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Về một điều kiện thác triển liên tục nghiệm phương trình vi phân ngẫu nhiên trong không gian Hilbert

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Ngày nhận bài: 09/11/2021; Ngày sửa bài: 15/11/2021;

Ngày nhận đăng: 15/12/2021; Ngày xuất bản: 30/06/2022

TÓM TẮT

Bài báo tập trung nghiên cứu khái niệm về nghiệm yếu và nghiệm tích phân (mild solutions) của phương trình vi phân ngẫu nhiên phi tuyến trên không gian Hilbert với hệ các toán tử phụ thuộc thời gian, không bị chặn. Chúng tôi đưa ra một điều kiện để hai khái niệm nghiệm tích phân và nghiệm yếu ở trên là trùng nhau và đồng thời nghiên cứu thác triển liên tục nghiệm tích phân trên các không gian Hilbert. Dạng phương trình và các khái niệm về nghiệm chúng tôi nghiên cứu bắt nguồn trong lĩnh vực toán công nghiệp.

Từ khóa: *Nghiệm yếu phương trình vi phân ngẫu nhiên, hệ tiến hoá không thuần nhất phụ thuộc thời gian, nghiệm giải tích yếu.*

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https://doi.org/10.52111/qnjs.2022.16303

QUY NHON UNIVERSITY
JOURNAL OF SCIENCE

A note on continuously extended solutions of stochastic differential equations on Hilbert spaces

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Received: 09/11/2021; Revised: 15/11/2021;

Accepted: 15/12/2021; Published: 30/06/2022

ABSTRACT

We study about mild solutions and weak solutions of non-linear stochastic differential equations (SDEs) in Hilbert spaces for the case of family of time-dependent and unbounded operators and get some conditions that weak solutions to become mild solutions and vice versa. We also study continuously extension of mild solutions on Hilbert spaces. Our equation and concept of solutions are arisen as a stochastic partial differential equation (SPDE) in industrial mathematics.

Keywords: *Weak solution of SDE, non-time homogeneous evolution systems, analytically weak solutions.*

1. INTRODUCTION

Let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space where the family $\mathcal{F}$ of subsets of Ω is a σ-algebra, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. It is always assumed that G, H are separable Hilbert spaces; the Q-Wiener process $W = (W(t))_{t\in[0,T]}, 0 < T < \infty$, is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ together with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and is valued in G. Consider the following stochastic differential equation

$$\begin{aligned} dX(t) &= \big(L(t)X(t) + F(t)\big)dt + AdW(t), \\ X(t_0) &= \xi, \quad 0 \leq t_0 \leq t \leq T, \end{aligned} \tag{1}$$

where for all $t \in [0, T]$ the linear operator $L(t) : D(L(t)) \subset H \longrightarrow H$ is closed and densely defined on H; the operator $A : G \longrightarrow H$ is linear and continuous, F is an H-valued process, pathwise Bochner integrable on $[0, T]$, and the initial value ξ is an $\mathcal{F}_{t_0}$-measurable random variable getting values in H.

There are many mathematician studying on the equation (1). DaPrato and Zabczyk[1] studied the case operators are independent in time. Instead of separable Hilbert sapces, Manthey and Zausinger, see,[2] constructed mild solutions to (1) in weighted L^p spaces. In,[3] Prevot and Roeckner considered for $L(t)$ coercive variational solutions to (1). Veraar and Zimmerschied[4] considered the case of the family $L(t)$ is uniformly sectorial in $[t_0, T]$. Baur, Grothaus, and Mai, see,[5] give some conditions for existence and uniqueness of analytically weak solution to (1) and apply these results to linearized versions of a non-linear stochastic partial differential algebraic equation arising in industrial mathematics, that leads to the time-dependent case with the state space is some Sobolev space.

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https://doi.org/10.52111/qnjs.2022.16303

In this paper, we continue to study on solutions in[5] and give some conditions that weak solutions become mild solutions and vice versa and study continuously extended (mild) solutions on some Hilbert spaces.

