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Một bất đẳng thức về độ chính xác của lượng tử

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TÓM TẮT

Độ chính xác của lượng tử là một vấn đề quan trọng trong lý thuyết thông tin lượng tử và lý thuyết hỗn độn lượng tử. Đại lượng này đo khoảng cách giữa các ma trận mật độ hay còn được hiểu là các trạng thái của lượng tử. Mặc dù đại lượng này không phải là một metric tuy nhiên nó có nhiều tính chất hữu dụng giúp xác định một metric trong không gian các ma trận mật độ. Trong bài báo này chúng tôi đưa ra và chứng minh một bất đẳng thức có tham số về độ chính xác của lượng tử. Hệ quả là với hai trạng thái lượng tử A và B sao cho $\frac{64}{81} \leq ||A-B||_1 \leq 16$, bất đẳng thức mà chúng tôi đưa ra là một trường hợp làm chặt hơn cho bất đẳng thức Fuchs-van de Graaf.

Từ khóa: *Thông tin lượng tử, hàm khoảng cách, trung bình nhân, độ chính xác.*

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An inequality for quantum fidelity

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ABSTRACT

Quantum fidelity is an important quantity in quantum information theory and quantum chaos theory. It is a distance measure between density matrices which are considered as quantum states. Although it is not a metric, it has many useful properties that can be used to define a metric on the space of density matrices. In this article, we prove a parameterized inequality for quantum fidelity. As a consequence, for quantum states A and B such that $\frac{64}{81} \leq ||A-B||_1 \leq 16$, our result is a refinement of the well-known Fuchs-van de Graaf 's inequality.

Keywords: *Quantum information, Function distances, Geometric mean, fidelity.*

1. INTRODUCTION

Let $\mathcal{H}$ be the $n-$dimensional Hilbert space $\mathbb{C}^n$. The inner product between two vectors x and y is written as $\left\langle x, y\right\rangle$ or as x^*y. We denote by $\mathcal{L}(\mathcal{H})$ the space of all linear operators on $\mathcal{H}$, and by $\mathbb{M}_n(\mathbb{C})$ (or simply $\mathbb{M}_n$) the algebra of $n \times n$ matrices over $\mathbb{C}$. Denote by I the identity matrix of $\mathbb{M}_n$.

Every element A of $\mathcal{L}(\mathcal{H})$ can be identified with its matrix with respect to the standard basis $\{e_j\}$ of $\mathbb{C}^n$. We use the symbol A for this matrix as well. We say A is *positive semidefinite*[1] if

$$\left\langle x, Ax\right\rangle \geq 0, \text{ for all } x \in \mathcal{H},$$

and *positive definite* if, in addition,

$$\left\langle x, Ax\right\rangle > 0, \text{ for all } x \neq 0.$$

It is clear that a positive semidefinite matrix is a positive definite matrix if only if it is invertible. For convenience, we use the term *positive* matrix for a positive semidefinite, or a positive definite, matrix. Sometimes, if we want to emphasize that the matrix is positive definite, we say that it is *strictly positive*. We use the notation $A \geq 0$ to mean that A is positive, and $A > 0$ to mean it is strictly positive. We denote by A^T the *transpose* of matrix A and the *adjoint matrix* A^* as the complex conjugate of the transpose A^T. If $A = A^*$ then we call A is Hermitian, we also denote $\mathbb{H}_n$ as the real subspace of $\mathbb{M}_n$ consisting of Hermitian matrices. For each $A \in \mathbb{M}_n$, we have $A^*A \geq 0$. Therefore, we can define the matrix $|A| = (A^*A)^{1/2}$ which is called the *absolute value* of A. This matrix can be also defined using functional calculus. We have a result[1], A is positive if only if $A = B^2$ for some positive matrix B. Such B is unique. We write $B = A^{1/2}$ or $B = \sqrt{A}$ and call it the (positive) square root of A. Evidently, A is strictly positive if only if B is strictly positive.

