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Chính quy mêtric và Ứng dụng - Một tổng quan

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TÓM TẮT

Lý thuyết chính quy mêtric và các tính chất liên quan là một công cụ hữu hiệu trong các lĩnh vực Giải tích biến phân và Tối ưu. Trong vài chục năm gần đây, rất nhiều công trình nghiên cứu của nhiều nhà toán học đã đóng góp vào lý thuyết này, trên cả hai phương diện lý thuyết và ứng dụng. Mục đích của bài báo tổng quan này nhằm trình bày một số phát triển nổi bật gần đây, trong đó nhấn mạnh chính vào những đóng góp của nhóm chúng tôi trong lý thuyết này. Đặc biệt, là một số ứng dụng của tính chính quy mêtric trong nghiên cứu sự hội tụ của phương pháp Newton giải phương trình suy rộng. Ngoài ra, chúng tôi áp dụng tính chính quy mêtric để thu được một phiên bản tổng quát của nguyên lý lồi cho ánh xạ đa trị.

Từ khóa: *Tính chính quy mêtric, đối đạo hàm, tính ổn định nhiễu, tính lặp Newton, lồi suy rộng.*

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A survey on the metric regularity and applications

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ABSTRACT

The theory of metric regularity and related topics plays an important role in variational analysis and have many applications in optimization. In recent decades, it has attracted the study of many researchers in both the theoretical aspects and applications. In the survey paper, we present some recent developments, with emphasis on the results established by our research group in the recent decade in this theory and its applications in the study of the Newton-type methods for solving generalized equations. In addition, based on the metric regularity, we establish a new result on a generalization of the convex principle for paraconvex multimaps.

Keywords: *Metric regularity, coderivative, perturbation stability, Newton iteration, paraconvex.*

1. INTRODUCTION

The study of many mathematical problems originated from practical applications, such as optimization and complementarity problems, variational inequalities, as well as models in equilibrium problems, control theory and design problems, leads to consider inclusions of the type:

$$\text{Find} \quad x \in X \text{ s.t. } y \in F(x) \quad \text{for given } y \in Y, \tag{1.1}$$

here, X, Y are metric spaces and $F : X \rightrightarrows Y$ is a set-valued mapping (also called multimap or multifunction) describing the model under consideration. These inclusions are usually called generalized equations, due to the pioneering work of Robinson.[1,2] The existence of solutions as well as the behavior and stability of the solutions of (1.1) are important principal topics and have attracted many authors working in the fields of variational analysis and optimization. The readers are referred to the monographs,[3−12] to some recent contributions[13−15] and the references therein.

One of the key ingredients to deal with the existence as well as the stability of the solutions of (1.1) is the metric regularity and related properties. Historically, this property goes back to the celebrated Banach open mapping theorem and latter to the Lyusternik theorem (see[5,9,10,16−27] and the references given therein). Recently, many important applications of this property have been found and investigated, especially in the study of stability of variational systems as well as convergence analysis of some algorithms, e.g., as the Newton type methods (see[28−31]). More recently, some generalized metric regularity concepts have been introduced and studied, due to the point of view of applications. For example, in the papers[32,33], some variants of relative metric regularity have used in convergence analysis of some optimization algorithms. An important notion of extended met-

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ric regularity is the directional metric regularity (see[34−39]). One can be referred to[6,34,35] and references given therein for applications of directional metric regularity in sensitivity analysis and optimization.

In this survey paper, we presents some contributions in the recent decade of our research group to the theory of metric regularity and some applications. This survey is far to be exhaustive on this subject. We recommend the recent book and the survey papers by Ioffe[9,40,41] for the excellent accounts on the recent developments of the theory of metric regularity and divers applications in variational analysis and optimization. Outline of the paper is as follows. In Section 2, we recall the notion of the metric regularity and related notions; some classical results on the metric regularity; some variational characterizations and the perturbation stability of this property. Section 3 is devoted to the relative metric regularity, in which we present some very recent results concerning the directional metric regularity relative to a cone. In the final section, some applications to the Newton methods for generalized equations and to the convex principle for multimaps are reported.

2. METRIC REGULARITY OF MULTIMAPS

We recall firstly some basic notations and notions from set-valued analysis. Throughout the paper, for a metric space X endowed with metric d, denote by $B(x,\rho)$ and $B[x,\rho]$ the open and the closed ball centered at $x \in X$ with radius $\rho > 0$, respectively. The distance function to a subset $C \subseteq X$ is denoted by $d(x,C) := \inf_{u\in C} d(x,u)$. By a set-valued mapping (or a multimap) $F : X \rightrightarrows Y$, it means a correspondance from X to $\mathcal{P}(Y)$, the set of the (possibly empty) subsets of Y. Given a multimap F, the graph of F, the domain of F are the sets gph $F := \{(x,y) \in X \times Y \ : \ y \in F(x)\}$ and Dom $F := \{x \in X \ : \ F(x) \neq \emptyset\}$, respectively. Inverse of F is denoted by $F^{-1} : Y \rightrightarrows X$, and defined by

$$x \in F^{-1}(y) \iff y \in F(x).$$

A multimap between normed spaces is called a convex (respectively closed) multimap if its graph is convex (respectively closed) graph in the product space.

2.1. Metric regularity: Classical results

Consider an operator equation defined by

$$f(x) = y, \tag{2.1}$$

where $f : X \to Y$ is a mapping acting between metric spaces X, Y .
In practice, one finds out an approximate solution rather than an exact one. The error of some approximate solution x is the quantity

$$d(x, f^{-1}(y)) = \inf\{d(x,u) : \ f(u) = y\}.$$

Naturally, the distance $d(y, F(x))$ is used to judge approximate solutions. One seeks so an error estimate of the form

$$d(x, f^{-1}(y)) \leq \kappa d(y, f(x)) \tag{2.2}$$

for all (x,y) in a suitable domain. If (2.2) is satisfied for (x,y) near a given $(\bar{x},\bar{y})$ with $\bar{y} = F(\bar{x})$, then F is called metrically regular at $\bar{x}$.

The metric inequality can be extended naturally to multimaps. For example, consider a system of inequalities:

$$g_i(x) \leq y_i, i = 1, ..., m. \tag{2.3}$$

This system of inequalities can be investigated via the generalized equation of the form: $y \in G(x)$, where,

$$G(x) := (g_i(x))_{i=1,\dots,m} + \mathbb{R}^m_+; \ \ y = (y_i)_{i=1,\dots,m}, \tag{2.4}$$

then $G : X \rightrightarrows \mathbb{R}^m$ is a multimap.

