Tính chính quy Milyutin của ánh xạ đa trị

Nguyễn Hữu Trọn[1,*], **Đào Ngọc Hân**[2]

[1]*Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam.*
[2]*Khoa Giáo dục Tiểu học và Mầm non, Trường Đại học Quy Nhơn, Việt Nam.*

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TÓM TẮT

Mục đích của bài báo là nghiên cứu những đặc trưng cho tính chính quy Milyutin của ánh xạ đa trị thông qua độ dốc địa phương và độ dốc toàn cục của hàm bao nửa liên tục dưới của hàm khoảng cách liên kết với ánh xạ đa trị đã cho. Bằng cách sử dụng các đặc trưng này, chúng tôi thu được tính ổn định nhiễu của chính quy Milyutin.

Từ khóa: *Tính chính quy mêtric, tính chính quy Milyutin, tính ổn định nhiễu, độ dốc.*

[*] *Tác giả liên hệ chính.*
Email: nguyenhuutron@qnu.edu.vn

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On the Milyutin regularity of set-valued mappings

Nguyen Huu Tron[1,*], **Dao Ngoc Han**[2]

[1]*Faculty of Mathematics and Statistics, Quy Nhon University, Vietnam.*
[2]*Faculty of Preschool and Primary Education, Quy Nhon University, Vietnam.*

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ABSTRACT

The aim of this paper is to study characterizations of Milyutin regularity of a set-valued mapping via the local and non-local slope of the lower semicontinuous envelope of the distance function associated with this set-valued mapping. By using of these characterizations, we get the stability under perturbation of the Milyutin regularity.

Keywords: *Metric regularity, Milyutin regularity, perturbation stability, slope.*

1. INTRODUCTION

The emergence of metric regularity is increasingly clear during last decades and considered one of the important concepts of variational analysis by its extensively applications in a large amount of mathematical areas. This property is studied by experts which obtained valuable results such as implicit and inverse function theorem and stability under small variations,.. It is also the basis for qualification conditions in various calculus rules and optimally criteria, etc. The reader is referred to many theoretical results on the metric regularity as well as its applications in works [1−21], and the references given therein.

It is also known that metric regularity is one of powerful tools to examine the solution existence of equations. For equations of the form $f(x) = y$, where $f : X \to Y$ is a single-valued function from a metric space X to metric space Y, the condition ensuring the existence of solutions is the surjectivity of f. As in nonlinear analysis, regularity of a strictly differentiable mapping at some point $\bar{x}$ is equivalent to its derivative at the point is onto.

However, variational analysis, especially optimization theory, appears the objects may lack of smoothness: non-differentiable functions at point of interest, set-valued mappings, etc. Thus, the condition on the surjectivity of the derivative mapping at the point may be failed. One way to overcome this problem is to give an upper estimation for the distance from a point x near a given solution $\bar{x}$ of the generalized equation $y \in F(x)$ to the solution set $F^{-1}(y)$ in terms of the residual $d(y, F(x))$. In applications, the residual is able to calculate or estimate easily, meanwhile the finding the exact solution set might be considerably more complicated. The map $F : X \rightrightarrows Y$ satisfying the above estimation is said to be local metric regularity around $(\bar{x}, \bar{y})$, it means that there exist some positive

[*]*Corresponding author.*
Email: Nguyenhuutron@qnu.edu.vn

numbers τ, δ, ρ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

for all $x \in B(\bar{x}, \delta)$ and $y \in B(\bar{y}, \rho)$.

Due to the crucial role of metric regularity theory in the areas of applied mathematics such as optimization, fixed point theory, convergence analysis of algorithms, economics, equilibrium, control theory, so on, many authors extended this property to non-local version. It found that the non-local regularity can be started from well-known Banach's contraction map principle. Extension of this principle on closed ball in a complete metric space established a connection between non-local regularity and fixed point of maps. This was first observed by Arutynov [22], Ioffe [7,15] and some years before Dmitruk, Milyutin, Osmolovskii [3] in connecting to the extremal problems. The reader is referred to the works [10,16,19,23–25] for these developments. Because of many applications in such practical problems, in recent papers [7,15], Ioffe presented a complete model of non-local regularity and its important applications. One of the most important properties in this type is Milyutin regularity. This type of regularity is associated with a regularity horizon function that is convenient to establish the criterion of regularity. In case of Milyutin regularity, there is almost no gap between necessary and sufficient conditons in regularity criteria, but with any regularity horizon function, this gap appears. The fact is that if the considered set is an open set, then the regularity horizon function is positive on it. Thus, dealing with Milyutin regularity, we do not need to be interseted in points outside the set. That is reason why the Milyutin regularity becomes important in non-local context.