The eigenvalues $\sigma_i(A)$ of $|A|$ are called *singular values* of A. If $A \in \mathbb{M}_n$, then the usual notation is

$$\sigma(A) = \Big(\sigma_1(A), ..., \sigma_n(A)\Big),$$

where

$$\sigma_1(A) \geq \sigma_2(A) \geq ... \geq \sigma_n(A).$$

Let denote by $\mathcal{D}_n$ the cone of positive definite matrices in $\mathbb{M}_n$. The space of density matrices is

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defined as

$$\mathcal{D}_n^1 = \{A \in \mathcal{D}_n : TrA = 1\}.$$

Definition 1.1. Let $A, B \in \mathcal{D}_n$ be positive semidefinite matrices. The *fidelity*[6,7] between two elements A and B is defined as

$$F(A, B) = ||\sqrt{A}\sqrt{B}||_1, \tag{1.1}$$

where $||.||_1$ is Schatten 1-norm (trace norm),

$$||A||_1 = Tr|A| = Tr\sqrt{AA^*}.$$

Alternatively, the trace norm of an operator (or a matrix) A can be expressed as the sum of its singular values, $||A||_1 = \sum_{i=1}^{n} \sigma_i(A)$.

In quantum theory, quantum fidelity is defined for density matrices, and it can be generalized to the set of positive semidefinite matrices. By (1.1), we have

$$F(A, B) = Tr\Big(A^{1/2}BA^{1/2}\Big)^{1/2}.$$

Many researchers have paid attentions on different distance functions on $\mathcal{D}_n$ in the past few years. One of the important distance functions is the Bures distance[12]

$$d_b(A, B) = \Big(Tr(A + B) - 2Tr((A^{1/2}BA^{1/2})^{1/2})\Big)^{1/2},$$

or

$$d_b(A, B) = (Tr(A + B) - 2F(A, B))^{1/2}.$$

When $A, B \in \mathcal{D}_n^1$, quantum fidelity have several important properties[3,5,7], which can be proved in the sense of unital C^*-algebras

(1) Bounds: $0 \leq F(A, B) \leq 1$. Furthermore $F(A, B) = 1$ iff $A = B$, while $F(A, B) = 0$ iff $supp(A) \perp supp(B)$.

(2) Symmetry: $F(A, B) = F(B, A)$.

(3) Unitary Invariance: $F(A, B) = F(UAU^*, UBU^*)$, for any unitary matrix U.

(4) Concavity: $F(A, tB + (1 - t)C) \geq tF(A, B) + (1-t)F(A, C)$, for $t \in [0, 1]$ and $A, B, C \in \mathcal{D}_n^1$.

(5) Multiplicativity: $F(A \otimes B, C \otimes D) = F(A, C) \cdot F(B, D)$, for A, B, C, and $D \in \mathcal{D}_n^1$.

(6) Joint concavity: $F(tA + (1 - t)B, tc + (1 - t)D) \geq tF(A, C) + (1 - t)F(B, D)$, for $t \in [0, 1]$ and A, B, C, and $D \in \mathcal{D}_n^1$.

In[12], the authors considered the function $f(X) = Tr(AX + BX^{-1})$ on $\mathcal{D}_n$. Using the Frechet derivative of the function $f(X)$ they showed that the geometric mean $X_0 = A^{-1}\sharp B = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$ is the only critical point of $f(X)$. Hence, $f(X)$ attains minimum at X_0:

$$\begin{aligned}\min_{X>0} f(X) = f(X_0) &= Tr\Big((A(A^{-1}\sharp B) + B(A\sharp B^{-1})\Big)\\ &= 2Tr(A^{1/2}BA^{1/2})^{1/2} = 2F(A, B).\end{aligned}$$

They also use the block matrix techniques to show the following: For positive definite matrices A and B,

(1) $F(A, B) = \min_{X>0} \sqrt{Tr(AX)Tr(BX^{-1})}$.

