end{bmatrix} = \begin{bmatrix} I_p & 0_{p,n-p} \end{bmatrix}$$

and denote

$$\begin{aligned} M_i &= C_i, H_{is} = 0_{1,n},\ i = 1,\dots,p, \\ &\quad s = 1,\dots,q, \quad (7) \\ M_j &= M_{j-p}A - \sum_{k=1}^{p} \alpha_j^k M_k, \quad (8) \\ H_{js} &= M_{j-p}A_s - \sum_{k=1}^{p} \beta_{js}^k M_k, \quad (9) \\ M_{i+1} &= M_i A - \sum_{k=1}^{p} \alpha_{i+1}^k M_k, \quad (10) \\ H_{(i+1)s} &= M_i A_s + H_{is}A - \sum_{k=1}^{p} \beta_{(i+1)s}^k M_k, \quad (11) \end{aligned}$$

where α_j^k, β_{js}^k $(j = p+1,\dots,2p)$, α_{i+1}^k and $\beta_{(i+1)s}^k$ $(i = 2p,\dots,n_z-1)$ are scalars will be determined. Let us now propose the following change of coordinates:

$$\begin{aligned} z_i(t) &= M_i x(t) + \sum_{s=1}^{q} H_{is} x(t-\tau_s) \\ &+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p} \nu_{is}^j M_j\Big) x(t-2\tau_s), \\ &\quad i = 1,2,\dots,n_z,\ t \geqslant 2\tau, \quad (12) \end{aligned}$$

where $\nu_{is}^j = 0$ are scalars will be determined later.

Theorem 2.1. *For some scalars α_i^j, β_{is}^j, γ_ℓ^m, $\gamma_{n_z}^r$, ν_{ks}^j, $(i = p+1, p+2,\dots,n_z, j = 1,2,\dots,p,\ s = 1,2,\dots,q,\ \ell = p+1,p+2,\dots,2p-1,\ m = 1,2,\dots,2p,\ r = p+1,p+2,\dots,n_z,\ k = 2p+1,2p+2,\dots,n_z)$, if the following equations hold*

$$\Big[M_i A - \sum_{j=p+1}^{2p} \gamma_i^j M_j\Big]_R = 0_{1,n-p}, \quad (13)$$

$$\Big[M_i A_s + H_{is}A - \sum_{j=p+1}^{2p} \gamma_i^j H_{js}\Big]_R = 0_{1,n-p}, \quad (14)$$

$$\Big[H_{is}A_s\Big]_R = 0_{1,n-p}, \quad (15)$$

$$\Big[H_{is}A_s + \Big(\sum_{j=1}^{p} \nu_{is}^j M_j\Big)A\Big]_R = 0_{1,n-p}, \quad (16)$$

$$\Big[H_{it}A_m + H_{im}A_t\Big]_R = 0_{1,n-p}, \quad (17)$$

$$\Big[\Big(\sum_{j=1}^{p} \nu_{is}^j M_j\Big)A_\ell\Big]_R = 0_{1,n-p}, \quad (18)$$

$$\Big[M_{n_z}A - \sum_{j=p+1}^{n_z} \gamma_{n_z}^j M_j\Big]_R = 0_{1,n-p}, \quad (19)$$

$$\Big[M_{n_z}A_s + H_{n_z s}A - \sum_{j=p+1}^{n_z} \gamma_{n_z}^j H_{js}\Big]_R = 0_{1,n-p}, \quad (20)$$

then (12) transforms (4)-(6) into the following form

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}u(t) + \sum_{s=1}^{q} \mathcal{B}_s^1 u(t-\tau_s) \\ &+ \sum_{s=1}^{q} \mathcal{B}_s^2 u(t-2\tau_s) + \mathcal{B}_d d(t) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{s=1}^{q} \mathcal{B}^1_{d_s} d(t-\tau_s) + \sum_{s=1}^{q} \mathcal{B}^2_{d_s} d(t-2\tau_s)\\
&+ \mathcal{B}_f f(t) + \sum_{s=1}^{q} \mathcal{B}^1_{f_s} f(t-\tau_s)\\
&+ \sum_{s=1}^{q} \mathcal{B}^2_{f_s} f(t-2\tau_s) + \Gamma y(t)\\
&+ \sum_{s=1}^{q} \Gamma^1_s y(t-\tau_s) + \sum_{s=1}^{q} \Gamma^2_s y(t-2\tau_s)\\
&+ \sum_{s=1}^{q} \Gamma^3_s y(t-3\tau_s) + \sum_{s=1}^{q} \sum_{s\neq \ell=1}^{q} \Gamma^{12}_{s\ell} y(t-\tau_s-\tau_\ell)\\
&+ \sum_{s=1}^{q} \sum_{s\neq \ell=1}^{q} \Gamma^{21}_{s\ell} y(t-2\tau_s-\tau_\ell),\ t \geqslant 3\tau, \qquad (21)
\end{aligned}
$$

$$
y(t) = \mathcal{C} z(t) + D_d d(t) + D_f f(t), \qquad (22)
$$

where

$$
\mathcal{A} = \begin{bmatrix} 0_{p,p} & I_p & 0_{p,(n_z-2p)} \\ 0_{p-1,p} & \mathcal{A}^{22}_{n_z} & 0_{p-1,(n_z-2p)} \\ 0_{n_z-2p,p} & 0_{n_z-2p,p} & I_{n_z-2p} \\ 0_{1,p} & \mathcal{A}^{42}_{n_z} & \mathcal{A}^{43}_{n_z} \end{bmatrix},
$$

$$
\mathcal{C} = \begin{bmatrix} I_p & 0_{p,(n_z-p)} \end{bmatrix},
$$

$\mathcal{A}^{22}_{n_z} \in \mathbb{R}^{(p-1)\times p}$, $\mathcal{A}^{42}_{n_z} \in \mathbb{R}^{1\times p}$, $\mathcal{A}^{43}_{n_z} \in \mathbb{R}^{1\times (n_z-2p)}$, $\mathcal{B} \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^1_s \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^2_s \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}_d \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^1_{d_s} \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^2_{d_s} \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}_f \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^1_{f_s} \in \mathbb{R}^{n_z\times 1}$, $\mathcal{B}^2_{f_s} \in \mathbb{R}^{n_z\times 1}$, $\Gamma \in \mathbb{R}^{n_z\times p}$, $\Gamma^1_s \in \mathbb{R}^{n_z\times p}$, $\Gamma^2_s \in \mathbb{R}^{n_z\times p}$, $\Gamma^3_s \in \mathbb{R}^{n_z\times p}$ *($s = 1, 2, \ldots, q$)*, $\Gamma^{12}_{s\ell} \in \mathbb{R}^{n_z\times p}$ *and* $\Gamma^{21}_{s\ell} \in \mathbb{R}^{n_z\times p}$ *are defined as below*

$$
\mathcal{A}^{22}_{n_z} = \begin{bmatrix} \gamma^{p+1}_{p+1} & \gamma^{p+2}_{p+1} & \cdots & \gamma^{2p}_{p+1} \\ \gamma^{p+1}_{p+2} & \gamma^{p+2}_{p+2} & \cdots & \gamma^{2p}_{p+2} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma^{p+1}_{2p-1} & \gamma^{p+2}_{2p-1} & \cdots & \gamma^{2p}_{2p-1} \end{bmatrix},
$$

$$
\mathcal{A}^{42}_{n_z} = \begin{bmatrix} \gamma^{p+1}_{n_z} & \gamma^{p+2}_{n_z} & \cdots & \gamma^{2p}_{n_z} \end{bmatrix},
$$

$$
\mathcal{A}^{43}_{n_z} = \begin{bmatrix} \gamma^{2p+1}_{n_z} & \gamma^{2p+2}_{n_z} & \cdots & \gamma^{n_z}_{n_z} \end{bmatrix},
$$

$$
\mathcal{B} = \begin{bmatrix} M_1 B \\ M_2 B \\ M_3 B \\ \vdots \\ M_{n_z} B \end{bmatrix},\ \mathcal{B}^1_s = \begin{bmatrix} 0_{p,1} \\ H_{(p+1)s} B \\ \vdots \\ H_{(n_z-1)s} B \\ H_{n_z s} B \end{bmatrix},
$$

$$
\mathcal{B}^2_s = \begin{bmatrix} 0_{2p,1} \\ \left(\sum_{j=1}^{p} \nu^j_{(2p+1)s} M_j\right) B \\ \vdots \\ \left(\sum_{j=1}^{p} \nu^j_{n_z s} M_j\right) B \end{bmatrix},
$$

$$
\mathcal{B}_d = \begin{bmatrix} M_1 B_d \\ M_2 B_d \\ M_3 B_d \\ \vdots \\ M_{n_z} B_d \end{bmatrix},\ \mathcal{B}^1_{d_s} = \begin{bmatrix} 0_{p,1} \\ H_{(p+1)s} B_d \\ \vdots \\ H_{(n_z-1)s} B_d \\ H_{n_z s} B_d \end{bmatrix},
$$

$$
\mathcal{B}^2_{d_s} = \begin{bmatrix} 0_{2p,1} \\ \left(\sum_{j=1}^{p} \nu^j_{(2p+1)s} M_j\right) B_d \\ \vdots \\ \left(\sum_{j=1}^{p} \nu^j_{n_z s} M_j\right) B_d \end{bmatrix},
$$

$$
\mathcal{B}_f = \begin{bmatrix} M_1 B_f \\ M_2 B_f \\ M_3 B_f \\ \vdots \\ M_{n_z} B_f \end{bmatrix},\ \mathcal{B}^1_{f_s} = \begin{bmatrix} 0_{p,1} \\ H_{(p+1)s} B_f \\ \vdots \\ H_{(n_z-1)s} B_f \\ H_{n_z s} B_f \end{bmatrix},
$$

$$
\mathcal{B}^2_{f_s} = \begin{bmatrix} 0_{2p,1} \\ \left(\sum_{j=1}^{p} \nu^j_{(2p+1)s} M_j\right) B_f \\ \vdots \\ \left(\sum_{j=1}^{p} \nu^j_{n_z s} M_j\right) B_f \end{bmatrix},
$$

