

## Bất đẳng thức kiểu Morrey cho các hàm có giá trị trung bình bằng 0

Nguyễn Văn Thành\*, Nguyễn Hữu Thuần, Nguyễn Đặng Thanh Giang,  
Đỗ Phương Oanh, Nguyễn Thị Hà Tiên, Đoàn Khánh Duy

*Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam*

*Ngày nhận bài: 15/06/2022; Ngày nhận đăng: 01/09/2022; Ngày xuất bản: 28/10/2022*

### TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu một bất đẳng thức kiểu Morrey cho các hàm Sobolev có giá trị trung bình bằng 0. Hằng số được dẫn ra ở trong bài báo này là tốt hơn so với hằng số hiện có trong một số tài liệu. Sau đó, chúng tôi nghiên cứu sâu hơn về một ứng dụng của bất đẳng thức kiểu Morrey này đối với sự hội tụ yếu của dãy nghiệm của phương trình  $p$ -Laplace với điều kiện biên Neumann khi  $p \rightarrow \infty$ .

**Từ khóa:** Không gian Sobolev, phương trình  $p$ -Laplace, bất đẳng thức kiểu Morrey.

---

\*Tác giả liên hệ chính.

Email: nguyenvanthanh@qnu.edu.vn

# A note on Morrey-type inequality for functions of mean value zero

Van Thanh Nguyen\*, Huu Thuan Nguyen, Dang Thanh Giang Nguyen  
Phuong Oanh Do, Thi Ha Tien Nguyen, Khanh Duy Doan

Department of Mathematics and Statistics, Quy Nhon University, Vietnam

Received: 15/06/2022; Accepted: 01/09/2022; Published: 28/10/2022

## ABSTRACT

In the present article, we study a Morrey-type inequality for Sobolev functions of mean value zero. The constant in our paper is smaller than the existing one in the literature. Then we study further an application of the Morrey-type inequality for the weak convergence of solutions to  $p$ -Laplace equations with a Neumann boundary condition as  $p \rightarrow \infty$ .

**Keywords:** Sobolev spaces,  $p$ -Laplace equations, Morrey-type inequality.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded, smooth domain of  $\mathbb{R}^n$ . This paper is concerned with a Morrey-type inequality for Sobolev functions of mean value zero in Sobolev space  $W^{1,p}(\Omega)$ . As usual, Sobolev spaces consist of  $L^p$  functions whose weak derivatives belong to Lebesgue spaces  $L^p$ . These spaces provide one of the most useful settings for the analysis of partial differential equations. It is known that Sobolev spaces equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p$$

are Banach spaces, where  $\|F\|_p := (\int_{\Omega} |F|^p dx)^{1/p}$ . For more details on Lebesgue and Sobolev spaces, we refer to the books.<sup>1-4</sup>

The well-known Morrey inequality in  $\mathbb{R}^n$  (see, for example,<sup>1,3,5,6</sup>) states that if  $p > n$  then for all  $v \in W^{1,p}(\mathbb{R}^n)$  and all  $x, y \in \mathbb{R}^n$

$$|v(x) - v(y)| \leq \bar{C}_{p,n} |x - y|^{1-n/p} \left( \int_{\mathbb{R}^n} |\nabla v|^p dx \right)^{\frac{1}{p}}, \quad (1)$$

where  $\bar{C}_{p,n}$  is a positive constant depending only on  $p$  and  $n$ .