(2) $F(A, B) = \max_{X>0}\{|TrX| : A \geq XB^{-1}X^*\}$.

One of the most important inequalities of quantum fidelity is Fuchs de Graaf's inequality[6,11].

Theorem 1.1. *(Fuchs-van de Graaf's inequality) For two density matrices A and* $B \in \mathcal{D}_n^1$*, we have*

$$1 - \frac{1}{2}||A - B||_1 \leq F(A, B) \leq \sqrt{1 - \frac{1}{4}||A - B||_1^2}. \tag{1.2}$$

Equivalently,

$$2 - 2F(A, B) \leq ||A - B||_1 \leq 2\sqrt{1 - F(A, B)^2}. \tag{1.3}$$

The above inequality provides an upper bound and lower bound of quantum fidelity. It is also a tight relationship between different distances between A and B.

The proof of the right inequality of (1.3) is based on Uhlmann's theorem[4] while the proof of the left inequality of (1.3) based on the following result[6].

Lemma 1.1. *Let $A, B \in \mathcal{D}_n$ be positive semidefinite matrices. It holds that*

$$||A - B||_1 \geq ||\sqrt{A} - \sqrt{B}||_2^2,$$

where $||.||_2$ is the Shcatten 2-norm,

$$||A||_2 = \Big(\sum_{i=1}^{n} \sigma_i^2(A)\Big)^{1/2}.$$

It is worth mentioning that it is difficult to improve the Fuchs-van de Graaf inequality. In[11], the authors established a lower bound for $F(A, B)$ as follows.

Let $\lambda_0 = \lambda_{\max}(B^{-1/2}AB^{-1/2})$, where $\lambda_{\max}(X)$ is used to denote the maximum eigenvalue of the matrix X. Then

$$F(A, B) \geq 1 - \frac{1}{2}\frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0} + 1}||A - B||_1.$$

2. MAIN RESULTS

In this section we establish an estimate for the trace-norm of the difference for two density matrices A and B and the fidelity of A and the convex combination $tA + (1 - t)B, t \in [0, 1]$ of A and B.

Before presenting the main result, let's recall the following well-known inequality[8,9,10]

$$d_b(A, B) \leq d_1^{1/2}(A, B),$$

where $A, B \in \mathcal{D}_n$. This inequality was first proved in C^*-algebra setting by Araki in[10]. However, we can prove this inequality by another way as follows.

By Lemma (1.1), we have

$$\begin{aligned} ||A - B||_1 \geq ||\sqrt{A} - \sqrt{B}||_2^2 & \\ &= Tr(\sqrt{A} - \sqrt{B})^2 \\ &= Tr(A + B - 2\sqrt{A}\sqrt{B}) \\ &\geq TrA + TrB - 2F(A, B) \\ &= d_b^2(A, B), \end{aligned}$$

where the last inequality follows from the fact that

$$F(A, B) = Tr(A^{1/2}BA^{1/2})^{1/2} \geq Tr(A^{1/2}B^{1/2}),$$

which is the consequence of the famous Araki-Lieb-Thirring inequality[15].

Theorem 2.1. *Let $A, B \in \mathcal{D}_n^1$ and $t \in [0, 1]$. Then*

$$\sqrt{F(A, tAa + (1 - t)B} \geq 1 - \frac{1}{4}(1 - \sqrt{t})||A - B||_1^{1/2}.$$

Proof. Firstly, let us recall the Jensen inequality for trace. Let f be a continuous and concave function on an interval J and m be a natural number. Then for self-adjoint matrices $X_1, \cdots, X_m$ with spectra in J,

$$Tr\Big(f\Big(\sum_{i=1}^{m} A_i^* X_i A_i\Big)\Big) \geq Tr\Big(\sum_{i=1}^{m} A_i^* f(X_i) A_i\Big),$$

where $A_1, \cdots, A_m$ satisfy $\sum_{i=1}^{m} A_i^* A_i = I$.