In this paper, we will characterize Milyutin regularity via non-local and local slope as well as its applications in establishing the stability of the Milyutin regularity under pertubation.

The organization of paper is as follows. In Section 2, we give some useful notations and definitions such as openness, pseudo-Lipschitz, metric regularity, Milyutin metric regularity and their equivalence. We establish the non-local and local slope characterizations for the Milyutin regularity on fixed sets in Section 3. Section 4 is dedicated for the stability of the Milyutin regularity under suitable pertubation.

2. PRELIMINARIES

In this section, we present some essential definitions and properties that will be used throughout this paper. Let X and Y be metric spaces endowed with metrics both denoted by $d(\cdot,\cdot)$. Let $F : X \rightrightarrows Y$ is a set-valued mapping. We use the notations $\text{gph}F := \{(x, y) \in X \times Y \ : \ y \in F(x)\}$ for the graph of F, $\text{dom}F := \{x \in X \ : \ F(x) \neq \emptyset\}$ for the domain of F and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse is defined by $F^{-1}(y) := \{x \in X \ : \ y \in F(x)\}$, $y \in Y$ and satisfies

$$(x, y) \in \text{gph}F \iff (y, x) \in \text{gph}F^{-1}.$$

Given $\bar{x} \in X$, $r > 0$, we denote by $B(\bar{x}, r), \overline{B}(\bar{x}, r)$, the open and closed balls with center $\bar{x}$ and radius $r > 0$, respectively.

In recent works [7,8], Ioffe studied a nonlinear non-local regularity model of set-valued mapping on a box $U \times V$ of $X \times Y$. In this paper, we suggest a new version of this property which is slightly different from the mentionned one but on a set of $W \subset X \times Y$. Let W be a subset of $X \times Y$ and a function $\gamma : X \to \mathbb{R}_+$ which is positive on $P_X W$ and a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 2.1. Let X, Y be metric spaces, W be a subset of $X \times Y$ and let $F : X \rightrightarrows Y$ be a set-valued mapping. F is said to be (μ, γ)-*metrically regular* on W if there exists a positive real numbers r, κ such that

$$d(x, F^{-1}(y)) \leq \kappa\mu(d(y, F(x))) \qquad (1)$$

for all $(x, y) \in W$ with $0 < r\mu(d(y, F(x))) < \gamma(x)$, where $\gamma(x) > 0$, for all $x \in P_X W$.

Next, we introduce equivalent versions of the regularity such as (γ, μ)-Hölder property and (γ, μ)-openness of set-valued mappings.

Definition 2.2. Let X, Y be metric spaces, W be a subset of $X \times Y$ and let $F : X \rightrightarrows Y$ be a set-valued mapping. F is (γ, μ)-Hölder on W if there are $r, \kappa > 0$ such that

$$d(y, F(x)) \leq \kappa\mu(d(x, u))$$

holds for all $(x, y) \in W, y \in F(u)$ and $0 < r\mu(d(x, u)) < \gamma(x)$.

Definition 2.3. Let X, Y be metric spaces, W be a subset of $X \times Y$ and let $F : X \rightrightarrows Y$ be a set-valued mapping. F is (γ, μ)-open on W if there are $r, \kappa > 0$ such that the conclusion

$$B(F(x), \mu(rt)) \cap W_x \subset F(B(x, \kappa rt))$$

fullfils for all $x \in P_X W$ and $0 < t < \gamma(x)$.

In the case W is a box $U \times V$, one gets simpler versions of the regularity above as the defintions below.