Let us now recall the notion of metric regularity.

Definition 1. *A multimap (set-valued mapping) $F : X \rightrightarrows Y$ is said to be metrically regular at $(\bar{x}, \bar{y})$ $(\bar{y} \in F(\bar{x}))$ if there are $\kappa, \delta > 0$ such that*

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x))$$

for all $(x, y) \in B((\bar{x}, \bar{y}), \delta)$. The infimum of such κ denoted by $\operatorname{Reg} F(\bar{x}, \bar{y})$, and is called the regular modulus of F at $(\bar{x}, \bar{y})$.

In the linear case, the metric regularity is closely related to the Banach open mapping principle for bounded linear operators between Banach spaces, restated as follows.

Theorem 2. *(Banach open mapping principle) Let X, Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$, the space of bounded linear operators from X to Y. If the operator A is surjective, that is, $\operatorname{Im} A = Y$, then A is an open mapping,that is, there is $r > 0$ such that $rB_Y \subseteq A(B_X)$. The upper bound of such r is called the Banach constant of A :*

$$C(A) = \inf\{\|A^* y^*\| : \ \|y^*\| = 1\}.$$

Moreover, the following inequality holds

$$d(x, A^{-1}(y)) \leq C(A)^{-1} \|Ax - y\|$$

for all $(x, y) \in X \times Y$.

The Banach open mapping principle tells us that a bounded linear operator between Banach spaces is (locally or equivalently globally) metrically regular if and only if it is surjective. This principle was extended to continuously differentiable mappings by Lyusternik as follows.

Theorem 3. *(Lyusternik) Let X, Y be Banach spaces; and let $f : X \to Y$ be a continuously differentiable mapping at $\bar{x} \in X$ with $f(\bar{x}) := \bar{y}$. Then f is metrically regular at $\bar{x}$ if and only if $Df(\bar{x})$ is onto: If $\operatorname{Im} Df(\bar{x}) = Y$.*

The metric regularity is strongly connected to the Robinson and Mangasarian-Fromovitz constraint qualifications in Mathematical Programming. Consider $F := f - C$, where $f : X \to Y$ is a mapping of $\mathcal{C}^1$ class and $C \subseteq Y$ is a nonempty closed convex subset. Given $(\bar{x}, 0) \in \operatorname{gph} F$, then F is metrically regular at $(\bar{x}, 0)$ *if* the Robinson constraint qualification (RCQ) is satisfied:

$$0 \in \operatorname{int}[f(\bar{x}) + Df(\bar{x})X - C].$$

In particular, for systems of equality and inequality (2.4), one has the equivalence:

$$\text{(RCQ)} \Leftrightarrow (MFCQ) \ \text{(Mangasarian-Fromovitz constraint qualification).}$$

In the case of convex multimaps, a necessary and sufficient condition was given by Robinson-Ursescu, stated in the following theorem.

Theorem 4. *(Robinson-Ursescu) Given Banach spaces X, Y, a closed and convex multimap $F : X \rightrightarrows Y$ (F has a closed and convex graph), F is metrically regular at $(x_0, y_0) \in \operatorname{gph} F$ if and only if $y_0 \in \operatorname{int}(\operatorname{Im} F)$.*

Next we recall the two notions of the openness at a linear rate and of Lipschitz-like (or Aubin) property of multimaps.

Definition 5. *Let $F : X \rightrightarrows Y$ be a multimap acting between metric spaces X, Y and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.*

(a) F is said to be open at a linear rate around $(\bar{x}, \bar{y})$ if there exist $s, \varepsilon > 0$ such that

$$B(y, ts) \subseteq F(B(x, t)),$$

$\forall (x, y) \in B((\bar{x}, \bar{y}), \varepsilon) \cap \operatorname{gph} F$. The supremum of such r denoted by $\operatorname{Sur} F(\bar{x}, \bar{y})$ and is called the rate of openness (or surjection) of F at $(\bar{x}, \bar{y})$.

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(b) *We say that F is Lipschitz-like (or Aubin) at $(\bar{x},\bar{y})$ if there exist $L,\varepsilon>0$ such that*

$$d(y,F(x))\le Ld(x,u),$$

$\forall x\in B(\bar{x},\varepsilon)$, $(u,y)\in B((\bar{x},\bar{y}),\varepsilon)\cap \operatorname{gph} F$. *The supremum of such L is the Lipschitz rate (or modulus) of F at $(\bar{x},\bar{y})$, and is denoted by* $\operatorname{Lip} F(\bar{x},\bar{y})$.

The equivalence of these notions to the metric regularity was given independently by several authors (see, e.g., Borwein-Zuang[42], Kruger[43], Penot[44], Ioffe[45]).

Proposition 6. *For a multimap $F: X\rightrightarrows Y$ between metric spaces X,Y, and for $(\bar{x},\bar{y})\in$* $\operatorname{gph} F$, *the following three assertions are equivalent.*

(a) *F is metrically regular around $(\bar{x},\bar{y})$;*

(b) *F is open at linear rate around $(\bar{x},\bar{y})$;*

(c) *F^{-1} is Lipschitz-like around $(\bar{x},\bar{y})$.*

Moreover, one has the equality between modulus

$$\operatorname{Reg} F(\bar{x},\bar{y})=\operatorname{Lip} F^{-1}(\bar{y},\bar{x})=\frac{1}{\operatorname{Sur} F(\bar{x},\bar{y})}.$$

2.2 Characterization of the metric regularity via strong slopes

Let $f: X\to\mathbb{R}\cup\{+\infty\}$ be an extended real-valued function defined on a metric space. We make use of the notations: $\operatorname{dom} f:=\{x\in X: f(x)<+\infty\}$, the domain of f; $\limsup\limits_{y\to x,y\neq x}$, $\liminf\limits_{y\to x,y\neq x}$ mean that the limits superior/inferior are taken as $y\to x$ and $y\neq x$; while $\limsup\limits_{y\to x}$, $\liminf\limits_{y\to x}$ allow the case $y=x$. The symbol $[f(x)]_+$ stands for $\max(f(x),0)$. Recall that the local slope of a lower semicontinuous function f at $x\in\operatorname{dom} f$ is denoted by $|\nabla f|(x)$ and defined by $|\nabla f|(x)=0$ if x is a local minimum of f; otherwise

$$|\nabla f|(x)=\limsup_{y\to x,y\neq x}\frac{f(x)-f(y)}{d(x,y)}.$$

For $x\notin\operatorname{dom} f$, we set $|\nabla f|(x)=+\infty$. The non-local slope of f is defined by

$$|\Gamma f|(x):=\sup_{u\neq x}\frac{[f(x)-f(u)]_+}{d(x,u)}.$$

For $x\notin\operatorname{dom} f$, we set $|\Gamma f|(x)=+\infty$.