2. PRELIMINARIES

From now on, we always assume that $(G, \langle\cdot,\cdot\rangle_G)$ and $(H, \langle\cdot,\cdot\rangle_H)$ are separable Hilbert spaces and $\|\cdot\|_G := \sqrt{\langle\cdot,\cdot\rangle_G}$ and $\|\cdot\|_H := \sqrt{\langle\cdot,\cdot\rangle_H}$ are the corresponding norms generated by the inner products, and $0 < T < \infty$. Let $(L(G,H), \|\cdot\|_{L(G,H)})$ be the space of all bounded linear operators from G to H together with the operator norm $\|\cdot\|_{L(G,H)}$; and $L(H) := L(H,H)$. Assume that the linear operator $L : D(L) \subset G \longrightarrow H$ is densely defined on G. We also denote $(L^*, D(L^*))$ the Hilbert adjoint operator of unbounded operator $(L, D(L))$ for the case $G \equiv H$, see.[6] In the application, we shall use concepts e.g. *stable family of operators, part of an operator in some subspace, invariant and admissible subspaces* as in.[7]-[9] The measurability of $L(G,H)$-valued functions will be considered as in.[1]

Definition 2.1. Let H be a Banach space and $L(H)$ be the space of linear bounded operators in H. A family $(S(t))_{t\geq 0} \subset L(H)$ is called a semi-group on H if

(i) $S(0) = Id$;

(ii) $S(t + r) = S(t)S(r)$ for all $t, r \geq 0$.

One concerns about a property of the family $(S(t))_{t\geq 0}$ at the "origin" $t = 0$ that $S(t)$ "converges" to Id as t decreases to 0. If the convergence is in the uniform topology on $L(H)$, i.e., $\lim_{t\downarrow 0} \|S(t) - Id\|_{L(H)} = 0$, then the family $(S(t))_{t\geq 0}$ is called a uniformly continuous semi-group. If it happens with the strong topology on $L(H)$, i.e. for all $u \in H$, $\lim_{t\downarrow 0} \|S(t)u - u\|_H = 0$, then $(S(t))_{t\geq 0}$ is called a *strongly continuous semi-group*; or is called shortly as a "semi-group of class C_0" or "C_0-semi-group". Of course, uniformly continuous semi-groups are also C_0-semi-groups.

Definition 2.2. A map $\Phi : \Omega \longrightarrow L(G,H)$ is called *strongly measurable* if for arbitrary $v \in G$ the function $\Phi v : (\Omega, \mathcal{F}) \longrightarrow (H, \mathcal{B}(H))$ is measurable. And $\Phi : \Omega \longrightarrow L(G,H)$ is said to be *Bochner integrable* if for all $v \in G$, the (H-valued) function Φv is Bochner integrable and there exists a linear bounded operator $\Psi \in L(G,H)$ such that

$$\int_\Omega \Phi(\omega) v \mathbb{P}(d\omega) = \Psi v, \ v \in G.$$

The operator Ψ is called the Bochner integral of Φ and is denoted by $\int_\Omega \Phi(\omega)\mathbb{P}(d\omega)$ or $\int_\Omega \Phi d\mathbb{P}$.

3. CONTINUOUSLY EXTENDED SOLUTIONS OF SPDEs

In this section, we continue to study the concepts mild and weak solutions as in[5] for nonlinear equations and will give conditions that weak solutions become mild solutions and vice versa. Moreover, we also study continuously extended mild solutions on Hilbert spaces.

We consider again the following equation on a separable Hilbert space H

$$\begin{cases} dX(t) = \big(L_0 X(t) + L_1(t)X(t) + F(t, X(t))\big)dt + \\ \qquad + A(t, X(t))dW(t), \quad \tau \leq t \leq T \\ X(\tau) = \xi \in H, \end{cases} \tag{2}$$

where $W := (W(t))_{t\geq\tau}$ is an H-valued Q-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a normal filtration $(\mathcal{F}_t)_{t\geq\tau}$.

Assumption 3.1. We assume that

(i) $(L_0, D(L_0))$ is the generator of a C_0-semi-group $(S(t))_{t\geq 0}$ on a Hilbert space H,

(ii) $L_1(t) \in L(H)$, $\forall\, t \in [\tau, T]$,

(iii) The function $F : [0,T] \times \Omega \times H \longrightarrow H$ is a measurable function on measurable spaces $(\Omega_T \times H, \mathscr{P}_T \times \mathcal{B}(H))$ and $(H, \mathcal{B}(H))$.

(iv) The function $A : [0,T] \times \Omega \times H \longrightarrow L_2^0$ is measurable on spaces $(\Omega_T \times H, \mathscr{P}_T \times \mathcal{B}(H))$ and $(L_2^0, \mathcal{B}(L_2^0))$, respectively.