Now, let  $d_{\Omega}$  denote the distance function to the boundary  $\partial\Omega$ , that is

$$d_{\Omega}(x) := \inf_{y \in \partial\Omega} |x - y|, \quad x \in \bar{\Omega}.$$

Taking an arbitrary  $y \in \partial\Omega$  in (1) one arrives at the following pointwise inequality, for all  $(x, v) \in \bar{\Omega} \times W_0^{1,p}(\Omega)$ ,

$$|v(x)| \leq \bar{C}_{p,n} (d_{\Omega}(x))^{1-n/p} \|\nabla v\|_p, \quad (2)$$

where  $\|\cdot\|_p$  stands for the standard norm of  $L^p(\Omega)$ . Passing to the maximum value in the left-hand side of (2) we arrive at the well-known Morrey-Sobolev inequality

$$\|v\|_{\infty} \leq \bar{C}_{p,n,\Omega} \|\nabla v\|_p, \quad \forall v \in W_0^{1,p}(\Omega), \quad (3)$$

where the constant  $\bar{C}_{p,n,\Omega}$  depends only on  $p, n$  and  $\Omega$ . The above inequality is devoted to Sobolev functions vanishing on the boundary and useful for studying partial differential equations involving a Dirichlet boundary condition. The counterpart for the Morrey-Sobolev inequality (3) for Sobolev functions of mean value zero can be written as

$$\|u\|_{\infty} \leq C_{p,n,\Omega} \|\nabla u\|_p, \quad (4)$$

for all Sobolev function  $u \in W^{1,p}(\Omega)$  of mean value zero, that is, it satisfies  $\int_{\Omega} u dx = 0$ .

In<sup>7</sup>, the authors make use of this inequality (for a smooth and convex domain  $\Omega$ ) in studying limits as  $p \rightarrow \infty$  of solutions to  $p$ -Laplace equations coupled with a Neumann boundary condition.

Such a constant  $C_{p,n,\Omega}$  has been implicitly mentioned in<sup>6</sup> and explicitly given in<sup>8</sup> Lemma B.1.16

\*Corresponding author.

Email: nguyenvanthanh@qnu.edu.vn

(see also<sup>9</sup>). Finding a smaller constant for the Morrey-type inequality (4) is an interesting issue.

In this short paper, we provide a better estimate for the Morrey-type inequality (4) compared with the constant given in<sup>8</sup> Lemma B.1.16. Furthermore, its application in studying the weak convergence of solutions to  $p$ -Laplace equations with a Neumann boundary condition as  $p \rightarrow \infty$  is also considered in detail.

The article is organized as follows. In section 2, we give a new constant  $C$  of the Morrey-type inequality (4) for Sobolev functions of mean value zero. Then, in section 3, we derive the key bound of the gradients of solutions to  $p$ -Laplace equations coupled with a Neumann boundary condition. As a consequence, we obtain the weak convergence of solutions in Sobolev spaces as  $p \rightarrow \infty$ .

## 2. MORREY-TYPE INEQUALITY FOR SOBOLEV FUNCTIONS OF MEAN VALUE ZERO

**Theorem 1.** *Let  $\Omega$  be a Lipschitz and convex domain of  $\mathbb{R}^n$  and  $p > n$ . Then every Sobolev function  $u \in W^{1,p}(\Omega)$  of mean value zero, i.e.  $\int_{\Omega} u dx = 0$ , obeys the following inequality*

$$\|u\|_{L^\infty} \leq \frac{d^{n+1-n/p}}{|\Omega| (n+1-n/p)} \omega_n^{1-1/p} \left( \frac{p-1}{p-n} \right)^{1-1/p} \|\nabla u\|_{L^p}, \quad (5)$$

where  $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$  is the surface area of the unit sphere in  $\mathbb{R}^n$ ,  $\Gamma$  is the gamma function,  $|\Omega|$  is Lebesgue measure of  $\Omega$  and  $d = \text{diam}(\Omega)$  is the diameter of  $\Omega$ .

*Proof.* Set

$$C(d, n, p) := \frac{d^{n+1-n/p}}{n+1-n/p} \omega_n^{1-1/p} \left( \frac{p-1}{p-n} \right)^{1-1/p}.$$

We divide the proof into three steps.