It is well-known in the literature that if X is a normed space and f is Fréchet differentiable at x then $|\nabla f|(x)=\|f'(x)\|$. Obviously, one always has the relation $|\nabla f|(x)\le|\Gamma f|(x)$ for all $x\in X$.

Recall the lower semicontinuous envelope $(x,y)\mapsto\varphi^F(x,y):=\varphi^F_y(x)$ of the function $(x,y)\mapsto d(y,F(x))$ defined by, for $(x,y)\in X\times Y$,

$$\varphi^F_y(x):=\liminf_{(u,v)\to(x,y)} d(v,F(u))=\liminf_{u\to x} d(y,F(u)).$$

For the simplification of the notation, when one works only with a given mapping F, one denotes $\varphi_y:=\varphi^F_y$. The following theorem established by Ngai-Tron-Théra[46] gives an estimate of the regularity rate $\operatorname{Reg} F(\bar{x},\bar{y})$ via strong slopes of the functions φ_y.

Theorem 7. *(Ngai-Tron-Théra[46]) Let X be a complete metric space and let Y be a metric space. For a multimap $F: X\rightrightarrows Y$, and for given $(\bar{x},\bar{y})\in$* $\operatorname{gph} F$, *one has*

$$\operatorname{Reg} F(\bar{x},\bar{y})^{-1}=\operatorname{Sur} F(\bar{x},\bar{y})\ge\liminf_{(x,y)\to(\bar{x},\bar{y}),y\notin F(x)}|\nabla\varphi_y|(x).$$

When Y is a normed space (or more general, a smooth manifold, or a length metric space), the equality holds

$$\operatorname{Reg} F(\bar{x},\bar{y})^{-1}=\liminf_{(x,y)\to(\bar{x},\bar{y}),y\notin F(x)}|\nabla\varphi_y|(x).$$

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2.3 Coderivative characterizations

Firstly we recall the main definitions and results from Variational Analysis necessary and used in the sequel. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function defined on a Banach space X, the *Fréchet (regular) subdifferential* of f at $\bar{x} \in \mathrm{dom} f$ is defined by

$$\partial f(\bar{x}) = \left\{ x^\star \in X^\star : \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^\star, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

When f is a convex function, the Fréchet subdifferential coincides with the subdifferential in the sense of convex analysis. When $\bar{x} \notin$ dom f, one sets $\partial f(\bar{x}) = \emptyset$. We shall also work on an interesting subclass of Banach spaces, called Asplund spaces being spaces such that on which every convex continuous function is generically Fréchet differentiable. Notice that any space with a Fréchet smooth renorming (and therefore any reflexive space) is Asplund. It is well-known that a Banach space is Asplund if and only if each of its separable subspaces has a separable dual.

In the Asplund setting, the Fréchet subdifferential enjoys a fuzzy sum rule which was firstly proved by Fabian[48] (see also[10,Thm.2.33]).

Given a nonempty closed set $C \subseteq X$, the *indicator function* associated to C is the function ι_C defined by $\iota_C(x) = 0$, when $x \in C$ and $\iota_C(x) = \infty$ otherwise. The *Fréchet normal cone* to C at $\bar{x}$ is the set $N(C, \bar{x}) := \partial \iota_C(\bar{x})$ if $\bar{x} \in C$, and $N(C, \bar{x}) := \partial \iota_C(\bar{x}) = \emptyset$ if $\bar{x} \notin C$.

The *limiting subdifferential* (or also called the Mordukhovich subdifferential) is defined by

$$\partial_{\mathcal{L}} f(\bar{x}) = \{x^\star \in X^\star : \exists x_k \in \bar{x},\ f(x_k) \to f(\bar{x}), \text{ and } \exists x_k^\star \in \partial f(x_k),\ x_k^\star \stackrel{\star}{\to} x^\star\}.$$

The *limiting normal cone* $N_{\mathcal{L}}(C, \bar{x})$ to a closed set C is defined through the indicator function of the set:

$$N_{\mathcal{L}}(C, \bar{x}) := \partial_{\mathcal{L}} \delta_C(\bar{x}).$$

Given a normal cone mapping $\mathbb{N}$, it is associated with a set-valued mapping $F : X \rightrightarrows Y$ a coderivative $D^\star_{\mathbb{N}} : Y^\star \rightrightarrows X^\star$ by the formula

$$D^\star_{\mathbb{N}} F(x, y)(y^\star) := \{x^\star \in X^\star \mid (x^\star, -y^\star) \in \mathbb{N}(\text{gph } F, (x, y))\}. \tag{2.5}$$

For further the properties and calculus rules for the Fréchet and liniting subdifferentials as well as coderivatives, see[10,49–51] and the references given therein. In what follows, the Fréchet coderivative of F will be denoted by $D^\star_{\mathcal{F}} F$, or simply by $D^\star F$, while the limiting coderivative is noted by $D^\star_{\mathcal{L}} F$.

The following theorem gave an estimate for the slope of φ_y via the coderivative of the multimap in question.

Theorem 8. *(Ngai-Tron-Théra[46]) Let $F : X \rightrightarrows Y$ be a closed multimap acting between Asplund spaces X and Y. Then for any $(x, y) \in X \times Y$ with $y \notin F(x)$, one has the following estimate*

$$|\nabla \varphi_y|(x) \geq \tau(x, y), \tag{2.6}$$

where $\tau(x, y)$ is defined by

$$\tau(x,y) := \liminf_{\xi \downarrow 0} \left\{ \|x^\star\| : \begin{array}{l} x^\star \in \widehat{D}^\star F(u,v)(y^\star),\ \|y^\star\| = 1, \\ (u,v) \in \text{gph } F,\ u \in B(x,\xi), \\ \|y - v\| \leq \varphi_y(x) + \xi, \\ |\langle y^\star, v - y\rangle - \varphi_y(x)| \leq \xi \end{array} \right\}.$$

This theorem yields immediately the following corollary.