$$
\Gamma = \begin{bmatrix} \alpha^1_{p+1} & \cdots & \alpha^p_{p+1} \\ \vdots & \cdots & \vdots \\ \alpha^1_{2p} & \cdots & \alpha^p_{2p} \\ \Gamma(p+1,1) & \cdots & \Gamma(p+1,p) \\ \vdots & \cdots & \vdots \\ \Gamma(2p-1,1) & \cdots & \Gamma(2p-1,p) \\ \alpha^1_{2p+1} & \cdots & \alpha^p_{2p+1} \\ \vdots & \cdots & \vdots \\ \alpha^1_{n_z} & \cdots & \alpha^p_{n_z} \\ \Gamma(n_z,1) & \cdots & \Gamma(n_z,p) \end{bmatrix},
$$

where

$$
\begin{aligned}
\Gamma(i,j) &= \Big[M_i A - \sum_{k=p+1}^{2p} \gamma^k_i M_k\Big]_{Lj},\\
\Gamma(n_z,j) &= \Big[M_{n_z} A - \sum_{k=p+1}^{n_z} \gamma^k_{n_z} M_k\Big]_{Lj}
\end{aligned}
$$

$$\Gamma_s^1 = \begin{bmatrix} \beta^1_{(p+1)s} & \dots & \beta^p_{(p+1)s} \\ \vdots & \vdots & \dots \\ \beta^1_{(2p)s} & \dots & \beta^p_{(2p)s} \\ \Gamma_s^1(p+1,1) & \dots & \Gamma_s^1(p+1,p) \\ \vdots & \vdots & \vdots \\ \Gamma_s^1(2p-1,1) & \dots & \Gamma_s^1(2p-1,p) \\ \beta^1_{(2p+1)s} & \dots & \beta^p_{(2p+1)s} \\ \vdots & \dots & \vdots \\ \beta^1_{n_z s} & \dots & \beta^p_{n_z s} \\ \Gamma_s^1(n_z,1) & \dots & \Gamma_s^1(n_z,p) \end{bmatrix},$$

where

$$\Gamma_s^1(i,j) = \Big[M_i A_s + H_{is}A - \sum_{k=p+1}^{2p} \gamma_i^k H_{ks}\Big]_{Lj},$$

$$\Gamma_s^1(n_z,j) = \Big[M_{n_z} A_s + H_{n_z s}A - \sum_{k=p+1}^{n_z} \gamma_{n_z}^k H_{ks}\Big]_{Lj},$$

$$\Gamma_s^2 = \begin{bmatrix} 0_{p,1} & \dots & 0_{p,1} \\ \Big[H_{(p+1)s}A_s\Big]_{L1} & \dots & \Big[H_{(p+1)s}A_s\Big]_{Lp} \\ \vdots & \dots & \vdots \\ \Big[H_{(2p-1)s}A_s\Big]_{L1} & \dots & \Big[H_{(2p-1)s}A_s\Big]_{Lp} \\ \Gamma_s^2(2p,1) & \dots & \Gamma_s^2(2p,p) \\ \Gamma_s^2(2p+1,1) & \dots & \Gamma_s^2(2p+1,p) \\ \vdots & \dots & \vdots \\ \Gamma_s^2(n_z-1,1) & \dots & \Gamma_s^2(n_z-1,p) \\ \Gamma_s^2(n_z,1) & \dots & \Gamma_s^2(n_z,p) \end{bmatrix},$$

$$\Gamma_s^2(2p,j) = \Big[H_{(2p)s}A_s - \sum_{k=1}^{p} \nu^k_{(2p+1)s} M_k\Big]_{Lj},$$

$$\Gamma_s^2(i,j) = \Big[H_{is}A_s + \Big(\sum_{k=1}^{p} \nu^k_{is} M_k\Big)A - \sum_{k=1}^{p} \nu^k_{(i+1)s} M_k\Big]_{Lj},$$

$$\begin{aligned}\Gamma_s^2(n_z,j) = \Big[&H_{n_z s}A_s + \Big(\textstyle\sum_{k=1}^{p} \nu^k_{n_z s} M_k\Big)A \\ &-\gamma_{n_z}^{2p+1}\Big(\textstyle\sum_{k=1}^{p} \nu^k_{(2p+1)s} M_k\Big) - \dots \\ &-\gamma_{n_z}^{n_z}\Big(\textstyle\sum_{k=1}^{p} \nu^k_{n_z s} M_k\Big)\Big]_{Lj}, \\ &j = 1,2,\dots,p,\end{aligned}$$

$$\Gamma_s^3 = \begin{bmatrix} 0_{2p,1} \\ \Big[\Big(\sum_{k=1}^{p} \nu^k_{(2p+1)s} M_k\Big)A_s\Big]_{Lj} \\ \vdots \\ \Big[\Big(\sum_{k=1}^{p} \nu^k_{n_z s} M_k\Big)A_s\Big]_{Lj} \end{bmatrix}_{j=\overline{1,p}},$$

$$\Gamma_{s\ell}^{12} = \begin{bmatrix} 0_{p,p} \\ \Big[H_{(p+1)s}A_\ell\Big]_L \\ \Big[H_{(p+2)s}A_\ell\Big]_L \\ \vdots \\ \Big[H_{n_z s}A_\ell\Big]_L \end{bmatrix},$$

$$\Gamma_{s\ell}^{21} = \begin{bmatrix} 0_{2p,p} \\ \Big[\Big(\sum_{k=1}^{p} \nu^k_{(2p+1)s} M_k\Big)A_\ell\Big]_L \\ \Big[\Big(\sum_{k=1}^{p} \nu^k_{(2p+2)s} M_k\Big)A_\ell\Big]_L \\ \vdots \\ \Big[\Big(\sum_{k=1}^{p} \nu^k_{n_z s} M_k\Big)A_\ell\Big]_L \end{bmatrix}.$$

Proof. By taking the derivatives of (12), we obtain the following

$$\begin{aligned}\dot z_i(t) &= M_i \dot x(t) \\ &= M_i(Ax(t) + \sum_{s=1}^{q} A_s x(t-\tau_s). \\ &+Bu(t) + B_d d(t) + B_f f(t)) \\ &= M_i A x(t) + M_i \sum_{s=1}^{q} A_s x(t-\tau_s) \\ &+M_i Bu(t) + M_i B_d d(t) + M_i B_f f(t) \\ &= z_{i+p}(t) + \sum_{j=1}^{p} \alpha^j_{i+p} y_j(t) + \\ &+\sum_{s=1}^{q}\sum_{j=1}^{p} \beta^j_{(i+p)s} y_j(t-\tau_s) \\ &+M_i Bu(t) + M_i B_d d(t) \\ &+M_i B_f f(t), i = 1,\dots,p, \end{aligned} \tag{23}$$

$$\begin{aligned}\dot z_i(t) &= M_i \dot x(t) + \sum_{s=1}^{q} H_{is}\dot x(t-\tau_s) \\ &= M_i A x(t) + \sum_{s=1}^{q}(M_i A_s + H_{is}A) \\ &\times x(t-\tau_s) + M_i Bu(t) \\ &+M_i B_d d(t) + M_i B_f f(t) \\ &+\sum_{s=1}^{q} H_{is}A_s x(t-2\tau_s) \\ &+\sum_{s=1}^{q} H_{is}\Big[\sum_{s\neq \ell=1}^{q} A_\ell x(t-\tau_s-\tau_\ell)\Big]\end{aligned}$$

$$+\sum_{s=1}^{q} H_{is}Bu(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{is}B_d d(t-\tau_s)+\sum_{s=1}^{q} H_{is}B_f f(t-\tau_s)$$
$$=\sum_{j=p+1}^{2p}\gamma_i^j z_j(t)+\Big[M_iA-\sum_{j=p+1}^{2p}\gamma_i^j M_j\Big]_L$$
$$\times[y_1(t)\ \ y_2(t)\ \ \dots\ \ y_p(t)]^T$$
$$+\Big[\sum_{s=1}^{q} M_iA_s+H_{is}A-\sum_{j=p+1}^{2p}\gamma_i^j H_{js}\Big]_L$$
$$\times[y_1(t-\tau_s)\ \ \dots\ \ y_p(t-\tau_s)]^T$$
$$+\Big[\sum_{s=1}^{q} H_{is}A_s\Big]_L[y_1(t-2\tau_s)\dots\ y_p(t-2\tau_s)]^T$$
$$+\Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q} H_{is}A_\ell\Big]_L[y_1(t-\tau_s-\tau_\ell)$$
$$y_2(t-\tau_s-\tau_\ell)\ \ \dots\ \ y_p(t-\tau_s-\tau_\ell)]^T$$
$$+M_iBu(t)+M_iB_d d(t)+M_iB_f f(t)$$
$$+\sum_{s=1}^{q} H_{is}Bu(t-\tau_s)+\sum_{s=1}^{q} H_{is}B_d d(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{is}B_f f(t-\tau_s),$$
$$i=p+1,\dots,2p-1, \tag{24}$$

$$\dot z_{2p}(t)=M_{2p}\dot x(t)+\sum_{s=1}^{q} H_{(2p)s}\dot x(t-\tau_s)$$
$$=z_{2p+1}(t)+\sum_{j=1}^{p}\alpha_{2p+1}^j y_j(t)$$
$$+\sum_{s=1}^{q}\sum_{j=1}^{p}\beta_{(2p+1)s}^j y_j(t-\tau_s)$$
$$+\Big[\sum_{s=1}^{q} H_{(2p)s}A_s-\Big(\sum_{j=1}^{p}\nu_{(2p+1)s}^j M_j\Big)\Big]_L$$
$$\times[y_1(t-2\tau_s)\ \ \dots\ \ y_p(t-2\tau_s)]^T$$
$$+\Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q} H_{(2p)s}A_\ell\Big]_L[y_1(t-\tau_s-\tau_\ell)$$
$$\dots y_p(t-\tau_s-\tau_\ell)]^T$$
$$+M_{2p}Bu(t)+M_{2p}B_d d(t)+M_{2p}B_f f(t)$$
$$+\sum_{s=1}^{q} H_{(2p)s}Bu(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{(2p)s}B_d d(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{(2p)s}B_f f(t-\tau_s), \tag{25}$$

$$\dot z_i(t)=M_i\dot x(t)+\sum_{s=1}^{q} H_{is}\dot x(t-\tau_s)$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)\dot x(t-2\tau_s)$$
$$=M_iAx(t)+\sum_{s=1}^{q}(M_iA_s+H_{is}A)x(t-\tau_s)$$
$$+\sum_{s=1}^{q}(H_{is}A_s+\sum_{j=1}^{p}\nu_{is}^j M_j)A)x(t-2\tau_s)$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)A_s x(t-3\tau_s)$$
$$+\sum_{s=1}^{q} H_{is}\Big[\sum_{s\neq\ell=1}^{q} A_\ell x(t-\tau_s-\tau_\ell)\Big]$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)\Big[\sum_{s\neq\ell=1}^{q} A_\ell x(t-2\tau_s-\tau_\ell)\Big]$$
$$+M_iBu(t)+M_iB_d d(t)+M_iB_f f(t)$$
$$+\sum_{s=1}^{q} H_{is}Bu(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{is}B_d d(t-\tau_s)$$
$$+\sum_{s=1}^{q} H_{is}B_f f(t-\tau_s)$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)Bu(t-2\tau_s)$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)B_d d(t-2\tau_s)$$
$$+\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{is}^j M_j\Big)B_f f(t-2\tau_s). \tag{26}$$