**Step 1.** Fix any  $x \in \mathbb{R}^n$ . Let us prove that for any  $\mathbb{R}^n$ -valued measurable function  $W \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ , we have for all  $p > n$

$$\int_{B(0,d)} \int_0^1 |W(x+tz)| |z| dt dz \leq C(d, n, p) \|W\|_{L^p(B(x,d))}, \quad (6)$$

where  $B(y, d)$  is the Euclidean ball of radius  $d$  and center  $y$  in  $\mathbb{R}^n$ . To this aim, we make use of the change of variables in polar coordinates by the bijection  $\Phi : B(0, d) \setminus \{0\} \rightarrow (0, d] \times \partial B(0, 1)$  defined as  $\Phi(z) := (r, z') = \left( |z|, \frac{z}{|z|} \right)$ . More precisely, one

has

$$\int_{B(0,d)} g(z) dz = \int_0^d r^{n-1} \int_{\partial B(0,1)} g(rz') dS(z') dr. \quad (7)$$

Now applying (7) with  $g(z) := \int_0^1 |W(x+tz)| |z| dt$

on  $B(0, d)$  for the second line below, and  $g(z) := |W(x+z)| |z|^{1-n}$  on  $B(0, r)$  for the fifth, we arrive the following estimate

$$\begin{aligned} & \int_{B(0,d)} \int_0^1 |W(x+tz)| |z| dt dz \\ &= \int_0^d r^{n-1} \int_{\partial B(0,1)} \int_0^1 |W(x+trz')| r dt dS(z') dr \\ &= \int_0^d r^{n-1} \int_{\partial B(0,1)} \int_0^r |W(x+\tau z')| d\tau dS(z') dr \quad (\text{set } \tau = rt) \\ &= \int_0^d r^{n-1} \int_0^r \tau^{n-1} \int_{\partial B(0,1)} |W(x+\tau z')| \tau^{1-n} dS(z') d\tau dr \\ &= \int_0^d r^{n-1} \int_{B(0,r)} |W(x+z)| |z|^{1-n} dz dr \\ &\leq \int_0^d r^{n-1} \left( \int_{B(0,r)} |W(x+z)|^p dz \right)^{1/p} \left( \int_{B(0,r)} |z|^{q(1-n)} dz \right)^{1/q} dr. \end{aligned} \quad (8)$$

Here, for the last line, we used Hölder's inequality with  $q = \frac{p}{p-1}$ .

By  $r \leq d$ , observe that

$$\left( \int_{B(0,r)} |W(x+z)|^p dz \right)^{1/p} \leq \|W\|_{L^p(B(x,d))}.$$

On the other hand, using again (7) with  $g(z) = |z|^{q(1-n)}$  we can compute explicitly

$$\begin{aligned} \int_{B(0,r)} |z|^{q(1-n)} dz &= \int_0^r \tau^{n-1} \int_{\partial B(0,1)} \tau^{q(1-n)} dS(z') d\tau \\ &= \omega_n \int_0^r \tau^{\frac{1-n}{p-1}} d\tau \\ &= \frac{\omega_n}{\frac{1-n}{p-1} + 1} r^{\frac{1-n}{p-1} + 1} = \omega_n \frac{p-1}{p-n} r^{\frac{p-n}{p-1}}. \end{aligned}$$

It follows that

$$\begin{aligned} & \int_{B(0,d)} \int_0^1 |W(x+tz)| |z| dt dz \\ & \leq \|W\|_{L^p(B(x,d))} \left( \omega_n \frac{p-1}{p-n} \right)^{1/q} \int_0^d r^{n-1} \left( r^{\frac{p-n}{p-1}} \right)^{1/q} dr \\ & = \|W\|_{L^p(B(x,d))} \left( \omega_n \frac{p-1}{p-n} \right)^{1/q} \int_0^d r^{n-n/p} dr \\ & = \|W\|_{L^p(B(x,d))} \left( \omega_n \frac{p-1}{p-n} \right)^{1-1/p} \frac{d^{n+1-n/p}}{n+1-n/p}. \end{aligned}$$