Corollary 9. *With the assumptions as in the preceding theorem, for any $(\bar{x}, \bar{y}) \in$ gph F, one has*

$$\liminf_{(x,y) \to (\bar{x},\bar{y}), y \notin F(x)} |\nabla \varphi_y|(x) = \lim_{\varepsilon \to 0} \inf\{\|x^*\| : x^* \in D^* F(u,v)(y^*), (u,v) \in B((\bar{x},\bar{y}), \varepsilon),\ \|y^*\| = 1\}.$$

In view of this corollary, Theorem 7 implies immediately the following characterization of the metric regularity through the coderivatives.

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Theorem 10. *(Ioffe[52]) With the assumptions as in the preceding corollary, one has*

$$\begin{aligned}&\mathrm{Sur}\, F(\bar{x},\bar{y}) = \\ &\lim_{\varepsilon\to 0}\inf\{\|x^*\|:\ x^*\in D^*F(u,v)(y^*),\\ &\qquad (u,v)\in B((\bar{x},\bar{y}),\varepsilon),\ \|y^*\|=1\}.\end{aligned}$$

In the case when X, Y are finite dimensional, one obtains the following nice point-based characterization due to Mordukhovich (e.g.,[10]).

Theorem 11. *(Mordukhovich[25]) Suppose that X, Y are finite dimensional spaces, and $F: X \rightrightarrows Y$ is a closed multimap. For given $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$, F is metrically regular at $(\bar{x},\bar{y})$ if and only if*

$$\mathrm{Ker}\, D^\star_{\mathcal{L}}F(\bar{x},\bar{y}) = \{0\},$$

where

$$\mathrm{Ker}\, D^\star_{\mathcal{L}}F(\bar{x},\bar{y}) = \{y^*\in Y^*:\ \ 0\in D^\star_{\mathcal{L}}F(\bar{x},\bar{y})(y^*)\}.$$

2.4 Stability of the metric regularity and regularity radius

For a mapping $f: X\to Y$, let us denote

$$\mathrm{Lip}\, f = \sup_{x_1,x_2\in X, x_1\neq x_2}\frac{d(f(x_1),g(x_2))}{d(x_1,x_2)};$$

$$\mathrm{Lip}\, f(\bar{x}) = \limsup_{x_1,\, x_2\to\bar{x}, x_1\neq x_2}\frac{d(f(x_1),f(x_2))}{d(x_1,x_2)},$$

called the Lipschitz modulus of f on X and near $\bar{x}$, respectively.
The regularity radius of a multimap F at $(\bar{x},\bar{y})\in \mathrm{gph}\, F$ is defined by

$$\begin{aligned}\mathrm{Rad}\, F(\bar{x},\bar{y}) = \inf_{f:X\to Y}\{&\mathrm{Lip}\, f(\bar{x}):\ F+f \text{ fails to be}\\ &\text{metrically regular at } (\bar{x},\bar{y}+f(\bar{x}))\}.\end{aligned}$$

The following relation between the regularity modulus and the regularity radius was established in [53,54,40]:

$$\mathrm{Rad}\, F(\bar{x},\bar{y}) \geq \frac{1}{\mathrm{Reg}\, F(\bar{x},\bar{y})}. \tag{2.7}$$

Precisely,

Theorem 12. *(Ioffe[54], Dontchev-Lewis-Rockafellar[53]) Let $F: X\rightrightarrows Y$ be a closed multimap from a completed metric space X to a normed space Y. Assume that F is metrically regular at $(\bar{x},\bar{y})\in \mathrm{gph}\, F$ with $\mathrm{reg}\, F(\bar{x},\bar{y}) := \tau$. Then for any locally Lipschitz mapping $f: X\to Y$ at $\bar{x}$ with Lipschitz constant $L\in(0,\tau^{-1})$, one has*

$$\mathrm{Reg}\,(F+f)(\bar{x},\bar{y}+f(\bar{x})) \leq 1/(\tau^{-1}-L).$$

Note that, due to Dontchev, Lewis, and Rockafellar[53] and also to Ioffe[54], that the equality holds if one of the following conditions is satisfied:

- X and Y are finite dimensional spaces;
- $F: X\to Y$ is a single-valued mapping.

The following theorem due to Ngai[55] shows the validity of the equality in (2.7) holds for the case of multimaps under suitable assumptions.

Theorem 13. *(Ngai[55]) Let $F: X\rightrightarrows Y$ be a closed multimap acting from a completed metric space X to a uniformly convex space Y. Assume that F is metrical regular around $(\bar{x},\bar{y})\in \mathrm{gph}\, F$ with $\mathrm{reg}\, F(\bar{x},\bar{y}) = \tau\in(0,+\infty)$. Then the equality in (2.7) holds under one of the two following conditions:*

(i) F is upper semicontinuous around $\bar{x}$, $F(u)$ is convex for u near $\bar{x}$ and either $F(\bar{x})$ is singleton or $\bar{y}$ is an interior point of $F(\bar{x})$;

(ii) Y is a finite dimensional space.

3. RELATIVE METRIC REGULARITY

Given a subset W of $X\times Y$ and a point $(x,y)\in X\times Y$, define the following set:

$$W_x := \{z\in Y:\ \ (x,z)\in V\}$$

$$\text{and}\quad W_y := \{u\in X:\ \ (u,y)\in V\}.$$

Definition 14. *(Ioffe[38]) For $W \subset X \times Y$, a multimap $F : X \rightrightarrows Y$ is said to be* metrically regular relative to W at $(\bar{x}, \bar{y}) \in W \cap \operatorname{gph} F$ with a modulus $\tau > 0$, *if there is $\delta > 0$ such that*

$$d\big(x, F^{-1}(y) \cap \operatorname{cl} W_y\big) \leq \tau d(y, F(x)) \qquad (3.1)$$

whenever $(x, y) \in \big(B(\bar{x}, \delta) \times B(\bar{y}, \delta)\big) \cap W$ and $d(y, F(x)) < \delta$.

We shall denote $\operatorname{reg}_W F(\bar{x}, \bar{y})$, the infinum over all $\tau > 0$ such that (3.1) is verified.

Given a cone $C \subseteq Y$ in a normed linear space Y, for $\delta > 0$, let us set

$$C(\delta) := \{v \in Y : \ d(v, C) \leq \delta \|v\|\}.$$

and

$$W_F(C, \delta) := \big\{(x, y) \in X \times Y : \ y \in F(x) + C(\delta)\big\}.$$

Let us recall the definition of the metric regularity with respect to a cone.