From (10)-(11), we have for $i = 2p$, $2p+1,\dots,n_z-1$,

$$M_iAx(t)=M_{i+1}x(t)+\sum_{j=1}^{p}\alpha_{i+1}^j y_j(t), \tag{27}$$

$$(M_iA_s + H_{is}A)x(t-\tau_s) = H_{(i+1)s}x(t-\tau_s) + \sum_{j=1}^{p}\beta^j_{(j+1)s}y_j(t-\tau_s),\ s = 1,2,\ldots,q. \quad (28)$$

Substituting (27)-(28) into (26) and using (15) - (18), we obtain

$$\begin{aligned}
\dot{z}_i(t) &= z_{i+1}(t) + \sum_{j=1}^{p}\alpha^j_{i+1}y_j(t) \\
&+ \sum_{s=1}^{q}\sum_{j=1}^{p}\beta^j_{(i+1)s}y_j(t-\tau_s) + \Bigg[\sum_{s=1}^{q}H_{is}A_s \\
&+ \Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big)A - \Big(\sum_{j=1}^{p}\nu^j_{(i+1)s}M_j\Big)\Bigg]_L \\
&[y_1(t-2\tau_s)\ \ \ldots\ \ y_p(t-2\tau_s)]^T \\
&+ \Big[\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big)A_s\Big]_L[y_1(t-3\tau_s) \\
&y_2(t-3\tau_s)\ \ \ldots\ \ y_p(t-3\tau_s)]^T \\
&+ \Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q}H_{is}A_\ell\Big]_L[y_1(t-\tau_s-\tau_\ell) \\
&y_2(t-\tau_s-\tau_\ell)\ \ \ldots\ \ y_p(t-\tau_s-\tau_\ell)]^T \\
&+ \Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big)A_\ell\Big]_L \\
&\times[y_1(t-2\tau_s-\tau_\ell)\ \ \ldots\ \ y_p(t-2\tau_s-\tau_\ell)]^T \\
&+ M_iBu(t) + M_iB_dd(t) + M_iB_ff(t) \\
&+ \sum_{s=1}^{q}H_{is}Bu(t-\tau_s) + \sum_{s=1}^{q}H_{is}B_dd(t-\tau_s) \\
&+ \sum_{s=1}^{q}H_{is}B_ff(t-\tau_s) + \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big) \\
&Bu(t-2\tau_s) + \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big)B_dd(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{is}M_j\Big)B_ff(t-2\tau_s), \\
&i = 2p+1,\ldots,n_z-1. \quad (29)
\end{aligned}$$

From (26), we have

$$\begin{aligned}
\dot{z}_{n_z}(t) &= M_{n_z}Ax(t) \\
&+ \sum_{s=1}^{q}(M_{n_z}A_s + H_{n_zs}A)x(t-\tau_s) \\
&+ \sum_{s=1}^{q}(H_{n_zs}A_s + \sum_{j=1}^{p}\nu^j_{n_zs}M_j)A)x(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)A_sx(t-3\tau_s) \\
&+ \sum_{s=1}^{q}H_{n_zs}\Big[\sum_{s\neq\ell=1}^{q}A_\ell x(t-\tau_s-\tau_\ell)\Big] \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)\Big[\sum_{s\neq\ell=1}^{q}A_\ell x(t-2\tau_s-\tau_\ell)\Big] \\
&+ M_{n_z}Bu(t) + M_{n_z}B_dd(t) + M_{n_z}B_ff(t) \\
&+ \sum_{s=1}^{q}H_{n_zs}Bu(t-\tau_s) + \sum_{s=1}^{q}H_{n_zs}B_dd(t-\tau_s) \\
&+ \sum_{s=1}^{q}H_{n_zs}B_ff(t-\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)Bu(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)B_dd(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)B_ff(t-2\tau_s). \quad (30)
\end{aligned}$$

From conditions (19) - (20), equation (30) becomes

$$\begin{aligned}
\dot{z}_{n_z}(t) &= \sum_{j=p+1}^{n_z}\gamma^j_{n_z}z_j(t) + M_{n_z}Bu(t) \\
&+ M_{n_z}B_dd(t) + M_{n_z}B_ff(t) \\
&+ \sum_{s=1}^{q}H_{n_zs}Bu(t-\tau_s) \\
&+ \sum_{s=1}^{q}H_{n_zs}B_dd(t-\tau_s) \\
&+ \sum_{s=1}^{q}H_{n_zs}B_ff(t-\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)Bu(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)B_dd(t-2\tau_s) \\
&+ \sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu^j_{n_zs}M_j\Big)B_ff(t-2\tau_s)
\end{aligned}$$

$$+\Big[M_{n_z}A-\sum_{j=p+1}^{n_z}\gamma_{n_z}^{j}M_j\Big]_L[y_1(t)\quad\dots\quad y_p(t)]^T$$
$$+\Big[\sum_{s=1}^{q}M_{n_z}A_s+H_{n_zs}A-\sum_{j=p+1}^{n_z}\gamma_{n_z}^{j}H_{js}\Big]_L$$
$$[y_1(t-\tau_s)\ \ y_2(t-\tau_s)\ \ \dots\ \ y_p(t-\tau_s)]^T$$
$$+\Bigg[\sum_{s=1}^{q}H_{n_zs}A_s+\Big(\sum_{j=1}^{p}\nu_{n_zs}^{j}M_j\Big)A$$
$$-\sum_{j=p+1}^{n_z}\gamma_{n_z}^{j}\Big(\sum_{k=1}^{p}\nu_{js}^{k}M_k\Big)\Bigg]_L$$
$$\times[y_1(t-2\tau_s)\quad\dots\quad y_p(t-2\tau_s)]^T$$
$$+\Big[\sum_{s=1}^{q}\Big(\sum_{j=1}^{p}\nu_{n_zs}^{j}M_j\Big)A_s\Big]_L[y_1(t-3\tau_s)$$
$$\dots\ \ y_p(t-3\tau_s)]^T$$
$$+\Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q}H_{n_zs}A_\ell\Big]_L[y_1(t-\tau_s-\tau_\ell)$$
$$y_2(t-\tau_s-\tau_\ell)\dots\ \ y_p(t-\tau_s-\tau_\ell)]^T$$
$$+\Big[\sum_{s=1}^{q}\sum_{s\neq\ell=1}^{q}\Big(\sum_{j=1}^{p}\nu_{n_zs}^{j}M_j\Big)A_\ell\Big]_L$$
$$[y_1(t-2\tau_s-\tau_\ell)\quad\dots\quad y_p(t-2\tau_s-\tau_\ell)]^T.$$

Finally, (23)-(25), (29) and (31) can now be expressed in the form (21)-(22). This completes the proof of Theorem 2.1.

In the following, we will solve unknowns in Theorem 2.1. For this, we first express (13)-(18) into the following compact form:

$$\chi_{n_z}X_{n_z}=Y_{n_z},\tag{31}$$

where

$$\chi_{n_z}=\begin{bmatrix}\chi_{n_z}^{1} & \chi_{n_z}^{2} & \dots & \chi_{n_z}^{p+2}\end{bmatrix},$$
$$X_{n_z}=\begin{bmatrix}X_{n_z}^{1} & X_{n_z}^{2} & \dots & X_{n_z}^{p+2}\end{bmatrix}^T,$$
$$Y_{n_z}=\begin{bmatrix}Y_{n_z}^{1} & \dots & Y_{n_z}^{p+q-1} & Y_{n_z}^{p+q}\end{bmatrix},$$

where, $\chi_{n_z}^{i}$, $X_{n_z}^{i}$ $(i=1,2,\dots,p+2)$ are as defined below

$$\chi_{n_z}^{1}=\begin{bmatrix}\chi_{n_z}^{11} & \chi_{n_z}^{12}\end{bmatrix},\tag{32}$$

$$\chi_{n_z}^{11}=\begin{bmatrix}\alpha_{p+1}^{1} & \dots & \alpha_{p+1}^{p}\\ \gamma_{p+1}^{p+1} & \dots & \gamma_{p+1}^{2p}\end{bmatrix},$$
$$\chi_{n_z}^{12}=\Big[\ \beta_{(p+1)1}^{1}\ \dots\ \beta_{(p+1)1}^{p}\ \ \beta_{(p+1)2}^{1}\ \dots$$
$$\beta_{(p+1)2}^{p}\dots\ \beta_{(p+1)q}^{1}\ \dots\ \beta_{(p+1)q}^{p}\ \Big],$$

$$\chi_{n_z}^{2}=\begin{bmatrix}\chi_{n_z}^{21} & \chi_{n_z}^{22}\end{bmatrix},\tag{33}$$

$$\chi_{n_z}^{21}=\begin{bmatrix}\alpha_{p+2}^{1} & \dots & \alpha_{p+2}^{p} & \\ \dots & \gamma_{p+2}^{p+1} & \dots & \gamma_{p+2}^{2p}\end{bmatrix},$$
$$\chi_{n_z}^{12}=\begin{bmatrix}\beta_{(p+2)1}^{1} & \dots & \beta_{(p+2)1}^{p} & \dots\\ \beta_{(p+2)q}^{1} & \dots & \beta_{(p+2)q}^{p} & \end{bmatrix},$$
$$\vdots$$
$$\chi_{n_z}^{p-1}=\begin{bmatrix}\chi_{n_z}^{(p-1)1} & \chi_{n_z}^{(p-1)2}\end{bmatrix},\tag{34}$$

