TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Một số tiêu chuẩn mới về ổn định hóa đối với hệ tuyến tính từng phần hai mô hình có trễ thời gian

Nguyễn Hoàng Tường Vy[1], **Lê Quang Thuận**[2,*]

[1] *Bộ môn Công nghệ thông tin, Trường Đại học FPT, Campus Quy Nhơn, Việt Nam*
[2] *Khoa Toán và Thống kê, Trường Đại học Quy Nhơn, Việt Nam*

Ngày nhận bài: 24/07/2024; Ngày sửa bài: 31/10/2024;
Ngày nhận đăng: 04/11/2024; Ngày xuất bản: 28/02/2025

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu tính ổn định và ổn định hóa của các hệ tuyến tính từng phần hai mô hình có trễ thời gian bằng cách sử dụng các phiếm hàm Lyapunov-Krasovskii trơn. Những đóng góp mới của bài báo bao gồm: (1) thiết lập một tiêu chuẩn ổn định mới dựa trên phiếm hàm Lyapunov-Krasovskii trơn để đảm bảo tính ổn định tiệm cận của hệ điều khiển trong trường hợp không có điều khiển đầu vào và (2) đề xuất một điều kiện đủ cho sự tồn tại một điều khiển ngược tuyến tính trạng thái để ổn định tiệm cận hệ thống khi có điều khiển đầu vào. Cuối cùng, một số ví dụ số được chọn lọc để minh họa cho tính hiệu quả của phương pháp đã đề xuất.

Từ khóa: *Hệ tuyến tính từng phần có trễ thời gian, phiếm hàm Lyapunov-Krasovskii bậc hai từng phần, ổn định hóa hệ hai mô hình.*

*Tác giả liên hệ chính.
Email: lequangthuan@qnu.edu.vn*

QUY NHON UNIVERSITY
JOURNAL OF SCIENCE

New stabilization criteria for time-delayed bimodal piecewise linear systems

Nguyen Hoang Tuong Vy[1], **Le Quang Thuan**[2,*]

[1]*Information Technology Division, FPT University, Quy Nhon Campus, Vietnam*
[2]*Department of Mathematics and Statistics, Quy Nhon University, Vietnam*

Received: 24/07/2024; Revised: 31/10/2024;
Accepted: 04/11/2024; Pubished: 28/02/2025;

ABSTRACT

In this paper, we study the stability and stabilization of time-delayed bimodal piecewise linear systems via smooth Lyapunov-Krasovskii functionals. The main contributions of the paper are twofold: (1) a new stability criterion based on the proposed smooth Lyapunov-Krasovskii functional is derived to guarantee asymptotic stability in the case of zero inputs and (2) an interesting condition is proposed to design linear state feedback controllers to stabilize the system which is less conservative than those previously reported in the literature. Finally, some numerical examples illustrate the effectiveness of proposed methods.

Keywords: *Time-delayed piecewise linear systems, piecewise quadratic Lyapunov-Krasovskii functionals, bimodal system stabilization.*

1. INTRODUCTION

In recent decades, piecewise affine (PWA) systems have received much attention in the field of system and control theory. Each PWA system can be seen as a switching one that is characterized by a finite collection of affine time-invariant dynamics together with a state-dependent switching law that is ruled by a polyhedral partition of the state space.[1] PWA systems also form an important subclass of hybrid systems and they can be found in several engineering applications: power converters, robotics, relay control systems, etc. PWA systems are also interesting models to be used for approximating complex nonlinear dynamics. Analysis and design of PWA systems are therefore important as a first step to establish hybrid control theory.