Definition 15. *F is called metrically regular relative to C at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if there is $\delta > 0$ such that F is metrically regular relative to $W := W_F(C, \delta)$ at $(\bar{x}, \bar{y})$.*

Due to Definition 14, we see that F is relatively metrically regular with respect to a cone C at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with a modulus $\tau > 0$, if there is $\varepsilon > 0$ such that

$$d\big(x, F^{-1}(y) \cap \operatorname{cl} W_{F,y}(C, \delta)\big) \leq \tau d(y, F(x)) \qquad (3.2)$$

for all $(x, y) \in \big(B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)\big) \cap W_F(C, \delta)$ with $d(y, F(x)) < \varepsilon$.

We consider the lower semicontinuous envelope relatively to W of the function $x \mapsto d(y, F(x))$, which is defined as follows.

$$\varphi_{F,W}(x, y) := \begin{cases} = \liminf\limits_{\operatorname{cl} W_y \ni u \to x} d(y, F(u)) & \text{if } x \in \operatorname{cl} W_y \\ +\infty & \text{otherwise.} \end{cases} \qquad (3.3)$$

Note that obviously $\varphi_{F,W}(x, y) \geq 0$ and $\varphi_{F,W}(x, y) \leq d(y, F(x))$ for every $(x, y) \in \operatorname{cl} W_y \times Y$.

The following result established by Ngai-Théra[39], gave a slope characterization of the relative metric regularity.

Theorem 16. *Let $F : X \rightrightarrows Y$ be a closed multimap from a completed metric space X to a metric one Y. For $(\bar{x}, \bar{y}) \in \operatorname{gph} F \cap W$, $W \subset X \times Y$ and $\tau \in (0, +\infty)$, consider the following assertions, one has (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c).*

(a) F is metrically regular relative to W at $(\bar{x}, \bar{y})$ with modulus τ;

(b) There are $\alpha, \beta > 0$ such that

$$|\Gamma \varphi_{F,W}(\cdot, y)|(x) \geq \tau^{-1}$$

for any $(x, y) \in B(\bar{x}, \alpha) \times B(\bar{y}, \alpha)$ with $\varphi_{F,W}(x, y) \in (0, \beta)$.

(c) There are $\alpha, \beta > 0$ such that

$$|\nabla \varphi_{F,W}(\cdot, y)|(x) \geq \tau^{-1}$$

for any $(x, y) \in B(\bar{x}, \alpha) \times B(\bar{y}, \alpha)$ with $\varphi_{F,W}(x, y) \in (0, \beta)$.

As the usual metric regularity, the relatively metric regularity with respect to a cone also possesses the following perturbation stability.

Theorem 17. *(Ngai-Tron-Théra, 2019[56]) Let $F : X \rightrightarrows Y$ be a closed multimap from a completed X to a normed space Y. For a nonempty cone $C \subseteq Y$, if F is relatively metrically regular at $(\bar{x}, \bar{y})$ with a modulus $\tau > 0$ with respect to C, then for any locally lipschitz mapping $g : X \to Y$ around $\bar{x}$ with a sufficiently small Lipschitz constant, the multimap $F + g$ is also relatively metrically regular with respect to C at $(\bar{x}, \bar{y} + g(\bar{x}))$.*

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The next result is a characterization in terms of coderivatives of the relative metric regularity. For this, associated to $F : X \rightrightarrows Y$, we define the multimap $G : X \rightrightarrows Y \times Y$ as follows

$$G(x) = F(x) \times F(x), \quad x \in X.$$

Theorem 18. *(Ngai - Tron - Théra,[56]) Let $F : X \rightrightarrows Y$ be a closed multimap between Asplund spaces X, Y. For $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a nonempty cone $C \subseteq Y$, suppose that F has convex values around $\bar{x}$. Then F is relatively metrically regular with respect to C with modulus $\tau \le m^{-1}$ at $(\bar{x}, \bar{y})$, provided*

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(\bar{x},\bar{y},\bar{y})\\ \delta\downarrow 0^+}} d(0, D^\star_{\mathcal{F}} G(x, y_1, y_2)(T(C, \delta))) > m > 0, \tag{3.4}$$

Recall that A multimap $F : X \rightrightarrows Y$ is *partially sequentially normally compact* (PSNC, shortly,[10]) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if for all sequence $\{(x_k, y_k, x_k^\star, y_k^\star)\}_{n\in\mathbb{N}} \subset \operatorname{gph} F \times X^\star \times Y^\star$ verifying

$$(x_k, y_k) \to (\bar{x}, \bar{y}),\ x_k^\star \in D^\star_{\mathcal{F}} F(x_k, y_k)(y_k^\star),$$
$$y_k^\star \overset{w^\star}{\to} 0, \|x_k^\star\| \to 0,$$

one has $\|y_k^\star\| \to 0$ as $k \to \infty$. Mention that (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ holds automatically for any multimap when Y is a finite dimensional space. Under condition (PSNC), we obtain a point-based sufficient condition for the relative metric regularity as follows.

Corollary 19. *(see[56]) With the assumptions as in Theorem 18, assume in addition that F^{-1} is PSNC at $(\bar{x}, \bar{y}, \bar{y})$. If*

$$d_\star(0, D^\star_{\mathcal{L}} G(\bar{x}, \bar{y}, \bar{y})(T(C, 0))) > 0,$$

then F is relatively metrically regular with respect to C around $(\bar{x}, \bar{y})$.

In particular for the case $F(x) := g(x) - D$, here $D \subseteq Y$, a closed convex subset, $g : X \to Y$ is a continuous map near a given point $\bar{x} \in X$ such as $g(\bar{x}) \in D$, one obtains the following corollary.

Corollary 20. *(see[56]) Let X, Y and $C \subseteq Y$ be as before. Let $D \subseteq Y$ be a closed convex subset and $g : X \to Y$ be a continuous map around $\bar{x} \in X$ with $d_0 := f(\bar{x}) \in D$. Then the multimap $F(x) := g(x) - D$, $x \in X$ is relatively metrically regular with respect to C around $(\bar{x}, \bar{y})$ with a modulus $\tau = m^{-1}$, provided*

$$\liminf_{\substack{(x,d_1,d_2)\to(\bar{x},d_0,d_0)\\ \delta\downarrow 0^+}} d_\star(0, D^\star_{\mathcal{F}} f(x)(T(C,\delta) \cap N(D, d_1) \times N(K, d_2)) > 0, \tag{3.5}$$

4. APPLICATIONS

4.1. Newton methods for solving generalized equations

4.1.1. Newton iteration for equations and the Kantorovich theorem

We recall firstly the classical Newton algorithm for solving equations associate to smooth mappings beween Banach spaces. Consider the equation

$$\text{find } x \in X \text{ such that } \ f(x) = 0,$$

where X, Y are Banach spaces and $f : X \to Y$ is a continuously differentiable map. The Newton method for solving this equation consists of the iterations:

$$x_{k+1} = x_k - Df(x_k)^{-1} f(x_k),$$

here x_0 is a started point, and $Df(x_k)$ is invertible for all k. In the other works, the regular zeros (i.e., at which the derivative is invertible) of f are the fixed points of the following Newton operator:

$$N_f(x) = x - Df(x)^{-1} f(x).$$

Generally, when $Df(x)$ is not necessarily invertible, but assumed just to be surjective, the Newton operator is given by

$$N_f(x) = x - Df(x)^+ f(x),$$

here $Df(x)^+$ denotes the Moore-Penrose generalized inverse which coincides with the usual inverse $Df(x)^{-1}$ when it exists.
The following local and non-local quadratic convergence results are due to Kantorovich (1948) (see[57–59]).