(v) There exists a positive constant C such that $\forall u, u_1, u_2 \in H, t \in [0,T]$, and $\forall \omega \in \Omega$ we have

$$\|F(t,\omega;u_1) - F(t,\omega;u_2)\| + \\ + \|A(t,\omega;u_1) - A(t,\omega;u_2)\|_{L_2^0} \le C\|u_1 - u_2\|$$

and

$$\|F(t,\omega;u)\|^2 + \|A(t,\omega;u)\|_{L_2^0}^2 \le C^2(1 + \|u\|^2).$$

(vi) and the initial value ξ is $\mathcal{F}_\tau$-measurable H-valued random variable.

Definition 3.1. Suppose that $(X(t))_{\tau \le t \le T}$ is a random process getting values in H. $(X(t))_{\tau \le t \le T}$ is a mild solution of (2) if it is predictable with square integrable trajectories satisfying

$$\begin{aligned} X(t) = & S(t-\tau)\xi + \int_\tau^t S(t-r)L_1(r)X(r) + \\ & + S(t-r)F(r, X(r))dr + \\ & + \int_\tau^t S(t-r)A(r, X(r))dW(r) \quad \mathbb{P}\text{−a.s.} \end{aligned} \tag{3}$$

Remark 1. Since the trajectories of process $(X(t))_{\tau \le t \le T}$ are Bochner square integrable $\mathbb{P}$-a.s., together with Assumption 3.1, the integrals in (3) do exist.

Definition 3.2. Suppose that $(X(t))_{\tau \le t \le T}$ is a random process getting values in H. If $(X(t))_{\tau \le t \le T}$ is predictable process, its trajectories are square integrable and satisfying that $\forall v \in D(L_0^*)$ and $\forall t \in [\tau, T]$ we have

$$\begin{aligned} &\langle X(t), v\rangle = \langle \xi, v\rangle + \\ &+ \int_\tau^t \langle X(s), L_0^* v\rangle + \langle L_1(s)X(s) + F(s, X(s)), v\rangle ds + \\ &+ \int_\tau^t \langle A(s, X(s))dW(s), v\rangle \quad \mathbb{P}-\text{a.s.}; \end{aligned}$$

then $(X(t))_{\tau \le t \le T}$ is a weak solution of (2).

Remark 2. (i) Since $(L_0, D(L_0))$ is the generator of a C_0-semi-group $(S(t))_{t \ge 0}$ on Hilbert space H, the operator $(L_0^*, D(L_0^*))$ generates the C_0-semi-group $(S^*(t))_{t \ge 0}$. Hence, obviously that de domain $D(L_0^*)$ of L_0^* is dense in H.

(ii) For the case of linear equation with additive noise as in,[5] i.e. $F \equiv 0$ and A does not depend on time t and process X, Definition 3.2 is really coincide to[5][Def. 2.4]. Indeed, since for every $t \in [0,T]$ the operator $L_1(t)$ is bounded and $L(t) = L_0 + L_1(t)$, we have $D(L^*(t)) = D(L_0^*)$ and $L^*(t) = L_0^* + L_1^*(t)$ for all $t \in [0,T]$. Hence, for all $t \in [0,T], h \in D(L_0^*)$ we have

$$\langle X(t), L^*(t)h\rangle = \langle X(t), L_0^* h\rangle + \langle L_1(t)X(t), h\rangle.$$

For the existence of mild solution of (2), we recall the following result.

Theorem 3.1. *Under the assumption 3.1, the equation (2) has a unique mild solution. Moreover, it has a continuous modification.*

Proof. See[1][Theo. 7.4]

As a condition that weak solutions and mild solutions of (2) are equivalent, we have the following result.

Theorem 3.2. *Under the assumption 3.1, a predictable process $(X(t))_{0 \le t \le T}$ getting values in H is a weak solution of equation (2) if and only if it is a mild solution.*

Proof. Frieler and Knoche in[10] considered the time-independent non-linear stochastic differential equations with multiplicative noises. In[10][Prop. 2.10], the authors give some conditions that weak solutions to become mild solutions and vice versa. The proving Theorem 3.2 is followed by some following steps.

(i) Fix any $t \in [0,T]$ in Assumption 3.1, i.e. we reduce to the time-independent cases, the assumptions in[10][Prop. 2.10] are satisfied.