Definition 2.4. F is said to be γ-metrically regular on (U, V) if there is a $\tau, \kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

provided $x \in U, y \in V$ and $\kappa d(y, F(x)) < \gamma(x)$. In the case of $r \equiv \kappa$ we denote by $\text{reg}_\gamma F(U|V)$ the lowest bound of such τ. If no such τ exists, set $\text{reg}_\gamma F(U|V) = \infty$. We shall call $\text{reg}_\gamma F(U|V)$ the modulus (of rate) of γ-metric regularity of F on (U, V).

Definition 2.5. F is said to have δ-pseudo-Lipschitz property on (U, V) if there is a λ such that

$$d(y, F(x)) \leq \lambda d(x, u)$$

if $x \in U, y \in V, \lambda d(x, u) < \delta(y)$ and $y \in F(u)$. Denote by $\text{lip}_\delta F(U|V)$ the lower bound of such λ. If no such λ exists, set $\text{lip}_\delta F(U|V) = \infty$. We shall call $\text{lip}_\delta F$ the δ-Lipschitz modulus of F such on (U, V).

Definition 2.6. F is said to be γ-open (or γ-covering) at a linear rate on (U, V) if there is a $r > 0$ such that

$$B(F(x), rt) \cap V \subset F(B(x, t)),$$

if $x \in U$ and $t < \gamma(x)$. Denote by $\text{sur}_\gamma F(U|V)$ the upper bound of such r. If no such r exists, set $\text{sur}_\gamma F(U|V) = 0$. We shall call $\text{sur}_\gamma F$ the modulus (or rate) of γ-openness of F on (U, V).

Proposition 2.7. *(see [15]) Let $F : X \rightrightarrows Y$ be a set-valued mapping defined between metric spaces X and Y, U, V be subsets of X and Y respectively, and $\gamma : X \to \overline{\mathbb{R}}$ be a extended-valued function which is positive on U. Then, the following three properties are equivalent*

(i) F is γ-open at a linear rate on (U, V);

(ii) F is γ-metrically regular on (U, V);

(iii) F^{-1} has γ-pseudo-Lipschitz property on (V, U).

Moreover, under the convention that $0 \cdot \infty = 1$, one has

$$sur_\gamma F(U|V) \cdot reg_\gamma F(U|V) = 1,$$

$$reg_\gamma F(U|V) = lip_\gamma F^{-1}(V|U).$$

By choosing the gauge function $\gamma(x)$ is $m(x) = d(x, X \setminus U)$ with U be an open subset of X, we get the Milyutin regularity of F on (U, V).

Definition 2.8. F is said to be Milyutin metrically regular on (U, V) if there is a $\tau, \kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

provided $x \in U, y \in V$ and $\kappa d(y, F(x)) < m(x)$.

A special case of this property is $U := B(\bar{x}, r), m(x) := r - d(x, \bar{x})$. When $V = Y$, we say that F is Milyutin regular on U. Then, we write $\text{reg}_m F(U)$ rather $\text{reg}_m F(U|Y)$. And, we say that F is globally regular if it is regular on $\text{Dom}F \times Y$ with $\gamma \equiv \infty$.

Remark 2.9. Unlike the local regularity, the domains U, V are essential elements of the definitions above. In the non-local case, the regular domain cannot be freely changed.

Example 2.10. Let $X = Y = \mathbb{R}$, $F(x) = \{x, 3\}$, $U = (0,1), V = (0,2)$. Then, F is 1-regular on (U, V) with modulus 1 but F is not γ-regular on (U, V') with $V' = (0,3)$ for any constant γ.

Indeed, one has that for $x \in U$, $y \in V$ such that $d(y, F(x)) < \gamma(x)$. So, $d(y, F(x)) < 1$. Thus, $d(y, F(x)) = |x - y|$. Therefore, $d(x, F^{-1}(y)) = |x - y| = d(y, F(x))$. It means that F is 1-metrically regular on $U \times V'$. However, F is not γ-regular on $U = (0,1), V' = (0,3)$ for any γ because for $x \in U$, there exists $t > 0$ such that $0 < x + t < 1$, then $(3 - t, 3) \subset B(F(x), t) \cap V'$ but $(3 - t, 3) \not\subseteq F(B(x, t)) = (x - t, x + t) \cup \{3\}$.