**Step 2.** We are now in a position to show the assertion of Theorem 1 for the case of smooth function  $u$ . In this case, set

$$W(x) := \begin{cases} \nabla u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Since  $\int_{\Omega} u(y) dy = 0$ , we get for any  $x \in \Omega$

$$\begin{aligned} u(x) &= \frac{1}{|\Omega|} \int_{\Omega} (u(x) - u(y)) dy \\ &= \frac{1}{|\Omega|} \int_{\Omega} \int_0^1 \langle \nabla u(x + t(y-x)), x-y \rangle dt dy \\ &\leq \frac{1}{|\Omega|} \int_{\Omega} \int_0^1 |\nabla u(x + t(y-x))| |x-y| dt dy \\ &\leq \frac{1}{|\Omega|} \int_{B(x,d)} \int_0^1 |W(x + t(y-x))| |x-y| dt dy \\ &= \frac{1}{|\Omega|} \int_{B(0,d)} \int_0^1 |W(x+tz)| |z| dt dz. \end{aligned}$$

Following Step 1, we obtain

$$\begin{aligned} |u(x)| &\leq \frac{C(d, n, p)}{|\Omega|} \|W\|_{L^p(B(x,d))} \\ &= \frac{C(d, n, p)}{|\Omega|} \|\nabla u\|_{L^p(\Omega)}, \end{aligned}$$

which completes the proof for smooth functions.

**Step 3.** For the general case, we make use a smooth approximation. Fix any  $u \in W^{1,p}(\Omega)$  satisfying  $\int_{\Omega} u dx = 0$ . Then there exists a sequence of smooth functions  $u_{\varepsilon}$  of mean value zero such that  $u_{\varepsilon}$  converges to  $u$  strongly in  $L^m(\Omega)$  ( $1 \leq m < \infty$ ) and  $u_{\varepsilon}$  converges to  $u$  strongly in  $W^{1,p}(\Omega)$  as  $\varepsilon \rightarrow 0$ . Passing to the limit as  $\varepsilon \rightarrow 0$ , from the inequality

$$\begin{aligned} \|u_{\varepsilon}\|_{L^m(\Omega)} &\leq |\Omega|^{\frac{1}{m}} \|u_{\varepsilon}\|_{L^{\infty}(\Omega)} \\ &\leq |\Omega|^{\frac{1}{m}} \frac{C(d, n, p)}{|\Omega|} \|\nabla u_{\varepsilon}\|_{L^p(\Omega)} \end{aligned}$$

we obtain

$$\|u\|_{L^m(\Omega)} \leq |\Omega|^{\frac{1}{m}} \frac{C(d, n, p)}{|\Omega|} \|\nabla u\|_{L^p(\Omega)}.$$

Letting  $m \rightarrow \infty$  we arrive to

$$\|u\|_{L^{\infty}(\Omega)} \leq \frac{C(d, n, p)}{|\Omega|} \|\nabla u\|_{L^p(\Omega)},$$

which completes the proof.  $\square$

**Remark 2.** The above Morrey-type inequality is inspired by the estimate (B.1.3) of<sup>8</sup> p. 556 for smooth functions of mean value zero. However, it is worth noting that the inequality given in Theorem 1 is a little bit sharper than the one in<sup>8</sup> p. 556, where the constant is explicitly stated by

$$C = \frac{d^{n+1-n/p}}{|\Omega|n} \omega_n^{1-1/p} \left( \frac{p-1}{p-n} \right)^{1-1/p}.$$

### 3. APPLICATIONS

Give a Lipschitz and convex domain  $\Omega$  of  $\mathbb{R}^n$ , we consider limits as  $p \rightarrow \infty$  of solutions  $u_p$  to the  $p$ -Laplace problems coupled with a Neumann boundary condition

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = f(x) & \text{in } \Omega \\ |\nabla u(x)|^{p-2} \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \quad (9)$$