(ii) We can repeat the proving of[10][Prop. 2.10] for Theorem 3.2 if the assumptions

$$\mathbb{P}\Big(\int_0^T \|F(X(t))\|dt < \infty\Big) = 1 \quad \text{and}$$
$$\mathbb{P}\Big(\int_0^T \|A(X(t))\|^2_{L_2(U,H)}dt < \infty\Big) = 1 \tag{4}$$

are replaced by

$$\mathbb{P}\Big(\int_0^T \|F(t,X(t))\|dt < \infty\Big) = 1 \text{ and}$$
$$\mathbb{P}\Big(\int_0^T \|A(t,X(t))\|^2_{L_2(U,H)}dt < \infty\Big) = 1. \tag{5}$$

Because, in the main calculation related to F and A as in[10] we just take derivatives on the C_0-semi-group $(S(t))_{t\geq 0}$. The condition (4) just guarantees that the related integrals do exist. Hence, the condition (5) is required for the time-dependent cases.

Consider a general stochastic equation on a separable Hilbert space H as following

$$\begin{cases} dX(t) = \big(L_0X(t) + L_1(t)X(t) + \\ \quad +F(t,X(t))\big)dt + A(t,X(t))dW(t), \\ X(\tau) = \xi \in H, \quad 0 \leq \tau \leq t \leq T. \end{cases} \tag{6}$$

together with Assumption 3.1. Let $(\tilde{H}, \langle\cdot,\cdot\rangle_{\tilde{H}})$ be a Hilbert space and H be a densely and continuously embedded space in $\tilde{H}$. Denote $\|\cdot\|_{\tilde{H}}$ the Hilbert norm on $\tilde{H}$ induced by the inner product $\langle\cdot,\cdot\rangle_{\tilde{H}}$.

Assumption 3.2. Assume that $\forall t \in [\tau, T]$ the operators $S(t)$ and $L_1(t)$ are in $L\big((H, \|\cdot\|_{\tilde{H}}), \tilde{H}\big)$. Moreover, for every $t \in [0,T]$ we assume that $F(t,\cdot)$ and $A(t,\cdot)$ have extensions $\tilde{F}(t,\cdot) : \tilde{H} \longrightarrow \tilde{H}$ and $\tilde{A}(t,\cdot) : \tilde{H} \longrightarrow L_2^0(H,\tilde{H})$, respectively, satisfying Assumption 3.1, (iii)-(v) on $\tilde{H}$.

Remark 3. Let $(\tilde{L}_0, D(\tilde{L}_0))$ be the generator of semi-group $(\tilde{S}(t))_{t\geq 0}$ on $\tilde{H}$. Then due to the uniqueness of the generator of an C_0-semi-group we have $\tilde{L}_0\big|_{D(L_0)} = L_0$.

We consider the equation on $\tilde{H}$

$$\begin{cases} d\tilde{X}(t) = \big(\tilde{L}_0\tilde{X}(t) + \tilde{L}_1(t)\tilde{X}(t) + F\big(t,\tilde{X}(t)\big)\big)dt + \\ \quad +\tilde{A}(t,\tilde{X}(t))dW(t) \\ \tilde{X}(\tau) = \xi \in \tilde{H}, \quad 0 \leq \tau \leq t \leq T. \end{cases} \tag{7}$$

Theorem 3.3. *Let Assumption 3.1 and Assumption 3.2 hold. Then each equation (6) and (7) has a unique mild solution with a continuous modification named* $(X(t))_{0\leq t\leq T}$ *and* $(\tilde{X}(t))_{0\leq t\leq T}$*, respectively, satisfying*

$$\sup_{t\in[\tau,T]} \mathbb{E}\big(\|X(t,\tau;u) - X(t,\tau;v)\|_H^2\big) \leq C_T\|u-v\|_H^2$$

and

$$\sup_{t\in[\tau,T]} \mathbb{E}\big(\|\tilde{X}(t,\tau;\tilde{u}) - \tilde{X}(t,\tau;\tilde{v})\|_{\tilde{H}}^2\big) \leq \tilde{C}_T\|\tilde{u}-\tilde{v}\|_{\tilde{H}}^2,$$

for some nonnegative constants $C_T, \tilde{C}_T$ *and* $\forall u, v \in H$ *and* $\forall \tilde{u}, \tilde{v} \in \tilde{H}$*. Moreover,* $(\tilde{X}(t,\tau;\xi))_{0\leq t\leq T}$ *is an extension process of* $(X(t,\tau;\xi))_{0\leq t\leq T}$ *on* $\tilde{H}$*; i.e.* $\forall\, t \in [\tau, T]$, $\xi \in H$*, we have*

$$\tilde{X}(t,\tau;\xi) = X(t,\tau;\xi) \quad \mathbb{P}-\text{a.s.}$$

As a consequence, we obtain the regularity of solution $(\tilde{X}(t,\tau;\xi))_{0\leq t\leq T}$ *of (7) on* H.