3. CHARACTERIZATIONS FOR MILYUTIN REGULARITY

Let X be a metric space and let $f : X \to \overline{\mathbb{R}}$ be a given function. As usual, Dom $f := \{x \in X : f(x) < +\infty\}$ denotes the domain of f. The symbol $[f(x)]_+$ stands for $\max(f(x), 0)$. Recall that the local slope $|\nabla f|(x)$ of a lower semicontinuous function f at $x \in \text{dom}f$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f; otherwise

$$|\nabla f|(x) = \limsup_{y \to x, y \neq x} \frac{f(x) - f(y)}{d(x, y)}.$$

For $x \notin \text{dom}f$, we set $|\nabla f|(x) = +\infty$. The non-local slope of f is defined by

$$|\Gamma f|(x) := \sup_{y \neq x} \frac{[f(x) - f(y)]_+}{d(x, y)}.$$

For $x \notin \text{dom}f$, we set $|\Gamma f|(x) = +\infty$.

It is easy to see that if X be a normed space and f is Fréchet differentiable at x then $|\nabla f|(x) = \|f'(x)\|$ and $|\nabla f|(x) \le |\Gamma f|(x)$ for all $x \in X$.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given as

$$f(x) := \begin{cases} x^2, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Since f attains the (global) minimum at $x = 0$, $|\nabla f|(0) = |\Gamma|(0) = 0$. For $x \neq 0$, f is differentiable, so we have $|\nabla f|(x) = 2x$ if $x > 0$ and $|\nabla f|(x) = 1$ if $x < 0$.

We notice that if F is any set-valued mapping, the distance function $d(\cdot, F(\cdot))$ is not generally a lower semicontinuous function. However, the tools of variational analysis often require the considered function to be lower semicontinuous. Therefore, instead of using the distance function, we often use the lower semicontinuous envelope $(x, y) \to \varphi_y(x)$ (which is always lower semicontinous) of the function $(x, y) \to d(y, F(x))$ defined by, for $(x, y) \in X \times Y$,

$$\varphi_y(x) = \liminf_{u \to x} d(y, F(u)).$$

We need two lemmas in the sequel.

Lemma 3.2. *Let $F : X \rightrightarrows Y$ be a closed multifunction, i.e., its graph is a closed set in $X \times Y$. Then, for each $y \in Y$,*

$$F^{-1}(y) = \{x \in X : \varphi_y(x) = 0\}.$$

Proof. Indeed, if $x \in F^{-1}(y)$, then $0 \le \varphi_y(x) \le d(y, F(x)) = 0$, so $\varphi_y(x) = 0$. Conversely, suppose $\varphi_y(x) = 0$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x such that $d(y, F(x_n))$ converges to 0. Then, we can find a sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \in F(x_n)$ and $d(y, v_n) \to 0$. Since the graph of F is closed, then $(x, y) \in \text{gph}F$, i.e. $x \in F^{-1}(y)$.

Lemma 3.3. *Let X, Y be metric spaces, W be a subset of $X \times Y$, let $F : X \rightrightarrows Y$ be a set-valued mapping. If there are positive reals r, κ such that*

$$d(x, F^{-1}(y)) \leq \kappa\varphi_y(x) \tag{2}$$

for all $(x, y) \in W$ with $0 < r\varphi_y(x) < \gamma(x)$ then F is γ-metrically regular on W. Conversely, if F is γ-metrically regular on the open set W of $X \times Y$ then (2) holds.

Proof. Because of $\varphi_y(x) \leq d(y, F(x))$ for all $(x, y) \in W$, if (2) satisfies then F is γ-metrically regular on W. Conversely, suppose that F is γ-metrically regular on open subset W of $X \times Y$. Let now $(x, y) \in W$ such that $0 < r\varphi_y(x) < \gamma(x)$, there exists a sequence $x_n \in X$ converging to x such that $d(y, F(x_n)) \to \varphi_y(x)$ when n tends to ∞. Because of the openness of $P_X W$ and $x \in P_X W$, $x_n \in P_X W$ for sufficiently large n. Thus, due to the regularity of F, and the continuity of γ, one has for sufficiently large n, $0 < rd(y, F(x_n)) < \gamma(x_n)$. It follows that

$$d(x_n, F^{-1}(y)) \leq \kappa d(y, F(x_n)).$$

Let n tends to ∞ in this inequality, one gets

$$d(x, F^{-1}(y)) \leq \kappa\varphi_y(x).$$

The proof is complete.