$$\chi_{n_z}^{(p-1)1}=\begin{bmatrix}\alpha_{2p-1}^{1} & \dots & \alpha_{2p-1}^{p}\\ \gamma_{2p-1}^{p+1} & \dots & \gamma_{2p-1}^{2p}\end{bmatrix},$$
$$\chi_{n_z}^{(p-1)2}=\begin{bmatrix}\beta_{(2p-1)1}^{1} & & \dots & \beta_{(2p-1)1}^{p}\\ \dots & & & \beta_{(2p-1)q}^{1}\\ \dots & \beta_{(2p-1)q}^{p} & & \end{bmatrix},$$

$$\chi_{n_z}^{p}=\begin{bmatrix}\chi_{n_z}^{p(2p)} & \chi_{n_z}^{p(2p+1)} & \dots & \chi_{n_z}^{pn_z}\end{bmatrix},\tag{35}$$

$$\chi_{n_z}^{p(2p)}=\begin{bmatrix}\beta_{(2p)1}^{1} & \dots & \beta_{(2p)1}^{p}\\ \dots & \beta_{(2p)q}^{1} & \\ \dots & \beta_{(2p)q}^{p} & \end{bmatrix},$$
$$\chi_{n_z}^{p(2p+1)}=\begin{bmatrix}\beta_{(2p+1)1}^{1} & \dots\\ \beta_{(2p+1)1}^{p} & \\ \dots & \beta_{(2p+1)q}^{1}\dots\\ \beta_{(2p+1)q}^{p} & \end{bmatrix},$$
$$\vdots$$
$$\chi_{n_z}^{pn_z}=\begin{bmatrix}\beta_{n_z1}^{1} & \dots & \beta_{n_z1}^{p} & & \\ & \dots & \beta_{n_zq}^{1} & \dots & \beta_{n_zq}^{p}\end{bmatrix},$$

$$\chi_{n_z}^{p+1}=\begin{bmatrix}\alpha_{2p}^{1} & \alpha_{2p}^{2} & \dots\\ \alpha_{2p}^{p} & \dots & \alpha_{n_z-1}^{1}\\ \alpha_{n_z-1}^{2} & \dots & \alpha_{n_z-1}^{p}\end{bmatrix},\tag{36}$$

$$\chi_{n_z}^{p+2}=\begin{bmatrix}\chi_{n_z(2p+1)}^{p+2} & & \\ \chi_{n_z(2p+2)}^{p+2} & \dots & \chi_{n_zn_z}^{p+2}\end{bmatrix},\tag{37}$$

$$\chi^{p+2}_{n_z(2p+1)} = \begin{bmatrix} \nu^1_{(2p+1)1} & \nu^2_{(2p+1)1} \\ \dots & \nu^1_{(2p+1)q} \\ \dots & \nu^p_{(2p+1)q} \end{bmatrix},$$

$$\chi^{p+2}_{n_z(2p+2)} = \begin{bmatrix} \nu^1_{(2p+2)1} & \nu^2_{(2p+2)1} \\ \dots & \nu^p_{(2p+2)1} \\ \dots & \nu^1_{(2p+2)q} \\ \dots & \nu^p_{(2p+2)q} \end{bmatrix},$$

$$\vdots$$

$$\chi^{p+2}_{n_z n_z} = \Big[\; \nu^1_{n_z 1} \quad \nu^2_{n_z 1} \quad \dots \quad \nu^p_{n_z 1} \quad \dots$$
$$\nu^1_{n_z q} \quad \nu^2_{n_z q} \quad \dots \quad \nu^p_{n_z q} \; \Big],$$

$$X^1_{n_z} = \Big[\; X^1_{n_z 1} \quad X^1_{n_z 2} \; \Big], \tag{38}$$

$$X^1_{n_z 1} = \Big[\; X^{11}_{n_z} \quad 0_{p(q+2),\frac{(q+2)(q+1)}{2}(n-p)}$$
$$\dots \quad 0_{p(q+2),\frac{(q+2)(q+1)}{2}(n-p)} \; \Big],$$

$$X^1_{n_z 2} = \Big[\; X^{11}_{n_z 2} \quad X^{12_2}_{n_z 2} \quad \dots \quad X^{12_q}_{n_z 2}$$
$$0_{p(q+2),(n_z-2p)q^2(n-p)} \; \Big],$$

$$X^{11}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)q(n-p)},$$
$$X^{12_2}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)(q-1)(n-p)},$$
$$\dots,$$
$$X^{12_q}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)(n-p)},$$

$$\vdots$$

$$X^{p-1}_{n_z} = \Big[\; X^{p-1}_{n_z 1} \quad X^{p-1}_{n_z 2} \; \Big], \tag{39}$$

$$X^{p-1}_{n_z 1} = \Big[\; 0_{p(q+2),\frac{(q+2)(q+1)}{2}(n-p)}$$
$$0_{p(q+2),\frac{(q+2)(q+1)}{2}(n-p)} \quad \dots \quad X^{11}_{n_z} \; \Big],$$

$$X^{p-1}_{n_z 2} = \Big[\; X^{(p-1)1}_{n_z 2} \quad X^{(p-1)2_2}_{n_z 2} \quad \dots$$
$$X^{(p-1)2_q}_{n_z 2} \quad 0_{p(q+2),(n_z-2p)q^2(n-p)} \; \Big],$$

$$X^{(p-1)1}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)q(n-p)},$$
$$X^{(p-1)2_2}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)(q-1)(n-p)},$$
$$\dots,$$
$$X^{(p-1)2_q}_{n_z 2} = 0_{p(q+2),(n_z-2p+1)(n-p)},$$

$$X^p_{n_z} = \Big[\; X^p_{n_z 1} \quad X^p_{n_z 2} \; \Big], \tag{40}$$

$$X^p_{n_z 1} = \Big[\; 0_{(n_z-2p+1)pq,\frac{(q+2)(q+1)}{2}(n-p)}$$
$$0_{(n_z-2p+1)pq,\frac{(q+2)(q+1)}{2}(n-p)} \quad \dots$$
$$0_{(n_z-2p+1)pq,\frac{(q+2)(q+1)}{2}(n-p)} \; \Big],$$

$$X^p_{n_z 2} = \Big[\; X^{pp}_{n_z} \quad X^{p(p+1)_2}_{n_z} \quad \dots \quad X^{p(p+1)_q}_{n_z}$$
$$0_{(n_z-2p+1)pq,(n_z-2p)q^2(n-p)} \; \Big],$$

$$X^{p+1}_{n_z} = \Big[\; X^{p+1}_{n_z 1} \quad X^{p+1}_{n_z 2} \; \Big], \tag{41}$$

$$X^{p+1}_{n_z 1} = \Big[\; 0_{(n_z-2p)p,\frac{(q+2)(q+1)}{2}(n-p)} \quad \dots$$
$$0_{(n_z-2p)p,\frac{(q+2)(q+1)}{2}(n-p)} \; \Big],$$

$$X^{p+1}_{n_z 2} = \Big[\; X^{(p+1)p}_{n_z} \quad X^{(p+1)(p+1)_2}_{n_z} \quad \dots$$
$$X^{(p+1)(p+1)_q}_{n_z} \quad 0_{(n_z-2p)p,(n_z-2p)q^2(n-p)} \; \Big],$$

$$X^{p+2}_{n_z} = \begin{bmatrix} X^{p+2}_{n_z(2p+1)1} & X^{p+2}_{n_z(2p+1)2} \\ X^{p+2}_{n_z(2p+2)1} & X^{p+2}_{n_z(2p+2)2} \\ \vdots & \vdots \\ X^{p+2}_{n_z n_z 1} & X^{p+2}_{n_z n_z 2} \end{bmatrix}, \tag{42}$$

$$X^{p+2}_{n_z(2p+1)1} = \Big[\; 0_{pq,\frac{(q+2)(q+1)}{2}(n-p)}$$
$$\dots \quad 0_{pq,\frac{(q+2)(q+1)}{2}(n-p)} \; \Big],$$

$$\vdots$$

$$X^{p+2}_{n_z n_z 1} = \Big[\; 0_{pq,\frac{(q+2)(q+1)}{2}(n-p)} \; ,$$
$$\dots \quad 0_{pq,\frac{(q+2)(q+1)}{2}(n-p)} \; \Big],$$

$$X^{p+2}_{n_z(2p+1)2} = \Big[\; X^{(p+2)p}_{n_z(2p+1)2} \; ,$$
$$\dots \quad X^{(p+2)(p+1)_q}_{n_z(2p+1)2} \quad X^{(p+2)(p+2)}_{n_z(2p+1)2} \; \Big],$$

$$X^{p+2}_{n_z(2p+2)2} = \Big[\; X^{(p+3)p}_{n_z(2p+2)2} \quad X^{(p+3)(p+1)_2}_{n_z(2p+2)2}$$
$$\dots \quad X^{(p+3)(p+1)_q}_{n_z(2p+2)2} \quad X^{(p+3)(p+2)}_{n_z(2p+2)2} \; \Big],$$

$$\vdots$$

$$X^{p+2}_{n_z n_z 2} = \Big[\; X^{np}_{n_z n_z 2} \quad X^{n(p+1)_2}_{n_z n_z 2} \; ,$$
$$\dots \quad X^{n(p+1)_q}_{n_z n_z 2} \quad X^{n(p+2)}_{n_z n_z 2} \; \Big],$$

where

$$X^{11}_{n_z} \in \mathbb{R}^{p(q+2)\times\frac{(q+2)(q+1)}{2}(n-p)},$$
$$X^{i(p+1)_\ell}_{n_z j 2} = 0_{pq,(n_z-2p+1)(q-\ell+1)(n-p)},$$
$$i = p+2,\dots,n, \;\; j = 2p+1,\dots,n_z,$$
$$\ell = 2,3,\dots,q,$$