Among the fundamental problems of system theory, the issues concerning stability and stabilization of PWA systems have been intensively studied for both cases: without and with time delays. For the first case, these problems are well-studied such as for general vector fields in the papers,[2,3] and for continuous ones in the papers.[4–6] With the appearance of time delays, there have been also existed many works developed over the past years, for instance, in the papers.[7,8] In the paper,[7] the authors investigated a class of piecewise time-delayed systems by using piecewise quadratic functions to derive stability criteria in term of LMIs and matrix equations. However, with these employed results, one can not solve the issue of state feedback controllers design to stabilize the systems. The paper[8] has proposed a method to design a piecewise linear state feedback controller to make the closed-loop system asymptotically stable. In this research direction, there are some restrictions. The first one is that from the LMIs combined with matrix equations that guarantees the continuity of Lyapunov functions for stability it is difficult to develop results about feedback controller designs. The second one is that the system under consideration requires non-Zeno behaviors. Note that the non-Zeno property has been established for continuous piecewise affine sys-

**Corresponding author.*
Email: lequangthuan@qnu.edu.vn

https://doi.org/10.52111/qnjs.2025.19102

tems without time-delays in the papers.[9–11] However, one can not obtain similar results for the case of time-delayed PWA systems. Therefore, checking for non-Zenoness of time-delayed PWA systems becomes an impossible task.

Motivated by the above mentioned challenges, we study the stabilization of time-delayed bimodal piecewise linear systems in this paper. Our approach also uses piecewise quadratic Lyapunov-Krasovskii functionals. However, the functionals are developed such that asymptotic stability works for more general solution concepts, *i.e.*, Carathéodory solutions, and the obtained LMIs can be employed to design a linear state feedback controller to stabilize the system. The main contributions are that by employing the special structure of such a functional, stability criteria will be derived for continuous bimodal time-delayed piecewise linear systems. Moreover, the derived LMIs can be employed to design a linear state feedback controller preserving continuity and stabilizing the system. It is worth to mention that there are only a few papers studying the stabilization of PWA systems by linear state feedback controllers taking quadratic Lyapunov function.[4] Finally, our approach is therefore hopefully generalize for more general multi-modal piecewise affine systems.

The rest of this paper is organized as follows. Section 2 introduces notations and preliminaries. In Section 3, we introduce time-delayed bimodal piecewise linear systems and present related preliminaries. This will be followed by stating and proving the main results of stability issue and stabilization of continuous time-delayed PWL systems in Section 4. The proposed theoretical results are validated by numerical examples in Section 5, before concluding the paper in Section 6.

2. NOTATIONS AND PRELIMINARIES

Denoted by $\mathbb{R}$ the set of all real numbers, $\mathbb{R}_+$ the set of all non-negative real numbers, and $\mathbb{R}^n_+$ the set of all n-tuple non-negative real numbers. The notation $\mathbb{R}^{n\times m}$ denotes the set of all real $n \times m$ matrices and the transpose of a real matrix $M \in \mathbb{R}^{n\times m}$ is denoted by M^T. The notation $\mathrm{He}(M)$ stands for the matrix $M + M^T$. A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is said to be positive definite, writing $Q > 0$, if $x^T Qx > 0$ for all non-zero $x \in \mathbb{R}^n$. We write $Q < 0$ if $-Q > 0$. For a positive definite matrix Q, the notation $\lambda(Q)$ stands for its the maximum eigenvalue. For $\tau > 0$, $\mathrm{C}([-\tau, 0], \mathbb{R}^n)$ denotes the normed space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ endowed with the norm

$$\|\varphi\|_{\mathrm{C}} := \max\{\varphi(s) \mid s \in [-\tau, 0]\}.$$

Also, $\mathrm{C}^1([-\tau, 0], \mathbb{R}^n)$ denotes the space of continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$.

Next, we introduce the following auxiliary results.

Lemma 2.1. *Let $P_1, P_2 \in \mathbb{R}^{n\times n}$ be symmetric matrices. The piecewise quadratic function*

$$F(x) = \begin{cases} x^T P_1 x \text{ if } c^T x \leqslant 0, \\ x^T P_2 x \text{ if } c^T x \geqslant 0 \end{cases}$$

is

a) *continuous if and only if there exist $h \in \mathbb{R}^n$ such that*

$$P_2 = P_1 + hc^T + ch^T. \tag{1}$$

b) *continuously differentiable if and only if there exist $\gamma \in \mathbb{R}$ such that*

$$P_2 = P_1 + \gamma cc^T. \tag{2}$$

Proof. a) On the one hand, the function F is continuous if and only if the following implication holds

$$c^T x = 0 \implies x^T (P_1 - P_2)x = 0.$$

On the other hand, this implication is equivalent to the existence of $h \in \mathbb{R}^n$ such that

$$\begin{aligned} x^T (P_1 - P_2)x &= 2(c^T x)(h^T x) \\ &= x^T (hc^T + ch^T)x \end{aligned}$$

for all $x \in \mathbb{R}^n$, *i.e.*, $P_1 - P_2 = hc^T + ch^T$. Therefore, F is continuous if and only if (1) holds for some $h \in \mathbb{R}^n$.