The following result establishes the necessary and sufficient condition for Milyutin regularity through the nonlocal slope of the lower envelope of distance function $(x, y) \to d(y, F(x))$.

Theorem 3.4. *Let X be a complete metric space and Y be a metric space, $U \subset X, V \subset Y$ be open subsets of X and Y, respectively and let $F : X \rightrightarrows Y$ be a closed set-valued mapping. Then, if any $x \in U$ and $y \in V$ with $0 < \tau\varphi_y(x) < m(x)$,*

$$|\Gamma\varphi_y|(x) > \tau^{-1},$$

then F is Milyutin regular on $U \times V$ with modulus τ. Conversely, if F is Milyutin regular on $U \times V$ with modulus τ, for $x \in U$ and $y \in V$ with $0 < \tau\varphi_y(x) < m(x)$,

$$|\Gamma\varphi_y|(x) \geq \tau^{-1}.$$

Proof. For the sufficient condition, take $x \in U$ and $y \in V$ with $0 < \varphi_y(x) < \tau m(x)$ such that

$$|\Gamma\varphi_y|(x) > \tau^{-1}. \tag{3}$$

We shall prove that F is Milyutin regular on $U \times V$ with modulus τ. Indeed, take $x \in U$ and $y \in V$ with $0 < \tau d(y, F(x)) < m(x)$. If follows that $0 < \varphi_y(x) < \tau^{-1}m(x)$. Fixing $y \in V$ and applying now the Ekeland's variational principle for the function $u \to f(u) := \varphi_y(u)$, one can find a point $z \in X$ satisfying the folowing conditions

(i) $d(z, x) \leq m(x)$;

(ii) $f(z) + \tau^{-1}d(x, z) \leq f(x)$;

(iii) $f(u) + \tau^{-1}d(u, z) > f(z), \forall u \neq z$.

We shall prove $f(z) = 0$. Suppose the contradiction that $f(z) > 0$. By (i), $z \in U$. By (ii),

$$\begin{aligned} f(z) &\leq f(x) - \tau^{-1}d(x, z) < \\ &< \tau^{-1}m(x) - \tau^{-1}d(x, z) \leq \tau^{-1}m(z). \end{aligned}$$

By (iii),

$$|\Gamma f|(z) \leq \tau^{-1}.$$

This contradicts to (3). So, $f(z) = 0$. It implies that $y \in F(z)$ From (ii), one obtains that

$$\tau^{-1}d(x, z) \leq f(x).$$

Thus,

$$\begin{aligned} \tau^{-1}d(x, F^{-1}(y)) &\leq \tau^{-1}d(x, z) \leq \\ &\leq f(x) \leq d(y, F(x)). \end{aligned}$$

Consequently,

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)).$$

One finishes the sufficient condition.

Conversely, suppose that F is Milyutin regular on $U \times V$ with modulus τ. Take now $x \in U, y \in V$ with $0 < \tau\varphi_y(x) < m(x)$. Subsequently, taking into account Lemma 3.3,

$$d(x, F^{-1}(y)) \leq \tau\varphi_y(x).$$

Then, for every $\varepsilon > 0$, there is $u \in F^{-1}(y)$ such that

$$d(x, u) < (\tau+\varepsilon)\varphi_y(x) = (\tau+\varepsilon)\varphi_y(x) - (\tau+\varepsilon)\varphi_y(u).$$

Note that $u \neq x$ and one gets

$$|\Gamma\varphi_y|(x) > (\tau + \varepsilon)^{-1}.$$

Let ε tend to 0, we obtain that

$$|\Gamma\varphi_y|(x) \geq \tau^{-1},$$

and we finish the necessary condition. The theorem is proved.