Theorem 21. *(Kantorovich-1948[58]) Let $f : X \to Y$ be a mapping of class $\mathcal{C}^2$. Let $\xi \in U$ be such that $f(\xi) = 0$ and the derivative of f at this point be surjective. For $r > 0$ with $B[\xi, \rho] \subseteq U$, set*

$$M(f, \xi, \rho) = \sup_{\|x-\xi\| \le \rho} \|Df(\xi)^+ D^2 f(x)\|.$$

If $2M(f, \xi, \rho)\rho \le 1$ then for all $x_0 \in B[\xi, \rho]$, then the Newton sequence $x_{k+1} = N_f(x_k)$ is completely defined and converges to ξ, with a quadratically convergence rate,

$$\|x_k - \zeta\| \le \left(\frac{1}{2}\right)^{2^k-1} \|x_0 - \zeta\|.$$

Set $\beta(f, x_0) = \|Df(x_0)^+ f(x_0)\|$ if $Df(x_0)$ is surjective, and $\beta(f, x_0) = +\infty$, otherwise.

Theorem 22. *(Kantorovich[58]) Let f: $X \to Y$ be a mapping of class $\mathcal{C}^2$. For $x_0 \in X$, suppose that the following conditions are satisfied:*

- *$Df(x_0)$ is surjective,*
- *$2\beta(f, x_0) \le \rho$,*
- *$2\beta(f, x_0)M(f, x_0, \rho) \le 1$.*

Then the Newton sequence $x_{k+1} = N_f(x_k)$ is well-defined and converges to some ξ with $f(\xi) = 0$, and one has the estimation

$$\|x_k - \xi\| \le 1.63281... \left(\frac{1}{2}\right)^{2^k-1} \|x_0 - \xi\|,$$

with

$$1.63281... = \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}}.$$

4.1.2. Newton type method of generalized equations

Given Banach spaces X, Y, consider the following generalized equation of the form of the sum of a single-valued map and a set-valued one:

$$0 \in f(x) + F(x), \quad x \in X \qquad (4.1)$$

here $f : X \to Y$ is a function of class C^1, and $F : X \rightrightarrows Y$ is a closed multimap.

The Newton type method for solving (4.1) can be described as follows (see[2,60,61]): Given a starting point x_0, the sequence (x_k) is iteratively defined in solving the auxiliary generalized equation: x_{k+1} is a suitable solution of

$$0 \in f(x_k) + Df(x_k)(x - x_k) + F(x) \quad \text{for } k = 0, 1, ... \qquad (4.2)$$

Equivalently,

$$x_{k+1} \in (Df(x_k) + F)^{-1}(Df(x_k) - f(x_k)).$$

This Newton type method means of the use of *"partial linearization"* of the single-valued part f.

The following theorem due to Adly-Ngai-Vu[62] gives a local version of convergence for Newton's iteration (4.2).

Theorem 23. *(Adly - Ngai - Vu[62]) Let $f : X \longrightarrow Y$ be a second-order continuously differentiable function on an open subset U of X and let $F : X \rightrightarrows Y$ be a closed multimap. Let $\xi \in U$ be a solution of (4.1) and set $\zeta := Df(\xi)(\xi) - f(\xi) \in Y$. Suppose that the multimap $\Phi = Df(\xi) + F$ is metrically regular on*

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$V = \mathbb{B}[\xi, r] \times \mathbb{B}[\zeta, \rho]$ *of* (ξ, ζ) *with modulus* $\tau > 0$ *with* $r > 0$, $\mathbb{B}[\xi, r] \subset U$. *Set*

$$M(\tau, \xi, r) := \tau \sup_{\|u-\xi\| \leqslant r} \left\| D^2 f(z) \right\|,$$

$$\delta = \min\{r, \rho, \tau\rho\}.$$

If $2M(\tau, \xi, r)r \leqslant 1$, *then for any* $x_0 \in \mathbb{B}[\xi, \delta]$, *the sequence* $(x_n)_{n \geqslant 0}$ *in* (4.2) *is well-defined and converges quadratically to* ξ. *More precisely, one has*

$$\|x_n - \xi\| \leqslant \left(\frac{1}{2}\right)^{2^n - 1} \|x_0 - \xi\|. \tag{4.3}$$

In order to establish a non-local of the Kantorovich theorem, we need the following definition of the metric regularity on a suitable domain.

Definition 24. *Let a multimap* $F : X \rightrightarrows Y$, $x_0 \in X$ *and positive constants* $r > 0$, $s > 0$, *we define*

$$V(F, x_0, r, s) = \{(x, y) \in X \times Y : x \in \mathbb{B}[x_0, r], \; d(y, F(x)) < s\}. \tag{4.4}$$

The multimap F *is called metrically regular on* $V(F, x_0, r, s)$ *with a modulus* $\tau > 0$ *if for all* $(x, y) \in V(F, x_0, r, s)$, *one has*

$$d\big(x, F^{-1}(y)\big) \leqslant \tau d\big(y, F(x)\big). \tag{4.5}$$

When the multimap F verifies Definition 24, denote the quantity $\operatorname{reg}(F, x_0, r, s)$, the infimum over all $\tau > 0$ satisfying (4.5). Otherwise, set $\operatorname{reg}(F, x_0, r, s) = \infty$.

The next result states a global convergence result of the Newton iterations. The assumptions for this are based on the classical Kantorovich's theorem (see[59]).