$X_{n_z}^{pp} \in \mathbb{R}^{(n_z-2p+1)pq\times(n_z-2p+1)q(n-p)}$,
$X_{n_z}^{p(p+1)_\ell} \in \mathbb{R}^{(n_z-2p+1)pq\times(n_z-2p+1)(q-\ell+1)(n-p)}$,
$\ell = 2, 3, \ldots, q$,
$X_{n_z}^{(p+1)(p+1)_\ell} \in \mathbb{R}^{(n_z-2p)p\times(n_z-2p+1)(q-\ell+1)(n-p)}$,
$\ell = 2, 3, \ldots, q$,
$X_{n_z}^{(p+1)p} \in \mathbb{R}^{(n_z-2p)p\times(n_z-2p+1)q(n-p)}$,
$X_{n_zj2}^{ip} \in \mathbb{R}^{pq\times(n_z-2p+1)q(n-p)}$,
$i = p+2, \ldots, n,\ j = 2p+1, \ldots, n_z$,
$X_{n_zj2}^{i(p+2)} \in \mathbb{R}^{pq\times(n_z-2p)q^2(n-p)}$,
$i = p+2, \ldots, n,\ j = 2p+1, \ldots, n_z$

are defined as below

$$X_{n_z}^{11} = \begin{bmatrix} X_{n_z1}^{11} & X_{n_z2}^{11} \end{bmatrix}, \tag{43}$$

where

$$X_{n_z1}^{11} = \begin{bmatrix} \sum & 0_{2p,q(n-p)} & 0_{2p,(q-1)(n-p)} \\ \Delta & \Omega^1 & \Phi_1^1 \end{bmatrix},$$

$$X_{n_z2}^{11} = \begin{bmatrix} 0_{2p,(q-2)(n-p)} & \ldots & 0_{2p,n-p} \\ \Phi_2^1 & \ldots & \Phi_{q-1}^1 \end{bmatrix}$$

$$X_{n_z}^{pp} = \begin{bmatrix} X_{n_z1}^{pp} & X_{n_z2}^{pp} \end{bmatrix}, \tag{44}$$

where

$$X_{n_z1}^{pp} = \begin{bmatrix} \Omega^1 & \Omega^2 \\ 0_{pq,q(n-p)} & \Omega^1 \\ 0_{pq,q(n-p)} & 0_{pq,q(n-p)} \\ \vdots & \vdots \\ 0_{pq,q(n-p)} & 0_{pq,q(n-p)} \end{bmatrix},$$

$$X_{n_z2}^{pp} = \begin{bmatrix} \Omega^3 & \ldots & \Omega^{n_z-2p+1} \\ \Omega^2 & \ldots & \Omega^{n_z-2p} \\ \Omega^1 & \ldots & \Omega^{n_z-2p-1} \\ \vdots & \ddots & \vdots \\ 0_{pq,q(n-p)} & \ldots & \Omega^1 \end{bmatrix},$$

$$X_{n_z}^{p(p+1)_l} = \begin{bmatrix} X_{n_z1}^{p(p+1)_l} & X_{n_z2}^{p(p+1)_l} \end{bmatrix}, \tag{45}$$

where

$$X_{n_z1}^{p(p+1)_l} = \begin{bmatrix} \Phi_{\ell-1}^1 & \Phi_{\ell-1}^2 \\ 0_{pq,(q-\ell+1)(n-p)} & \Phi_{\ell-1}^1 \\ 0_{pq,(q-\ell+1)(n-p)} & 0_{pq,(q-\ell+1)(n-p)} \\ \vdots & \vdots \\ 0_{pq,(q-\ell+1)(n-p)} & 0_{pq,(q-\ell+1)(n-p)} \end{bmatrix},$$

$$X_{n_z2}^{p(p+1)_l} = \begin{bmatrix} \Phi_{\ell-1}^3 & \ldots & \Phi_{\ell-1}^{n_z-2p+1} \\ \Phi_{\ell-1}^2 & \ldots & \Phi_{\ell-1}^{n_z-2p} \\ \Phi_{\ell-1}^1 & \ldots & \Phi_{\ell-1}^{n_z-2p-1} \\ \vdots & \ddots & \vdots \\ 0_{pq,(q-\ell+1)(n-p)} & \ldots & \Phi_{\ell-1}^1 \end{bmatrix},$$

$$X_{n_z}^{(p+1)p} = \begin{bmatrix} X_{n_z1}^{(p+1)p} & X_{n_z2}^{(p+1)p} \end{bmatrix}, \tag{46}$$

where

$$X_{n_z1}^{(p+1)p} = \begin{bmatrix} 0_{p,q(n-p)} & \Xi^1 \\ 0_{p,q(n-p)} & 0_{p,q(n-p)} \\ \vdots & \vdots \\ 0_{p,q(n-p)} & 0_{p,q(n-p)} \end{bmatrix},$$

$$X_{n_z2}^{(p+1)p} = \begin{bmatrix} \Xi^2 & \ldots & \Xi^{n_z-2p} \\ \Xi^1 & \ldots & \Xi^{n_z-2p-1} \\ \vdots & \ddots & \vdots \\ 0_{p,q(n-p)} & \ldots & \Xi^1 \end{bmatrix},$$

$$X_{n_z}^{(p+1)(p+1)} = \begin{bmatrix} X_{n_z1}^{(p+1)(p+1)} & X_{n_z2}^{(p+1)(p+1)} \end{bmatrix}, \tag{47}$$

where

$$X_{n_z1}^{(p+1)(p+1)} = \begin{bmatrix} 0_{p,(q-\ell+1)(n-p)} & \Psi_{\ell-1}^1 \\ 0_{p,(q-\ell+1)(n-p)} & 0_{p,(q-\ell+1)(n-p)} \\ \vdots & \vdots \\ 0_{p,(q-\ell+1)(n-p)} & 0_{p,(q-\ell+1)(n-p)} \end{bmatrix},$$

$$X_{n_z2}^{(p+1)(p+1)} = \begin{bmatrix} \Psi_{\ell-1}^2 & \ldots & \Psi_{\ell-1}^{n_z-2p} \\ \Psi_{\ell-1}^1 & \ldots & \Psi_{\ell-1}^{n_z-2p-1} \\ \vdots & \ddots & \vdots \\ 0_{p,(q-\ell+1)(n-p)} & \ldots & \Psi_{\ell-1}^1 \end{bmatrix},$$

$$X^{(p+2)p}_{n_z} = \begin{bmatrix} X^{(p+2)p}_{n_z1} & X^{(p+2)p}_{n_z2} \end{bmatrix}, \quad (48)$$

where

$$X^{(p+2)p}_{n_z} = \begin{bmatrix} X^{(p+2)p}_{n_z(2p+1)2} \\ X^{(p+3)p}_{n_z(2p+2)2} \\ \vdots \\ X^{np}_{n_zn_z2} \end{bmatrix}$$

$$X^{(p+2)p}_{n_z1} = \begin{bmatrix} 0_{pq,q(n-p)} & \nabla \\ 0_{pq,q(n-p)} & 0_{pq,q(n-p)} \\ \vdots & \vdots \\ 0_{pq,q(n-p)} & 0_{pq,q(n-p)} \end{bmatrix},$$

$$X^{(p+2)p}_{n_z2} = \begin{bmatrix} 0_{pq,q(n-p)} & \dots & 0_{pq,q(n-p)} \\ \nabla & \dots & 0_{pq,q(n-p)} \\ \vdots & \ddots & \vdots \\ 0_{pq,q(n-p)} & \dots & \nabla \end{bmatrix},$$

$$X^{(p+2)(p+2)}_{n_z} = \begin{bmatrix} X^{(p+2)(p+2)}_{n_z1} & X^{(p+2)(p+2)}_{n_z2} \end{bmatrix}, \quad (49)$$

$$X^{(p+2)(p+2)}_{n_z} = \begin{bmatrix} X^{(p+2)(p+2)}_{n_z(2p+1)2} \\ X^{(p+3)(p+2)}_{n_z(2p+2)2} \\ \vdots \\ X^{n(p+2)}_{n_zn_z2} \end{bmatrix},$$

$$X^{(p+2)(p+2)}_{n_z1} = \begin{bmatrix} \Pi & 0_{pq,q^2(n-p)} \\ 0_{pq,q^2(n-p)} & \Pi \\ \vdots & \vdots \\ 0_{pq,q^2(n-p)} & 0_{pq,q^2(n-p)} \end{bmatrix},$$

$$X^{(p+2)(p+2)}_{n_z2} = \begin{bmatrix} \dots & 0_{pq,q^2(n-p)} \\ \dots & 0_{pq,q^2(n-p)} \\ \ddots & \vdots \\ \dots & \Pi \end{bmatrix}.$$

In (32), $Y^i_{n_z} \in \mathbb{R}^{1\times(2q+1+\frac{q(q-1)}{2})(n-p)}$ $(i = 1, 2, \dots, p-1)$, $Y^p_{n_z} \in \mathbb{R}^{1\times((n-p)^2 q}$, $Y^{p+1}_{n_z} \in \mathbb{R}^{1\times(n-p)^2(q-1)}$, $Y^{p+2}_{n_z} \in \mathbb{R}^{1\times(n-p)^2(q-2)}$, $\dots$, $Y^{(p+q-1)}_{n_z} \in \mathbb{R}^{1\times(n-p)^2}$, $Y^{p+q}_{n_z} = 0_{1,q^2(n_z-2p)(n-p)}$ are defined as follows

$$Y^i_{n_z} = \begin{bmatrix} Y^i_{n_z1} & Y^i_{n_z2} \end{bmatrix}, \quad (50)$$

where

$$Y^i_{n_z1} = \begin{bmatrix} X^2_i & \bar{Y}^1_{i1} & \dots & \bar{Y}^1_{iq} \\ Y^1_{i1} & \dots & Y^1_{iq} & \end{bmatrix}$$

$$Y^i_{n_z2} = \begin{bmatrix} Z^1_{i,12} & \dots & Z^1_{i,1q} \\ Z^1_{i,23} & \dots & Z^1_{i,(q-1)q} \end{bmatrix},$$

for $i = 1, 2, \dots, p-1$,

$$Y^p_{n_z} = \begin{bmatrix} Y^{p1}_{n_z} & Y^{p2}_{n_z} & \dots & Y^{p(n-p)}_{n_z} \end{bmatrix}, \quad (51)$$

$$Y^{p1}_{n_z} = \begin{bmatrix} Y^1_{p1} & Y^1_{p2} & \dots & Y^1_{pq} \end{bmatrix},$$

$$Y^{p2}_{n_z} = \begin{bmatrix} Y^2_{p1} & Y^2_{p2} & \dots & Y^2_{pq} \end{bmatrix},$$

$$\vdots$$

$$Y^{p(n-p)}_{n_z} = \begin{bmatrix} Y^{n-p}_{p1} & Y^{n-p}_{p2} & \dots & Y^{n-p}_{pq} \end{bmatrix},$$

$$Y^{p+1}_{n_z} = \begin{bmatrix} Y^{(p+1)1}_{n_z} & \dots & Y^{(p+1)(n-p)}_{n_z} \end{bmatrix}, \quad (52)$$