Using the local slope of the function mentioned above, one only gets the sufficient condition for the Milyutin regularity.

Theorem 3.5. *Let X be a complete metric space and Y be a metric space, $U \subset X, V \subset Y$ be open subsets of X and Y, respectively and let $F : X \rightrightarrows Y$ be a closed set-valued mapping. If for any $x \in U$ and $y \in V$ with $0 < \tau\varphi_y(x) < m(x)$,*

$$|\nabla\varphi_y|(x) > \tau^{-1}$$

then F is Milyutin regular on $U \times V$ with modulus τ.

Proof. The proof of this theorem is very simple with noting that $|\nabla f|(x) \leq |\Gamma f|(x)$ for all $x \in X$. Therefore, if $|\nabla\varphi_y|(x) > \tau^{-1}$ then $|\Gamma\varphi_y|(x) > \tau^{-1}$. So, by Theorem 3.4, one obtains the result.

However, the converse of this result is not true in general and the results above can be extended to more general case when one considers the regularity on an open subset W of $X \times Y$

Theorem 3.6. *Let X be a complete metric space and Y be a metric space, $W \subset X \times Y$ be an open subset of $X \times Y$ and let $F : X \rightrightarrows Y$ be a closed set-valued mapping. If for any $x \in P_X W$ and $y \in P_Y W$ with $0 < \tau\varphi_y(x) < m(x) = d(x, X \setminus P_X W)$,*

$$|\Gamma\varphi_y|(x) > \tau^{-1}$$

then, F is Milyutin regular on W with modulus τ. Conversely, if F is Milyutin regular on W with modulus τ for any $x \in P_X W$ and $y \in P_Y W$ with $0 < \tau\varphi_y(x) < m(x) = d(x, X \setminus P_X W)$,

$$|\Gamma\varphi_y|(x) \geq \tau^{-1}.$$

When F is Milyutin regular on W we get the following characterization.

Theorem 3.7. *Let X be a complete metric space and Y be a metric space, let $W \subset X \times Y$ be an open set. Let $F : X \rightrightarrows Y$ be a closed set-valued mapping. $m : P_X W \to \mathbb{R}_+$ is Lipschitz function on $P_X W$ with constant 1, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a gauge function. Then F is Milyutin regular on W if and only if*

$$\liminf_{\delta \downarrow 0} \left\{ |\Gamma\varphi_y|(x) : \begin{array}{l} \frac{d(x, W_y)}{m(x)} < \delta, \\ y \in P_Y W, \\ 0 < \frac{\varphi_y(x)}{m(x)} < \delta \end{array} \right\} > 0.$$

4. MILYUTIN REGULARITY UNDER PERTURBATION

Using Theorem 3.7, we can establish the stability of the Milyutin regularity under suitable perturbation.

Theorem 4.1. *Let X be a Banach space and Y be a normed space, U, V be open sets in X*

and Y, respectively. Let $F : X \rightrightarrows Y$ be a closed set-valued mapping. If F is Milyutin regular on $U \times V$ with modulus τ, $g : X \to Y$ is Lipschitz on U with constant $\lambda > 0$ satisfying $\tau\lambda < 1$ then $F + g$ is Milyutin regular on W, where $W := \{(x, y) \in X \times Y : x \in U, B(y - g(x), m(x)) \subset V\}$.

Proof. For in detail, see Ngai, Tron, Han[26].

5. CONCLUSIONS

In this paper, we established the characterizations for the Milyutin regularity of closed set-valued mappings defined on the complete metric spaces throughout the slopes of the lower semicontinuous envelope assiciated to this map. Then, by using obtained results in previous sections, we give the stability of the Milyutin regularity under perturbation.