Theorem 25. *(Adly-Ngai-Vu[62]) Let* $f : X \longrightarrow Y$ *and* $F : X \rightrightarrows Y$ *be maps acting between Banach spaces* X, Y. *Suppose that* f *is a* $\mathcal{C}^2$*−mapping on an open subset* $U \subset X$ *and* F *is a closed multimap. Consider problem (4.1), and define the following quantities*

$$\beta(\tau, x) := \tau d\big(0, (f + F)(x)\big),$$

$$M(\tau, x, \rho) := \tau \sup_{\|u-x\| \leqslant \rho} \left\| D^2 f(u) \right\|.$$

Let $x \in U$, $\delta \in (0, 1]$ *and* $\rho > 0$, $t > 0$ *such that the following statements hold.*

1. $G := Df(x) + F$ *is metrically regular on* $V := V(G, x, 4\rho, t)$ *with a modulus* $\tau > 0$,
2. $d\big(0, f(x) + F(x)\big) < s$,
3. $2\beta(\tau, x)M(\tau, x, \rho) \leqslant \delta$,
4. $2\eta\beta(\tau, x) \leqslant \rho$, *with* $\eta = \frac{1-\sqrt{1-\delta}}{\delta} = \frac{1}{1+\sqrt{1-\delta}}$.

Then the generalized equation (4.1) *has a solution* ξ *satisfying*

$$\|x - \xi\| \leqslant 2\eta\beta(\tau, x) \leqslant \rho. \tag{4.6}$$

Moreover, there exists a sequence (x_n) *generated by the Newton iterations* (4.2) *with the starting point* x *and converges to* ξ, *and the following error estimate holds*

- *if* $\delta < 1$, *then*

$$\|x_n - \xi\| \leqslant \frac{4\sqrt{1-\delta}}{\delta} \frac{\theta^{2^n}}{1 - \theta^{2^n}} \beta(\tau, x), \tag{4.7}$$

where $\theta = \frac{1-\sqrt{1-\delta}}{1+\sqrt{1-\delta}}$;

- *if* $\delta = 1$, *then*

$$\|x_n - \xi\| \leqslant 2^{-n+1} \beta(\tau, x). \tag{4.8}$$

When the mapping f is analytic, an extension of Smale's (α, γ)−theory to generalized equation was also extensively developed in[62].

4.1.3. Newton methods under relativemetric regularity

We consider now the case in which the multimap is not longer metrically regular but merely relatively metrically regular. For a cone $C \subseteq Y$, consider the inexact Newton iterations with respect to C defined as follows.

$$0 \in f(x_k) + Df(x_k)(x - x_k) + F(x) + \varepsilon_k C \cap \mathbb{S}_Y \tag{4.9}$$

for $k = 0, 1, ...$, where $\mathbb{S}_Y$ stands for the unit sphere in Y, and (ε_n) is a suitable sequence of positive reals, converging to zero.
To analyze the convergence of the iterative process (4.9), we established a non-local version of the relative metric regularity as was stated in the following theorem.

Theorem 26. *(Ngai [63]) Let $F : X \rightrightarrows Y$ be a closed multimap acting from a completed metric space X to a normed one Y. For a nonempty cone $C \subseteq Y$, suppose that $F : X \rightrightarrows Y$ is metrically regular on $V(F, \bar{x}, r, s)$ for some $r, s > 0$ with a modulus $\tau > 0$, relatively to C, i.e., there is a real $\delta \in (0, 1)$ such that for all $(x, y) \in V(F, \bar{x}, r, s) \cap W_F(C, \delta)$, one has*

$$d(x, F^{-1}(y) \cap \operatorname{cl} W_{F,y}(C, \delta)) \leq \tau d(y, F(x)). \tag{4.10}$$

Then with a map $f : X \to Y$ being locally Lipschitz on $B[\bar{x}, r]$ with a Lipschitz constant $L > 0$, the multimap $G := F + g$ is relatively metrically regular on $V(G, \bar{x}, \frac{r}{4}, R)$ with respect to C with modulus

$$\operatorname{Reg}_C(G, \bar{x}, \tfrac{r}{4}, R) \leq \kappa := \left(\tfrac{1-\beta(1+\theta)}{\tau(1+\beta(1+\theta))} - L \right)^{-1},$$

provided that

$$\beta \in (0, 1), 0 < \theta < \frac{\delta(1-\beta)}{1+\beta\delta}, \text{ and}$$

$$L < \frac{\theta(1-\theta)\beta}{\tau\left[1+\beta(1+\theta)\right](1+\theta)}, \ R := \min\left\{s, \frac{r}{5\kappa}\right\}.$$

The convergence result of the iterative process (4.9) is stated as follows.

Theorem 27. *(Ngai [63]) Given a nonempty cone $C \subseteq Y$, a function $f : X \longrightarrow Y$ between two Banach spaces X, Y being a $\mathcal{C}^2-$mapping on an open subset $U \subset X$ and a closed multimap $F : X \rightrightarrows Y$. Consider problem (4.1), and define the quantities*

$$\beta(\tau, x) := \tau d\big(0, (f + F)(x)\big),$$

$$M(\tau, x, \rho) := \tau \sup_{\|u-x\| \leqslant \rho} \left\| D^2 f(u) \right\|.$$

Let $x \in U$, $\delta \in (0, 1]$ and $\rho > 0$, $t > 0$, such that the following statements hold.

1. *$G = Df(x) + F$ is metrically regular on $V := V(G, x, 4r, s)$ relatively to C with a modulus $\tau > \operatorname{Reg}_C(G, x, 4r, s)$,*
2. *$0 \in f(x) + F(x) + C$,*
3. *$d\big(0, f(x) + F(x)\big) < t$,*
4. *$2\beta(\tau, x)M(\tau, x, \rho) \leqslant \delta$,*
5. *$2\eta\beta(\tau, x) \leqslant \rho$, with $\eta = \frac{1-\sqrt{1-\delta}}{\delta} = \frac{1}{1+\sqrt{1-\delta}}$.*

Then the generalized equation (4.1) has a solution $\xi \in U$ satisfying

$$\|x - \xi\| \leqslant 2\eta\beta(\tau, x) \leqslant r. \tag{4.11}$$

Moreover, there are constants $0 < b_1, b_2 < 1$ and $\varepsilon > 0$ such that for any sequence of positive reals (ε_n) satisfying

$$\varepsilon_0 \in (0, \varepsilon); \ \varepsilon_{n+1} \leq b_1 \varepsilon_n; \ \varepsilon_n^2 \leq b_2 \varepsilon_{n+1}, \ n \in \mathbb{N},$$

there exists a sequence (x_n) generated by the iterative process (4.9) with the starting point x and converges to ξ. In addition, if $\delta < 1$ and $\limsup_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^2} < +\infty$, then the convergence rate of such sequence (x_n) is quadratic.