For a fixed  $p > n$ , the equation (9) has unique solution  $u_p$  of mean value zero, that is  $\int_{\Omega} u_p dx = 0$ . This is a standard result in the field of calculus of variations and partial differential equations. In fact, consider the variational problem

$$\min_{u \in S_p} \left\{ \int_{\Omega} \frac{|\nabla u|^p}{p} dx - \int_{\Omega} u f dx \right\}, \quad (10)$$

where  $S_p := \left\{ u \in W^{1,p}(\Omega) : \int_{\Omega} u dx = 0 \right\}$ . The cost functional in (10) is lower semi-continuous, coercive and strictly convex on the non-empty convex set  $S_p$ . Therefore, there exists a unique minimizer  $u_p$  to (10), which is also a weak solution of problem (9), that is, it verifies

$$\int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi dx = \int_{\Omega} f \phi dx, \quad \forall \phi \in C^{\infty}(\overline{\Omega}). \quad (11)$$

In this section, we are interested in studying the behavior of solutions  $u_p$  as  $p \rightarrow \infty$ . More precisely, we will show that the sequence  $\{u_p\}$  converges weakly in Sobolev spaces to a 1-Lipschitz function  $u_{\infty}$  as  $p \rightarrow \infty$ .

#### 3.1. Bound of the gradients

Our aim is to prove that the  $L^p$ -norm of the gradients  $\nabla u_p$  is bounded independently of all  $p \geq n+1$ .

**Lemma 3.** Let  $u_p$  be a unique solution to (9) with  $\int_{\Omega} u_p dx = 0$ . Then there exists a positive constant  $C$  independent of  $p \geq n+1$  such that

$$\|\nabla u_p\|_{L^p(\Omega)} \leq C^{\frac{1}{p-1}} \text{ for all } p \geq n+1. \quad (12)$$

*Proof.* As a consequence of Theorem 1, there exists a positive constant  $C_\Omega$  independent of  $p \geq n + 1$  such that

$$\|u\|_{L^\infty(\Omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega)} \quad (13)$$

for all  $p \geq n + 1$  and all Sobolev functions  $u \in W^{1,p}(\Omega)$  of mean value zero, i.e.,  $\int_\Omega u dx = 0$ . In particular, applying for  $u = u_p$  we get

$$\|u_p\|_{L^\infty(\Omega)} \leq C_\Omega \|\nabla u_p\|_{L^p(\Omega)}, \text{ for all } p \geq n + 1. \quad (14)$$

On the other hand, using (11) with  $\phi = u_p$ , Hölder's inequality and (14), we obtain

$$\begin{aligned} \int_\Omega |\nabla u_p|^p dx &= \int_\Omega f u_p dx \leq \|f\|_{L^1(\Omega)} \|u_p\|_{L^\infty(\Omega)} \\ &\leq C_\Omega \|f\|_{L^1(\Omega)} \|\nabla u_p\|_{L^p(\Omega)}. \end{aligned}$$

It follows that

$$\|\nabla u_p\|_{L^p(\Omega)} \leq C^{\frac{1}{p-1}} \text{ for all } p \geq n + 1,$$

with  $C := C_\Omega \|f\|_{L^1(\Omega)}$  being independent of all  $p \geq n + 1$ .  $\square$

### 3.2. Weak convergence

As a consequence of the previous bound of the gradients, we obtain uniform convergence of  $u_p$  and weak convergence of the gradients  $\nabla u_p$ .