We have

$$\begin{aligned} &F(A, tA + (1 - t)B) \\ &= Tr[A^{1/2}(tA + (1 - t)B)A^{1/2}]^{1/2} \\ &= Tr[tA^2 + (1 - t)A^{1/2}BA^{1/2}]^{1/2} \\ &\geq Tr[tA + (1 - t)(A^{1/2}BA^{1/2})^{1/2}] \\ &= t + (1 - t)F(A, B), \end{aligned}$$

where the inequality is valid according to the concavity of the function $x \mapsto x^{1/2}$ and Jensen's trace inequality.

From $d_b(A, B) \leq d_1^{1/2}(A, B) = ||A - B||_1^{1/2}$, we have

$$\begin{aligned} & 1 - \frac{1}{4}(1 - \sqrt{t})||A - B||_1^{1/2} \\ \leq\ & 1 - \frac{1}{4}(1 - \sqrt{t})d_b(A, B) \end{aligned}$$

$$= 1 - \frac{1}{4}(1-\sqrt{t})\sqrt{(2 - 2Tr(A^{1/2}BA^{1/2})^{1/2})}$$
$$= 1 - (1-\sqrt{t})\sqrt{(1 - F(A,B))}.$$

Thus, it is necessary to prove

$$\sqrt{t + (1-t)F(A,B)} \geq 1 - (1-\sqrt{t})\sqrt{(1-F(A,B))}.$$

Since $0 \leq t \leq 1$, and $0 \leq F(A,B) \leq 1$, squaring both sides of this inequality, we have

$$\begin{aligned} & t + F - tF \geq 1 - 2(1-\sqrt{t})\sqrt{1-F} \\ & \qquad + (1-\sqrt{t})^2(1-F) \\ \Leftrightarrow\ & (t-1)(1-F) + 2(1-\sqrt{t})\sqrt{1-F} \\ & \qquad - (1-\sqrt{t})^2(1-F) \geq 0 \\ \Leftrightarrow\ & (1-F)[(t-1) - (1-\sqrt{t})^2] \\ & \qquad + 2(1-\sqrt{t})\sqrt{1-F} \geq 0 \\ \Leftrightarrow\ & 2(1-\sqrt{t})[\sqrt{1-F} - (1-F)] \geq 0. \end{aligned}$$

In the above transformations, F is used to denote for $F(A,B)$. The last inequality is evident because $0 \leq \sqrt{t} \leq 1$, and $0 \leq 1 - F(A,B) \leq 1$. □

Remark 2.1. For $t = 0$, with $||A-B||_1 \leq 16$ and from the theorem we have

$$F(A,B) \geq (1 - \frac{1}{4}||A-B||_1^{1/2})^2.$$

Let's compare the value $(1 - \frac{1}{4}||A-B||_1^{1/2})^2$ and the value $1 - \frac{1}{2}||A-B||_1$ on the left-hand-side part in the Fuchs-van de Graaf inequality. By a simple computation, if $||A-B||_1 \geq 64/81$ then we have

$$F(A,B) \geq (1 - \frac{1}{4}||A-B||_1^{1/2})^2 \geq 1 - \frac{1}{2}||A-B||_1.$$

Indeed, from the last inequality we have

$$\begin{aligned} & (1 - \frac{1}{4}||A-B||_1^{1/2})^2 \geq 1 - \frac{1}{2}||A-B||_1 \\ \Leftrightarrow\ & 1 - \frac{1}{2}||A-B||_1^{1/2} + \frac{1}{16}||A-B||_1 \\ & \qquad \geq 1 - \frac{1}{2}||A-B||_1 \end{aligned}$$

$$\Leftrightarrow\ \frac{||A-B||_1^{1/2}}{2}\Big(\frac{9}{8}||A-B||_1^{1/2} - 1\Big) \geq 0,$$

which is equivalent to that

$$||A-B||_1 \geq 64/81.$$

Therefore, the main result is a refinement of the Fuchs-van de Graaf inequality for a big set of quantum states A and B.

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