b) It can be seen that F continuously differentiable if and only if

$$c^T x = 0 \implies P_1 x = P_2 x. \tag{3}$$

Thus, it suffices to show the equivalence between (3) and (2). The implication "(2) ⇒ (3)" is obvious. To prove its converse, the condition (3) implies the existence of $p \in \mathbb{R}^n$ such that

$$P_1 - P_2 = pc^T.$$

Then, due to the symmetry of P_1, P_2, we further get $pc^T = cp^T$ and hence

$$p = \frac{(cp^T)c}{c^T c} = \frac{c(p^T c)}{c^T c} = \frac{p^T c}{c^T c} c = \gamma c$$

where $\gamma = \dfrac{p^T c}{c^T c} \in \mathbb{R}$. □

https://doi.org/10.52111/qnjs.2025.19102

Lemma 2.2. *For any vectors* $x, y \in \mathbb{R}^n$ *and positive definite matrix* $P \in \mathbb{R}^{n\times n}$*, one has*

$$2x^T y \leqslant x^T P x + y^T P^{-1} y.$$

Lemma 2.3. *Let* $X, Y \in \mathbb{R}^{m\times n}$*. Let* $F \in \mathbb{R}^{m\times m}$ *be a symmetric positive definite matrix. Then, one has*

$$X^T FY + Y^T FX \leqslant \gamma X^T FX + \gamma^{-1} Y^T FY$$

for any $\gamma > 0$.

Proof. It follows from the fact that

$$\left(\sqrt{\gamma}X - \sqrt{\gamma^{-1}}Y\right)^T F\left(\sqrt{\gamma}X - \sqrt{\gamma^{-1}}Y\right) \geqslant 0,$$

due to the positive definite property of matrix F. □

The last lemma is well-known in the field of linear control theory called S-lemma that provides a characterization of symmetric positive definite matrices using Schur complements.

Lemma 2.4 [12]**.** *Let* $Q \in \mathbb{R}^{m\times m}, R \in \mathbb{R}^{k\times k}, S \in \mathbb{R}^{m\times k}$*. The following statements are equivalent.*

a) $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$*;*

b) $R > 0$ *and* $Q - SR^{-1}S^T > 0$*;*

c) $Q > 0$ *and* $R - S^T Q^{-1} S > 0$*.*

3. TIME-DELAYED BIMODAL PIECEWISE LINEAR SYSTEMS

Consider the time-delayed bimodal piecewise linear systems with inputs

$$\dot{x}(t) = \begin{cases} A_1 x(t) + A_d x(t-\tau) + Bu(t) \\ \qquad \text{if } c^T x(t) < 0, \\ A_2 x(t) + A_d x(t-\tau) + Bu(t) \\ \qquad \text{if } c^T x(t) \geqslant 0, \end{cases} \tag{4a}$$

$$x(s) = \varphi(s), s \in [-\tau, 0] \tag{4b}$$

where $x \in \mathbb{R}^n$ is the state and $\dot{x}(t)$ denotes its derivative with respect to time t, $u \in \mathbb{R}^m$ is the input, the positive number τ is the time delay, the matrices $A_1, A_2, A_d \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^n$ are given. The initial function $\varphi(s)$ is in $\mathrm{C}([-\tau, 0], \mathbb{R}^n)$. For this work, the right-hand side of (4a) is assumed to be continuous; equivalently, there exists $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = ec^T. \tag{5}$$

Definition 3.1. Consider the system (4) for a given continuous input $u \in \mathrm{C}(\mathbb{R}_+, \mathbb{R}^m)$. A continuous function $x : [-\tau, \infty) \to \mathbb{R}^n$ is said to be a solution of system (4) for the initial function $\varphi \in \mathrm{C}([-\tau, 0], \mathbb{R}^n)$ if $x(s) = \varphi(s), \forall s \in [-\tau, 0]$, x is differentiable on $(0, \infty)$ and satisfies (4a) for all $t \geqslant 0$.