Proof. Under the above assumptions, the existence and uniqueness of mild solution is obtained, see[1][Theo. 7.4]. The two inequalities follow by[1][Theo. 9.1]. We prove now the remaining task. The mild solution of equation (6) and equation (7), respectively, are

$$X(t,\tau;\xi) = S(t-\tau)\xi + \\ + \int_\tau^t \big(S(t-r)L_1(r)X(r) + S(t-r)F(r,X(r))\big)dr \\ + \int_\tau^t S(t-r)A(r,X(r))dW(r), \tag{8}$$

and

$$\tilde{X}(t,\tau;\xi) = \tilde{S}(t-\tau)\xi + \\ + \int_\tau^t \big(\tilde{S}(t-r)\tilde{L}_1(r)\tilde{X}(r) + \tilde{S}(t-r)F(r,\tilde{X}(r))\big)dr \\ + \int_\tau^t \tilde{S}(t-r)\tilde{A}(r,\tilde{X}(r))dW(r). \tag{9}$$

Let $\xi \in H$. Following the continuous extension of linear operators as in Assumption 3.2, the mild solution $(X(t,\tau;\xi))_{\tau\le t\le T}$ as in (8) also satisfies (9). Note that the uniqueness of mild solutions of (7) is up to an equivalence, among the processes satisfying

$$\mathbb{P}\Big(\int_0^T \|X(r)\|^2_{\tilde{H}} dr < \infty\Big) = 1. \tag{10}$$

However, as in[1], if two mild solution $(X_1(t))_{\tau\le t\le T}$ and $(X_2(t))_{\tau\le t\le T}$ of (6) satisfying (10) then $\forall t \in [\tau, T]$ we have $\mathbb{P}\big(X_1(t) = X_2(t)\big) = 1$. Hence, for arbitrary $\xi \in H$ and $t \in \mathbb{R} : \tau \le t \le T$ we have

$$\tilde{X}(t,\tau;\xi) = X(t,\tau;\xi) \quad \mathbb{P}-\text{a.s.}$$

Definition 3.3. We called $(\tilde{X}(t))_{0\le t\le T}$ a continuously extended (mild) solution of (6) on $\tilde{H}$.

We consider the transition semi-groups corresponding to equations (6) and (7). Let $C_b(H)$ and $C_b(\tilde{H})$ be the space of bounded and continuous functionals on H and $\tilde{H}$, respectively. Let $\varphi \in C_b(H), \tilde{\varphi} \in C_b(\tilde{H})$ and $x \in H, \tilde{x} \in \tilde{H}$. Denote

$$P_{\tau,t}\varphi(x) := \mathbb{E}\big(\varphi(X(t,\tau;x))\big)$$

and

$$\tilde{P}_{\tau,t}\tilde{\varphi}(\tilde{x}) := \mathbb{E}\big(\tilde{\varphi}(\tilde{X}(t,\tau;\tilde{x}))\big).$$

Definition 3.4. The family $(P_{\tau,t})_{0\le\tau\le t\le T}$ is called a Feller evolution systems on $C_b(H)$ if for all $\varphi \in C_b(H)$ we have

(i) $P_{\tau,t}\varphi \in C_b(H)$ for all $0 \le \tau \le t \le T$ and

(ii) $P_{\tau,r}\big(P_{r,t}\varphi\big)(x) = P_{\tau,t}\varphi(x)$ for all $0 \le \tau \le r \le t \le T$ and for all $x \in H$.

If the two items as above satisfy for all $\varphi \in B_b(H)$, the space of bounded and measurable functionals on H, then the family $(P_{\tau,t})_{0\le\tau\le t\le T}$ is called a strong Feller evolution systems.

Theorem 3.4. *Let Assumption 3.1 and Assumption 3.2 hold. Then the families $(P_{\tau,t})_{0\le\tau\le t\le T}$ and $(\tilde{P}_{\tau,t})_{0\le\tau\le t\le T}$ are Feller evolution systems on $C_b(H)$ and $C_b(\tilde{H})$ respectively.*

Proof. The Feller property is followed Theorem 3.3 and "semi-group" property is proved by[1][Cor. 9.9].

Acknowledgements

This study is conducted within the framework of science and technology projects at institutional level of Quy Nhon University under the project code T2017.543.46.

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