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4.2. Convex principle for multimaps

In[64,65], Polyak has established an interesting convex principle with numerous applications in linear algebra, optimization and control theory. This convex principle states that a nonlinear image of a small ball in a Hilbert space is convex, provided the mapping is of class $\mathcal{C}^{1,1}$ and the center of the ball is a regular point of the mapping. In this final subsection, based on the metric regularity theory, we present a generalization of the Polyak convex principle for multimaps. Recall[66] that a multimap $F : X \rightrightarrows Y$ between normed spaces X, Y is paraconvex with a modulus $L > 0$ if for all $x_1, x_2 \in X$, $t \in [0,1]$,

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + + Lt(1-t)\|x_1 - x_2\|^2.$$

Obviously, a single-valued mapping of class $\mathcal{C}^{1,1}$ is paraconvex.

Theorem 28. *Let X be a Hilbert space and Y be a normed space. Let $F : X \rightrightarrows Y$ be a paraconvex multimap with modulus $L > 0$. For given $(\bar{x}, \bar{y}) \in \text{gph } F$, $\varepsilon_1, \varepsilon_2 > 0$, if F is metrically regular with a constant $\kappa > 0$ on $B(\bar{x}, \varepsilon_1) \times B(\bar{y}, \varepsilon_2)$, then for $0 < \delta < \delta_0 := \min\{\varepsilon, (2\kappa L)^{-1}\}$, the set $F(B[\bar{x}, \delta]) \cap B(\bar{y}, \varepsilon_2)$ is a convex set in Y.*

We need the following lemma.

Lemma 29. *Let X be a Hilbert spaces. For $\delta > 0$, $\bar{x} \in X$, $x_1, x_2 \in B[\bar{x}, \delta]$, $t \in (0,1)$, $x_t := tx_1 + (1-t)x_2$, one has*

$$B[x_t, \delta - \sqrt{\delta^2 - s}] \subseteq B[\bar{x}, \delta],$$

where $s := t(1-t)\|x_1 - x_2\|^2$.

Proof. Using the equality

$$\|ta + (1-t)b\|^2 = t\|a\|^2 + (1-t)\|b\|^2 - t(1-t)\|a - b\|^2,$$

validated in any Hilbert space X, $a, b \in X$, and $t \in \mathbb{R}$, one has the estimation

$$\begin{aligned}\|x_t - \bar{x}\|^2 &= t\|x_1 - \bar{x}\|^2 + (1-t)\|x_2 - \bar{x}\|^2 - \\ &\quad - t(1-t)\|x_1 - x_2\|^2 \\ &\le \delta^2 - t(1-t)\|x_1 - x_2\|^2 = \delta^2 - s.\end{aligned}$$

Hence for $\gamma := \delta - \sqrt{\delta^2 - s}$, $z \in B[x_t, \gamma]$, one has

$$\begin{aligned}\|z - \bar{x}\|^2 &\le (\|z - x_t\| + \|x_t - \bar{x}\|)^2 \\ &= \|z - x_t\|^2 + 2\|z - x_t\|\|x_t - \bar{x}\| + \|x_t - \bar{x}\|^2 \\ &\le \gamma^2 + 2\gamma\sqrt{\delta^2 - s} + \delta^2 - s = \delta^2,\end{aligned}$$

that is, $z \in B[\bar{x}, \delta]$, and the lemma is proved. □

Proof of Theorem 28. Let $y_1, y_2 \in F(B[\bar{x}, \delta]) \cap B(\bar{y}, \varepsilon_2)$, $t \in (0,1)$ be given. We want to show $y_t := ty_1 + (1-t)y_2 \in F(B[\bar{x}, \delta]) \cap B(\bar{y}, \varepsilon_2)$. As obviously $y_t \in B(\bar{y}, \varepsilon_2)$, for $y_1, y_2 \in B(\bar{y}, \varepsilon_2)$, $t \in (0,1)$, it suffices to show $y_t \in F(B[\bar{x}, \delta])$. Let $x_1, x_2 \in B[\bar{x}, \delta]$ be such that $y_i \in F(x_i)$, $i = 1, 2$. Then $x_t := tx_1 + (1-t)x_2 \in B[\bar{x}, \delta] \subseteq B(\bar{x}, \varepsilon_1)$, according to the metric regularity with constant κ of F on $B(\bar{x}, \varepsilon_1) \times B(\bar{y}, \varepsilon_2)$, one has

$$d(x_t, F^{-1}(y_t)) \le \kappa d(y_t, F(x_t)).$$

Since F is paraconvex with modulus L,

$$y_t \in tF(x_1) + (1-t)F(x_2) \subseteq F(x_t) + Lt(1-t)\|x_1 - x_2\|^2,$$

therefore, $d(y_t, F(x_t)) \le Lt(1-t)\|x_1 - x_2\|^2$, which along with the previous inequality yields

$$d(x_t, F^{-1}(y_t)) \le \kappa Lt(1-t)\|x_1 - x_2\|^2 = \kappa Ls.$$

As by assumption, $\kappa L < \delta/2$, one can find $z_t \in F^{-1}(y_t)$ such that $\|x_t - z_t\| \le \delta s/2$. Then by

$$\delta s/2 \le \delta - \sqrt{\delta^2 - s},$$

one has $z_t \in B[x_t, \delta - \sqrt{\delta^2 - s}]$, and thanks to Lemma 29, $z_t \in B[\bar{x}, \delta]$, consequently $y_t \in F(z_t) \subseteq F(B[\bar{x}, \delta])$. □

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5. CONCLUSIONS

The paper gives a survey on some recent contributions of our research group to the theory of metric regularity and its applications. Some important characterizations of the metric regularity via the strong slopes as well as the coderivatives have established. We have presented some extended variants of the notion of metric regularity and related properties. Regarding the radius of metric regularity which is an interesting topic in this theory, we have established a relationship between the regularity radius and the regularity modulus for general multimaps under some suitable assumptions. It remains still some open questions related to estimate the regularity radius for future studies. Many applications to numerical algorithms for solving optimization problems and generalized equations have found out recently, especially the applications to the Newton type methods have been investigated in this paper. We have used the metric regularity to obtain a new result related to the convexity of an image of balls through paraconvex multimaps. Many open problems related to the theory of metric regularity and applications are devoted to future researches.

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