$$Y^{(p+1)1}_{n_z} = \begin{bmatrix} Z^1_{p,12} & \dots & Z^1_{p,1q} \end{bmatrix},$$

$$Y^{(p+1)2}_{n_z} = \begin{bmatrix} Z^2_{p,12} & \dots & Z^2_{p,1q} \end{bmatrix},$$

$$\vdots$$

$$Y^{(p+1)(n-p)}_{n_z} = \begin{bmatrix} Z^{n-p}_{p,12} & \dots & Z^{n-p}_{p,1q} \end{bmatrix},$$

$$Y^{p+2}_{n_z} = \begin{bmatrix} Y^{(p+2)1}_{n_z} & \dots & Y^{(p+2)(n-p)}_{n_z} \end{bmatrix}, \quad (53)$$

$$Y^{(p+2)1}_{n_z} = \begin{bmatrix} Z^1_{p,23} & Z^1_{p,24} & \dots & Z^1_{p,2q} \end{bmatrix},$$

$$\vdots$$

$$Y^{(p+2)(n-p)}_{n_z} = \begin{bmatrix} Z^{n-p}_{p,23} & \dots & Z^{n-p}_{p,2q} \end{bmatrix},$$

$$\vdots$$

$$Y^{p+q-1}_{n_z} = \begin{bmatrix} Y^{(p+q-1)1}_{n_z} & \dots & Y^{(p+q-1)(n-p)}_{n_z} \end{bmatrix}, \quad (54)$$

$$Y^{(p+q-1)1}_{n_z} = Z^1_{p,(q-1)q},$$

$$Y^{(p+q-1)2}_{n_z} = Z^2_{p,(q-1)q},$$

$$\vdots$$

$$Y^{(p+q-1)(n-p)}_{n_z} = Z^{n-p}_{p,(q-1)q}.$$

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From (31), a solution for χ_{n_z} exists if and only if

$$\operatorname{rank}\begin{bmatrix} X_{n_z} \\ Y_{n_z} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X_{n_z} \end{bmatrix}. \tag{55}$$

Next, we express (19)-(20) into the following form

$$\zeta_{n_z} Z_{n_z} = T_{n_z}, \tag{56}$$

where

$$\zeta_{n_z} = \begin{bmatrix} \alpha^1_{n_z} & \ldots & \alpha^p_{n_z} & \gamma^{p+1}_{n_z} & \ldots & \gamma^{n_z}_{n_z} \end{bmatrix}, \tag{57}$$

$$Z_{n_z} = \begin{bmatrix} Z_{n_z1} & Z_{n_z2} & Z_{nz3} & \ldots & Z_{nz4} \end{bmatrix}, \tag{58}$$

$$Z_{n_z1} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ Z^1_{n_z}(2p+1, n-p) \\ Z^1_{n_z}(2p+2, n-p) \\ \vdots \\ Z^1_{n_z}(n_z-1, n-p) \\ Z^1_{n_z}(n_z, n-p) \end{bmatrix},$$

$$Z_{n_z2} = \begin{bmatrix} \overline{\mathcal{X}}^1_1 \\ \overline{\mathcal{X}}^1_1 \\ Z^2_{n_z}(2p+1, n-p) \\ Z^2_{n_z}(2p+2, n-p) \\ \vdots \\ Z^2_{n_z}(n_z-1, n-p) \\ Z^2_{n_z}(n_z, n-p) \end{bmatrix},$$

$$Z_{n_z3} = \begin{bmatrix} \overline{\mathcal{X}}^1_2 \\ \overline{\mathcal{X}}^1_2 \\ Z^3_{n_z}(2p+1, n-p) \\ Z^3_{n_z}(2p+2, n-p) \\ \vdots \\ Z^3_{n_z}(n_z-1, n-p) \\ Z^3_{n_z}(n_z, n-p) \end{bmatrix},$$

$$Z_{n_z4} = \begin{bmatrix} \overline{\mathcal{X}}^1_q \\ \overline{\mathcal{X}}^1_q \\ Z^{q+1}_{n_z}(2p+1, n-p) \\ Z^{q+1}_{n_z}(2p+2, n-p) \\ \vdots \\ Z^{q+1}_{n_z}(n_z-1, n-p) \\ Z^{q+1}_{n_z}(n_z, n-p) \end{bmatrix},$$

$$T_{n_z} = \begin{bmatrix} T^1_{n_z} & T^2_{n_z} & T^3_{n_z} & \ldots & T^{q+1}_{n_z} \end{bmatrix}. \tag{59}$$

In (58)-(59), $T^s_{n_z}$ and $Z^s_{n_z}(k, n-1)$ $(s = 2, 3, \ldots, q+1)$, $(k = 2p+1, 2p+2, \ldots, n_z)$ are defined as follows

$$T^1_{n_z} = X^{n_z-2p+2}_p - \sum_{j=1}^{p}\Big(\alpha^j_{2p} X^{n_z-2p+1}_j + \alpha^j_{2p+1} X^{n_z-2p}_j + \ldots + \alpha^j_{n_z-2} X^3_j + \alpha^j_{n_z-1} X^2_j\Big), \tag{60}$$

$$T^s_{n_z} = \bar{Y}^{n_z-2p+1}_{p(s-1)} - \sum_{j=1}^{p}\Big(\alpha^j_{2p} \bar{Y}^{n_z-2p}_{j(s-1)} + \alpha^j_{2p+1} \bar{Y}^{n_z-2p-1}_{j(s-1)} + \ldots + \alpha^j_{n_z-1} \bar{Y}^1_{j(s-1)} + \beta^j_{(2p+1)(s-1)} X^{n_z-2p}_j + \ldots + \beta^j_{n_z(s-1)} X^1_j\Big), \tag{61}$$

$$Z^1_{n_z}(k, n-p) = X^{k-2p+1}_p - \sum_{j=1}^{p}\Big(\alpha^j_{2p} X^{k-2p}_j + \alpha^j_{2p+1} X^{k-2p-1}_j + \ldots + \alpha^j_{k-1} X^1_j\Big), \tag{62}$$

$$Z^s_{n_z}(k, n-p) = \bar{Y}^{k-2p}_{p(s-1)} - \sum_{j=1}^{p}\Big(\alpha^j_{2p} \bar{Y}^{k-2p-1}_{j(s-1)} + \alpha^j_{2p+1} \bar{Y}^{k-2p-2}_{j(s-1)} + \ldots + \alpha^j_{k-1} \bar{X}^1_{j(s-1)} + \beta^j_{(2p+1)(s-1)} X^{k-2p-1}_j + \ldots + \beta^j_{(k-1)(s-1)} X^1_j\Big). \tag{63}$$

Since Z_{n_z} and T_{n_z} are two known constant matrices, it follows form (56) that, a solution for

ζ_{n_z} always exists if and only if

$$\operatorname{rank}\begin{bmatrix} Z_{n_z} \\ T_{n_z} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Z_{n_z} \end{bmatrix}. \quad (64)$$

Algorithm 1

Step 1: Check if condition (55) is satisfied or not. If so, obtain $\chi_{n_z} = Y_{n_z} X_{n_z}^{+}$.
Step 2: Substitute β_{ks}^{j} and α_{ℓ}^{j} into (58)-(59) and obtain Z_{n_z} and T_{n_z}. Check if condition (64) is satisfied or not. If so, obtain $\zeta_{n_z} = T_{n_z} Z_{n_z}^{+}$.
Step 4: From (7)-(11), obtain the state transformation (12) and (21)-(22).

3. THE DESIGN OF FAULT ESTIMATOR

Consider the following observer-based fault detection filter

$$\begin{aligned}
\dot{\hat{z}}(t) &= \mathcal{A}\hat{z}(t) + \mathcal{B}u(t) + \sum_{s=1}^{q} \mathcal{B}_s^1 u(t-\tau_s) \\
&+ \sum_{s=1}^{q} \mathcal{B}_s^2 u(t-2\tau_s) + \Gamma y(t) \\
&+ \sum_{s=1}^{q} \Gamma_s^1 y(t-\tau_s) + \sum_{s=1}^{q} \Gamma_s^2 y(t-2\tau_s) \\
&+ \sum_{s=1}^{q} \Gamma_s^3 y(t-3\tau_s) \\
&+ \sum_{s=1}^{q} \sum_{s\neq\ell=1}^{q} \Gamma_{s\ell}^{12} y(t-\tau_s-\tau_\ell) \\
&+ \sum_{s=1}^{q} \sum_{s\neq\ell=1}^{q} \Gamma_{s\ell}^{21} y(t-2\tau_s-\tau_\ell) \\
&+ L(y(t)-\hat{y}(t)),\ t \geqslant 3\tau, \quad (65)\\
\hat{y}(t) &= \mathcal{C}\hat{z}(t), \quad (66)\\
\hat{f}(t) &= V(y(t)-\hat{y}(t)), \quad (67)
\end{aligned}$$

where $\hat{z}(t) \in \mathbb{R}^{n_z}$, $\hat{y}(t) \in \mathbb{R}^{p}$, $\hat{f}(t) \in \mathbb{R}^{n_f}$ are the state, output and fault estimation vectors, respectively; L and V are observer gain matrices to be determined. Define

$$e(t) = z(t) - \hat{z}(t),\ r(t) = \hat{f}(t) - f(t).$$

We have the following estimation error system

$$\begin{aligned}
\dot{e}(t) &= (\mathcal{A} - L\mathcal{C})e(t) + \mathcal{B}_d d(t) \\
&+ \sum_{s=1}^{q} \mathcal{B}_{d_s}^1 d(t-\tau_s) + \sum_{s=1}^{q} \mathcal{B}_{d_s}^2 d(t-2\tau_s) \\
&+ \mathcal{B}_f f(t) + \sum_{s=1}^{q} \mathcal{B}_{f_s}^1 f(t-\tau_s) \\
&+ \sum_{s=1}^{q} \mathcal{B}_{f_s}^2 f(t-2\tau_s) \\
&- LD_d d(t) - LD_f f(t), \quad (68)\\
r(t) &= V\mathcal{C}e(t) + VD_d d(t) \\
&+ (VD_f - I)f(t). \quad (69)
\end{aligned}$$