**Proposition 4.** *Let  $u_p$  be a unique solution to (9) with  $\int_\Omega u_p dx = 0$ . Then, up to a subsequence,  $u_p$  converges uniformly on  $\bar{\Omega}$  to a limit function  $u_\infty \in W^{1,\infty}(\Omega)$  and  $\nabla u_p \rightharpoonup \nabla u_\infty$  weakly in  $L^m(\Omega)$  as  $p \rightarrow \infty$  for any  $1 \leq m < \infty$ . Moreover, the limit function  $u_\infty$  is 1-Lipschitz, that is,*

$$|\nabla u_\infty(x)| \leq 1 \text{ for a.e. in } \Omega.$$

*Proof.* Fix any  $m > n$ . Let  $p^* = \frac{p}{m}$ . Using Hölder's inequality with  $p^*$  and  $q^*$  satisfying  $\frac{1}{p^*} + \frac{1}{q^*} = 1$  and Lemma 3, we obtain

$$\begin{aligned} \|\nabla u_p\|_{L^m(\Omega)} &\left( \int_\Omega |\nabla u_p|^m dx \right)^{\frac{1}{m}} \\ &\leq \left( \int_\Omega |\nabla u_p|^{mp^*} dx \right)^{\frac{1}{mp^*}} \left( \int_\Omega dx \right)^{\frac{1}{mq^*}} \\ &= \|\nabla u_p\|_{L^p(\Omega)} |\Omega|^{\frac{1}{m} - \frac{1}{p}} \\ &\leq |\Omega|^{\frac{1}{m} - \frac{1}{p}} C^{\frac{1}{p-1}} \end{aligned} \quad (15)$$

for all  $p \geq \max\{n + 1, m\}$ , where  $C$  is a constant independent of  $p$  from Lemma 3. Observe that  $|\Omega|^{\frac{1}{m} - \frac{1}{p}} C^{\frac{1}{p-1}} \rightarrow |\Omega|^{\frac{1}{m}}$  as  $p \rightarrow \infty$ . Hence, the sequence of gradients  $\nabla u_p$  is bounded in  $L^m(\Omega)$  and so is  $\{u_p\}$  in  $W^{1,m}(\Omega)$  ( $u_p$  is of mean value zero).

Therefore, up to a subsequence,  $u_p$  converges uniformly on  $\bar{\Omega}$  to a limit function  $u_\infty$  and  $\nabla u_p \rightharpoonup \nabla u_\infty$  weakly in  $L^m(\Omega)$  as  $p \rightarrow \infty$ . Obviously, the weak convergence of  $\nabla u_p$  also holds true in  $L^m(\Omega)$  for any  $1 \leq m \leq n$ . Finally, taking the limit as  $p \rightarrow \infty$  in (15), we arrive to

$$\|\nabla u_\infty\|_{L^m(\Omega)} \leq |\Omega|^{\frac{1}{m}}. \quad (16)$$

Letting  $m \rightarrow \infty$  we obtain  $\|\nabla u_\infty\|_{L^\infty(\Omega)} \leq 1$ , which completes the proof.  $\square$

### REFERENCES

1. R. A. Adams and J.F. Fournier. *Sobolev spaces*, 2<sup>nd</sup> edition, Elsevier, Amsterdam, 2012.
2. H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2010.
3. G. Leoni. *A First Course in Sobolev Spaces*, 2<sup>nd</sup> edition, American Mathematical Society, Providence-Rhode Island, 2017.
4. W. Rudin. *Real and Complex Analysis*, 3<sup>rd</sup> edition, McGraw-Hill International Edition, Singapore, 1987.
5. G. Ercole and G. A. Pereira. An optimal pointwise Morrey-Sobolev inequality, *Journal of Mathematical Analysis and Applications*, **2020**, 489, Art. 124143.
6. L. C. Evans. *Partial Differential Equations*, 2<sup>nd</sup> edition, American Mathematical Society, Providence-Rhode Island, 2010.
7. J. Mazon, J. Rossi, and J. Toledo. Mass transport problems obtained as limits of  $p$ -Laplacian type problems with spatial dependence, *Advances in Nonlinear Analysis*, **2014**, 3(3), 133–140.
8. D. McDuff and D. Salamon. *J-holomorphic Curves and Symplectic Topology*, 2<sup>nd</sup> edition, American Mathematical Society, Providence-Rhode Island, 2012.
9. J. Weber. *Introduction to Sobolev Spaces*, Lecture Notes, Unicamp, 2018.