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REFERENCES

1. D. Azé. A unified theory for metric regularity of multifunctions, *J. Convex Anal.*, **2006**, *13* (2), 225-252.
2. J. M. Borwein, D. M. Zhuang. Verifiable necessary and sufficient conditions for openness and regularity for set-valued and single-valued mapps, *J. Math. Anal. Appl.*, **1988**, *134*, 441–459.
3. A. V. Dmitruk, A. A. Milyutin, N. P. Osmolovskii. Lyusternik's theorem and the theory of extrema, *Russian Math. Surveys*, **1980**, *35* (6), 11–51.
4. A. L. Dontchev, R. T. Rockafellar. *Implicit Functions and Solution Mappings: A View from Variational Analysis*, 2^{rd} edition, Springer Series in Operations Research and Financial Engineering, Springer New York, 2016.
5. M. Durea, R. Strugariu. Chain rules for linear openness in metric spaces. Applications to parametric variational systems, *Mathematical Programming*, **2014**, Series A, *143* (1), 147-176.
6. A. D. Ioffe. On regularity concepts in variational analysis, *J. Fixed Point Theory Appl*, **2010**, *8* (2), 339–363.
7. A. D. Ioffe. Regularity on fixed sets, *SIAM J. Optim.*, **2011**, *21* (4), 1345-1370.
8. A. D. Ioffe. Nonlinear regularity models, *Math. Program.*, **2013**, *139* (1), 223-242.
9. A. D. Ioffe. Metric regularity-a survey. Part II. Applications, *J. Aust. Math. Soc.*, **2016**, *101*, 376-417.
10. A. D. Ioffe. Metric regularity and subdifferential calculus, *Russian Mathematical Surveys*, **2000**, *55* (3), 501-558.
11. B. S. Mordukhovich. *Variational analysis and generalized differentiation. I. Basic theory*, Springer-Verlag, Berlin, 2006.
12. H. V. Ngai, N. H. Tron, M. Théra. Directional Holder metric regularity, *J. Optim. Theory Appl.*, **2016**, *171*, 785-819.
13. H. V. Ngai, N. H. Tron, M. Théra. Metric regularity of the sum of multifunctions and applications, *Journal of Optimization: Theory and Applications*, **2014**, *160* (2), 355–390.

14. J. -P. Penot. Metric regularity, openness and Lipschitzian behavior of multifunctions, *Nonlinear Anal.*, **1989**, *13* (6), 629–643.

15. A. D. Ioffe. *Variational Analysis of Regular Mappings: Theory and Applications*, Springer Monographs in Mathematics, Springer, 2017.

16. A. D. Ioffe. Towards variational analysis in metric spaces: Metric regularity and fixed points, *Math. Program.*, **2010**, *123*, 241-252.

17. M. Ivanov, N. Zlateva. On Characterizations of Metric Regularity of Multi-valued Maps, <https://arxiv.org/abs/1811.08243>, reprint, 2020.

18. A. Jourani, L. Thibault. Metric regularity and subdifferential calculus in Banach spaces, *Set Valued Anal.*, **1995**, *3* (1), 87–100.

19. P. Q. Khanh. An induction theorem and general open mapping theorem, *J. Math. Anal. Appl.*, **1986**, *118*, 519-534.

20. S. M. Robinson. Generalized equations and their solutions, part I: Basic theory, *Point-to-Set Maps and Mathematical Programming*, **1979**, *10*, 128–141.

21. S. M. Robinson. Generalized equations and their solutions, part II: Applications to nonlinear programming, *Optimality and Stability in Mathematical Programming*, **1982**, *19*, 200–221.

22. A. V. Arutyunov, E. R. Avakov, S. E. Zhukovskiy. Stability Theorems for Estimating the Distance to a Set of Coincidence Points, *SIAM J. Optim.*, **2015**, *25* (2), 807–828.

23. A. V. Arutyunov, E. R. Avakov, B. Gelman, A. Dmitruk, V. Obukhovski. Locally covering maps in metric spaces and coincidence points, *J. Fixed Point Theory Appl.*, **2009**, *5*, 106-127.

24. A. V. Arutyunov. Covering mappings in metric spaces and fixed points, *Dokl. Math.*, **2007**, *76*, 665-668.

25. A. L. Dontchev, H. Frankowska. Lusternik-Graves theorem and fixed points, *Proc. Amer. Math. Soc.*, **2011**, *139*, 521- 534.

26. H. V. Ngai, N. H. Tron, D. N. Han. Nonlinear regularity on fixed sets, **2020**, *In revision.*