Let us now denote

$$\begin{aligned}
\tilde{A} &= \mathcal{A} - L\mathcal{C},\ \tilde{C} = V\mathcal{C}, \\
\overline{B}_d &= \begin{bmatrix} \mathcal{B}_d & \mathcal{B}_{d_1}^1 & \dots & \mathcal{B}_{d_q}^1 & \mathcal{B}_{d_1}^2 & \dots & \mathcal{B}_{d_q}^2 \end{bmatrix}, \\
\overline{B}_f &= \begin{bmatrix} \mathcal{B}_f & \mathcal{B}_{f_1}^1 & \dots & \mathcal{B}_{f_q}^1 & \mathcal{B}_{f_1}^2 & \dots & \mathcal{B}_{f_q}^2 \end{bmatrix}, \\
\overline{I}_f &= \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}, \\
\overline{D}_d &= \begin{bmatrix} D_d & 0 & \dots & 0 \end{bmatrix}, \\
\overline{D}_f &= \begin{bmatrix} D_f & 0 & \dots & 0 \end{bmatrix}, \\
\tilde{B}_d &= \overline{B}_d - L\overline{D}_d, \tilde{B}_f = \overline{B}_f - L\overline{D}_f, \\
\tilde{D}_d &= V\overline{D}_d, \tilde{D}_f = V\overline{D}_f - \overline{I}_f, \\
\bar{d}(t) &= \begin{bmatrix} d(t) & d(t-\tau_1) & \dots & d(t-\tau_q) \\ d(t-2\tau_1) & \dots & d(t-2\tau_q) \end{bmatrix}^T \\
\bar{f}(t) &= \begin{bmatrix} f(t) & f(t-\tau_1) & \dots & f(t-\tau_q) \\ f(t-2\tau_1) & \dots & f(t-2\tau_q) \end{bmatrix}^T
\end{aligned}$$

Then we can rewrite system (68)-(69) as

$$\begin{aligned}
\dot{e}(t) &= \tilde{A}e(t) + \tilde{B}_d\bar{d}(t) + \tilde{B}_f\bar{f}(t), \quad (70)\\
r(t) &= \tilde{C}e(t) + \tilde{D}_d\bar{d}(t) + \tilde{D}_f\bar{f}(t). \quad (71)
\end{aligned}$$

In the following development, we will design an H_∞ fault estimatior, i.e. to find matrices L and V such that the error system (70)-(71) with $\bar{d}(t) = 0$ and $\bar{f}(t) = 0$ is asymptotically stable and, for given $\gamma > 0$, $\lambda > 0$ the following performance is satisfied

$$\begin{aligned}
||G_{r\bar{f}}||_\infty^{[-\bar{w},\bar{w}]} &< \gamma, \quad (72)\\
||G_{r\bar{d}}||_\infty &< \lambda, \quad (73)
\end{aligned}$$

where $G_{r\bar{f}}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}_f + \tilde{D}_f$ and $G_{r\bar{d}}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}_d + \tilde{D}_d$. Since $||\bar{d}(t)||_{L_2} = \sqrt{2q+1}||d(t)||_{L_2}$ and $||\bar{f}(t)||_{L_2} = \sqrt{2q+1}||f(t)||_{L_2}$. Note that, $||G_{r\bar{f}}||_\infty^{[-\bar{w},\bar{w}]} = \frac{1}{\sqrt{2q+1}}||G_{rf}||_\infty^{[-\bar{w},\bar{w}]}$ and $||G_{r\bar{d}}||_\infty = \frac{1}{\sqrt{2q+1}}||G_{rd}||_\infty$.

The following lemmas are essential for the proof of our results.

Lemma 3.1. *Consider a transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. Letting a symmetric Π be given, the following statements are equivalent:*

(a) The finite frequency inequality

$$\begin{bmatrix} G(iw) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(iw) \\ I \end{bmatrix} < 0, \ \forall |w| \leqslant \bar{w}. \quad (74)$$

(b) There exists Hermitian matrices $Q > 0$, P of appropriate dimmension such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P \\ P & \bar{w}^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0. \quad (75)$$

Lemma 3.2. *Consider a transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. Letting a symmetric Π be given, the following statements are equivalent:*

(a) The infinite frequency inequality

$$\begin{bmatrix} G(iw) \\ I \end{bmatrix}^* \prod \begin{bmatrix} G(iw) \\ I \end{bmatrix} < 0, \ \forall w \in \mathbb{R}. \quad (76)$$

(b) There exists a Hermitian matrix $P > 0$ of appropriate dimmension such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0. \quad (77)$$

Lemma 3.3. *Given a symmetric matrix Φ and two matrices Γ and Λ, there exists a decision matrix X, that satisfies*

$$\Phi + \Gamma X \Lambda + (\Gamma X \Lambda)^T < 0 \quad (78)$$

if and only if the following conditions are satisfied

$$\Gamma^{\perp}\Phi\Gamma^{\perp T} < 0, \ \ \Gamma^{T\perp}\Phi\Gamma^{T\perp T} < 0. \quad (79)$$

Theorem 3.1. *For given matrix $R = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, scarlars $\gamma > 0$, $\lambda > 0$, $\bar{w} > 0$, the error system (70)-(71) is aymptotically stable and performances (72)-(73) hold if there exist Hermitian matrices $Q > 0$, P, real matrices $W > 0$, X and V with appropriate dimmensions such that the following LMIs hold*

$$\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} < 0 \quad (80)$$

$$\begin{bmatrix} Q_3 & Q_4 \end{bmatrix} < 0 \quad (81)$$

where

$$Q_1 = \begin{bmatrix} -Q & P - W \\ * & \Omega \\ * & * \\ * & * \end{bmatrix},$$

$$\Omega = \bar{w}^2 Q + \mathcal{A}^T W + W\mathcal{A}$$
$$-\mathcal{C}^T X^T - X\mathcal{C}T$$

$$Q_2 = \begin{bmatrix} 0 & 0 \\ W\overline{B}_f - X\overline{D}_f & \mathcal{C}^T V^T \\ -\gamma^2 I & \overline{D}_f^T V^T - \overline{I}_f^T \\ * & -I \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} \mathcal{A}^T W + W\mathcal{A} - \mathcal{C}^T X^T - X\mathcal{C} \\ * \\ * \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} W\overline{B}_d - X\overline{D}_d & \mathcal{C}^T V^T \\ \overline{D}_d^T V^T & -\lambda^2 I \\ * & -I \end{bmatrix}.$$

Proof. We first prove that performance (72) is satisfied if LMI (80) holds. Applying Lemma 3.1 and letting $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, then inequality (72) becomces

$$G_{r\bar{f}}(iw)^* G_{r\bar{f}}(iw) - \gamma^2 I < 0, \ \forall |w| \leqslant \bar{w}.$$

That means performance (72) is satisfied if and only if there exists $n_z \times n_z$ Hermitian matrices $Q > 0$, P such that

$$\begin{bmatrix} \tilde{A} & \tilde{B}_f \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P \\ P & \bar{w}^2 Q \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}_f \\ I & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C} & \tilde{D}_f \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} \tilde{C} & \tilde{D}_f \\ 0 & I \end{bmatrix} < 0. \quad (82)$$

By denoting $Y = \begin{bmatrix} \tilde{A}^T & I & 0 \\ \tilde{B}^T & 0 & I \end{bmatrix}$ and

$$\Phi = \begin{bmatrix} -Q & P & 0 \\ P & \bar{w}^2 Q + \tilde{C}^T \tilde{C} & \tilde{C}^T \tilde{D}_f \\ 0 & \tilde{D}_f^T \tilde{C} & \tilde{D}_f^T \tilde{D}_f - \gamma^2 I \end{bmatrix},$$

(82) can be written as

$$Y \Phi Y^T < 0. \quad (83)$$

On the other hand, by denoting $M = \begin{bmatrix} -I & \tilde{A} & \tilde{B}_f \end{bmatrix}^T$ and $N = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T$, it can be seen that $Y = M^{\perp}$ and N is the null space of R. Hence, from Lemma 3.3 we see that (83) and inequality

$$N^T \Phi N < 0 \quad (84)$$

hold, if and only if there exists an $n_z \times n_z$ real matrix $W > 0$ such that

$$\Phi + MWR + (MWR)^T < 0. \quad (85)$$

We can express (85) in the following form

$$\begin{bmatrix} P_1 & P_2 \end{bmatrix} < 0 \quad (86)$$

where

$$P_1 = \begin{bmatrix} -Q & P - W \\ * & \bar{w}^2 Q + \tilde{A}^T W + W \tilde{A} + \tilde{C}^T \tilde{C} \\ * & * \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 \\ W \tilde{B}_f + \tilde{C}^T \tilde{D}_f \\ \tilde{D}_f^T \tilde{D}_f - \gamma^2 I \end{bmatrix}.$$

By letting $X = WL$ and using Schur[34], it follows that (86) is equivalent to (80).

Next, we will derive sufficient conditions ensuring the asymptotic stability of the error system (70) - (71) and the performance (73). By applying Lemma 3.2 and letting $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\lambda^2 I \end{bmatrix}$, inequality (73) becomces

$$G_{r\bar{d}}(iw)^* G_{r\bar{d}}(iw) - \lambda^2 I < 0, \ \forall w \in \mathbb{R}.$$

That means performance $||G_{r\bar{d}}||_\infty < \lambda$ is satisfied if and only if there exists a $n_z \times n_z$ real symmetric matrix $W > 0$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B}_d \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & W \\ W & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}_d \\ I & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C} & \tilde{D}_d \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} \tilde{C} & \tilde{D}_d \\ 0 & I \end{bmatrix} < 0, \quad (87)$$

which is equivalent to

$$\begin{bmatrix} \tilde{A}^T W + W \tilde{A} & W \tilde{B}_d & \tilde{C}^T \\ * & -\lambda^2 I & \tilde{D}_d^T \\ * & * & -I \end{bmatrix} < 0. \quad (88)$$

Let $X = WL$, it follows that (88) is equivalent to (81). If (81) holds, then $\tilde{A}^T W + W \tilde{A} < 0$, which implies that system (70)-(71) asymptotically stable. Note that, we can solve the following optimization problem in order to get optimal fault estimation:

$$\min \gamma, \quad s.t.(80), (81). \quad (89)$$

Remark 1. *The results of this paper can be extended to the case where the time delays* τ_s *($s = 1, 2, \dots, q$) are time varying, that is,* $\tau_s(t)$, $\tau_s^- \leq \tau_s(t) \leq \tau_s^+$ *for all* $t \geq 0$, *where* $\tau_s^- > 0$, $\tau_s^+ > 0$. *Indeed, we can approximate terms* $x_s(t - \tau_s(t))$ *by* $x_s(t - \tau_s^*)$, *where* $\tau_s^* = \frac{\tau_s^- + \tau_s^+}{2}$. *By this way, systems with time-varying delays can be reduced to systems with time-invariant delays of the form (4)-(6), where* $\tau_s(t)$ *are replaced by* τ_s^* *for all* $s = 1, 2, \dots, q$.

Remark 2. *Let us consider the case where the output vector of system (4)-(6) is delayed, that is,* $y(t) = Cx(t) + C_d x(t - \tau) + D_d d(t) + D_f f(t)$. *For this case, we introduce an integral output vector* $z(t) = \int_0^t y(\theta) d\theta$ *such that* $\dot{z}(t) = y(t) =$

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$Cx(t) + C_d x(t-\tau) + D_d d(t) + D_f f(t)$. By denoting $\zeta(t) = \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}$, we obtain an augmented system of the form (4)-(6), where state variable is $\zeta(t)$ and the output is not delayed. By this way, we can extend the results of this paper to systems with delayed outputs.

4. A NUMERICAL EXAMPLE

Let us consider the motivated fourth-order example in Section 1. According to Step 1 of the Algorithm 1, we obtain matrices X_5 and Y_5 from equations (38)-(54). Since rank $\begin{bmatrix} X_5 \\ Y_5 \end{bmatrix} = 34 =$ rank $\begin{bmatrix} X_5 \end{bmatrix}$, condition (55) is satisfied and we get

$$\begin{aligned}
\chi_5 &= \begin{bmatrix} \chi_5^1 & \chi_5^2 & \chi_5^3 & \chi_5^4 \end{bmatrix} \\
&= \begin{bmatrix} \chi_5^{11} & \chi_5^{12} & \chi_5^{25} & \chi_5^{26} & \chi_5^3 & \chi_5^4 \end{bmatrix}, \\
\chi_5^{11} &= \begin{bmatrix} \alpha_3^1 & \alpha_3^2 & \gamma_3^3 & \gamma_4^3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -0.8333 & -1 & -0.8333 \end{bmatrix}, \\
\chi_5^{12} &= \begin{bmatrix} \beta_{32}^1 & \beta_{32}^2 & \beta_{33}^1 & \beta_{33}^2 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -1.3333 & -3 & -3.3333 \end{bmatrix}, \\
\chi_5^{25} &= \begin{bmatrix} \beta_{41}^1 & \beta_{41}^2 & \beta_{42}^1 & \beta_{42}^2 \end{bmatrix} \\
&= \begin{bmatrix} -8 & -3 & 4 & 3 \end{bmatrix}, \\
\chi_5^{26} &= \begin{bmatrix} \beta_{51}^1 & \beta_{51}^2 & \beta_{52}^1 & \beta_{52}^2 \end{bmatrix} \\
&= \begin{bmatrix} -1.636 & 5.2082 & 0.5657 & -0.8032 \end{bmatrix}, \\
\chi_5^3 &= \begin{bmatrix} \alpha_4^1 & \alpha_4^2 \end{bmatrix} = \begin{bmatrix} -1.0561 & -2.6835 \end{bmatrix}, \\
\chi_5^4 &= \begin{bmatrix} \nu_{51}^1 & \nu_{51}^2 & \nu_{52}^1 & \nu_{52}^2 \end{bmatrix} = 0_{1,4}.
\end{aligned}$$

Next, in Step 2, by substituting β_{41}^1, β_{41}^2, β_{42}^1, β_{42}^2, β_{51}^1, β_{51}^2, β_{52}^1, β_{52}^2, α_4^1 and α_4^2 into (58)-(59) and obtain matrices Z_5 and T_5. Since rank $\begin{bmatrix} Z_5 \\ T_5 \end{bmatrix} = 2 =$ rank $\begin{bmatrix} Z_5 \end{bmatrix}$, condition (64) is satisfied. Hence, we obtain $\zeta_5 = \begin{bmatrix} \alpha_5^1 & \alpha_5^2 & \gamma_5^3 & \gamma_5^4 & \gamma_5^5 \end{bmatrix} = \begin{bmatrix} -1.2007 & -0.6244 & -1.2007 & -0.6244 & 0.0823 \end{bmatrix}$.

Then according to Step 3, we obtain

$$\begin{aligned}
z_1(t) &= x_1(t), \\
z_2(t) &= x_2(t), \\
z_3(t) &= -0.1667x_2(t) + 2x_1(t-\tau_1) \\
&+ 0.3333x_2(t-\tau_1) - x_3(t-\tau_1) \\
&- 2x_4(t-\tau_1) - x_1(t-\tau_2) - 0.6667x_2(t-\tau_2), \\
z_4(t) &= 2.0561x_1(t) + 3.6835x_2(t) + 6x_1(t-\tau_1) \\
&+ x_2(t-\tau_1) - 3x_3(t-\tau_1) - 3x_4(t-\tau_1) \\
&- x_1(t-\tau_2) - 0.6667x_2(t-\tau_2), \\
z_5(t) &= 2.8281x_1(t) + 2.2518x_2(t) \\
&+ 8.213x_1(t-\tau_1) + 1.3688x_2(t-\tau_1) \\
&- 4.1065x_3(t-\tau_1) - 3.1626x_4(t-\tau_1) \\
&- 4.1065x_1(t-\tau_2) - 2.7377x_2(t-\tau_2).
\end{aligned}$$

And then, a transformed system of the form (21)-(22) is obtained, where $\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, $\mathcal{B}_1^2 = \mathcal{B}_2^2 = \mathcal{B}_{d_1}^2 = \mathcal{B}_{d_2}^2 = \mathcal{B}_{f_1}^2 = \mathcal{B}_{f_2}^2 = 0_{5,1}$, $\Gamma_1^3 = \Gamma_2^3 = \Gamma_{12}^{21} = \Gamma_{21}^{21} = 0_{5,2}$ and

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -0.8333 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1.2007 & -0.6244 & 0.0823 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 1 \\ 2 \\ -0.3333 \\ 9.4230 \\ 7.3317 \end{bmatrix}, \mathcal{B}_1^1 = \begin{bmatrix} 0 \\ 0 \\ -8.3333 \\ -13 \\ -14.019 \end{bmatrix},$$

$$\mathcal{B}_2^1 = \begin{bmatrix} 0 \\ 0 \\ -2.3333 \\ -7 \\ -9.5818 \end{bmatrix}, \mathcal{B}_d = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.0333 \\ 0.9423 \\ 0.7332 \end{bmatrix},$$

$$\mathcal{B}_{d_1}^1 = \begin{bmatrix} 0 \\ 0 \\ -0.8333 \\ -1.3 \\ -1.4019 \end{bmatrix}, \mathcal{B}_{d_2}^1 = \begin{bmatrix} 0 \\ 0 \\ -0.2333 \\ -0.7 \\ -0.9582 \end{bmatrix},$$

$$\mathcal{B}_f = \begin{bmatrix} 1.1 \\ 1.2 \\ -0.2 \\ 6.6818 \\ 5.8131 \end{bmatrix}, \ \mathcal{B}_{f_1}^1 = \begin{bmatrix} 0 \\ 0 \\ -1.5 \\ -0.3 \\ 0.9109 \end{bmatrix},$$

$$\mathcal{B}_{f_2}^1 = \begin{bmatrix} 0 \\ 0 \\ -1.9 \\ -5.7 \\ -7.8023 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -1 & -0.8333 \\ -1.0561 & -2.6835 \\ 1.5467 & 2.7362 \\ -1.2007 & -0.6244 \\ 0.4748 & 1.3382 \end{bmatrix},$$

$$\Gamma_1^1 = \begin{bmatrix} -3 & -1.3333 \\ -8 & -3 \\ 17.6667 & 11.8333 \\ -1.636 & 5.2082 \\ 15.322 & 10.7619 \end{bmatrix},$$

$$\Gamma_2^1 = \begin{bmatrix} -3 & -3.3333 \\ 4 & 3 \\ -3.3333 & -2.1667 \\ 0.5657 & -0.8032 \\ -10.4279 & -9.5159 \end{bmatrix},$$

$$\Gamma_1^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.3333 & -2.6667 \\ 4 & -8 \\ 5.4753 & -10.9506 \end{bmatrix},$$

$$\Gamma_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3.3333 & 3.3333 \\ 10 & 10 \\ 13.6883 & 13.6883 \end{bmatrix},$$

$$\Gamma_{12}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -5.6667 & -5.6667 \\ -17 & -17 \\ -23.2701 & -23.2701 \end{bmatrix},$$

$$\Gamma_{21}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.3333 & 2.3333 \\ 7 & 7 \\ 9.5818 & 9.5818 \end{bmatrix}.$$

On the other hand, we have the following matrices

$$\overline{I}_f = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_d = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \\ -0.0333 & -0.8333 & -0.2333 & 0 & 0 \\ 0.9423 & -1.3 & -0.7 & 0 & 0 \\ 0.7332 & -1.4019 & -0.9582 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_f = \begin{bmatrix} 1.1 & 0 & 0 & 0 & 0 \\ 1.2 & 0 & 0 & 0 & 0 \\ -0.2 & -1.5 & -1.9 & 0 & 0 \\ 6.6818 & -0.3 & -5.7 & 0 & 0 \\ 5.8131 & 0.9109 & -7.8023 & 0 & 0 \end{bmatrix},$$

$$\overline{D}_d = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{D}_f = \begin{bmatrix} 0.3 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set $\lambda = 0.5$. Solving the optimization problem (89), we obtain $\gamma_{\min} = 0.7$ and

$$L = \begin{bmatrix} 15.1993 & -4.1548 \\ -5.8977 & 21.5871 \\ 53.7296 & -37.7022 \\ -47.5788 & 118.5215 \\ -35.4247 & 102.6286 \end{bmatrix},$$

$$V = \begin{bmatrix} 9.5164 & -4.7344 \end{bmatrix}.$$

5. CONCLUSION

A novel method has been proposed for computing state transformations of time-delay systems in this paper. The H_∞ fault estimation problem for time-delay systems has been re-formulated as the corresponding problem for linear time invariant systems. A numerical example has been given to demonstrate the obtained results. In the future, we will extend the results of this paper to address the problem of H_∞ fault estimation for interconnected systems with time-varying delays and unknown inputs.

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