Note that the existence and uniqueness of such a solution follow from the theory of non-homogeneous ordinary differential equations with continuous right-hand-sides. The corresponding solution is denoted by $x^u(t; \varphi)$. In the case that $u(t) \equiv 0$, it is simply denoted by $x(t; \varphi)$.

Remark 3.1. For $\tau = 0$, the system (4) boils down to bimodal PWA systems that is the main object studied in the paper[4] for stability and stabilization.

Definition 3.2. We consider the system (4) without inputs, *i.e.*, $u(t) \equiv 0$. The system (4) is said to be

a) stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|\varphi\|_{\mathrm{C}} < \delta \implies \|x(t)\| < \epsilon, \forall t \geqslant 0$.

b) asymptotically stable if it is stable and there is a positive number δ_1 such that

$$\|\varphi\|_{\mathrm{C}} < \delta_1 \implies \lim_{t\to\infty} \|x(t)\| = 0.$$

To study the stability of system (4), we will employ continuous functionals as follows.

Definition 3.3. A continuous function $w : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $w(0) = 0$ and $w(x) > 0$ for all $x \in \mathbb{R}^n$.

Definition 3.4. We say that a continuous functional $V : \mathrm{C}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ is positive definite if $V(0) = 0$ and there exists a positive-definite function $w : \mathbb{R}^n \to \mathbb{R}$ such that $w(\phi(0)) \leqslant V(\phi)$ for all $\phi \in \mathrm{C}([-\tau, 0], \mathbb{R}^n)$.

The following proposition yields a sufficient condition to ensure the asymptotic stability of system (4).

Proposition 3.2 [13]**.** Consider the system (4) without inputs. The system (4) is asymptotically stable if there exist a positive-definite functional $V(\phi)$ and a positive-definite function $w(x)$ such that the value of the functional along any selections x_t of solution $x(t)$ of the system is differentiable by t, and its time derivative satisfies the inequality

$$\frac{dV(x_t)}{dt} \leqslant -w(x(t)) \text{ for all } t \geqslant 0,$$

where $x_t(s) := x(t+s), s \in [-\tau, 0]$.

4. MAIN RESULTS

In this section, we will provide a novel method to design state feedback controllers for stabilizing system (4). To do so, a suitable class of smooth Lyapunov-Krasovskii functionals (LKF) is first introduced and discussed. By employing the proposed LKFs, a new criterion on asymptotic stability is derived only in term of linear matrix inequalities (LMIs). Then, these LMIs are used to design a linear state feedback controller to stabilize the system.

4.1. Lyapunov-Krasovskii functionals

In the literature of time-delayed piecewise affine systems, piecewise quadratic LKFs have been often used to study the stability of systems.[6–8] Such a functional is basically composed from two parts: a piecewise quadratic Lyapunov function and an integral functional defined on the space $C^1([-\tau,0],\mathbb{R}^n)$, with τ is the delay, as

$$V(\varphi) = V_1(\varphi(0)) + V_2(\varphi), \forall \varphi \in C^1([-\tau,0],\mathbb{R}^n), \tag{6}$$

where the functional $V_2 : C^1([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ is defined as

$$V_2(\varphi) = \int_{-\tau}^{0} \varphi^T(s)Q\varphi(s)ds + \int_{-\tau}^{0}\int_{\eta}^{0} \dot{\varphi}^T(s)R\dot{\varphi}(s)dsd\eta,$$

for some positive definite matrices Q, R and the quadratic piecewise Lyapunov function $V_1 : \mathbb{R}^n \to \mathbb{R}_+$ is defined corresponding with a given polyhedral subdivision $\{\mathcal{X}_i\}_{i=1}^k$ of $\mathbb{R}^n$; that is, $V_1(x) = x^T P_i x$ whenever $x \in \mathcal{X}_i$. The matrices P_i are often chosen in such a way that V_1 is positive definite and it is continuous across region boundaries[6–8]. In our point of view, the restriction when one uses this kind of Lyapunov functions is that the stability of solutions only can be applied for the systems whose trajectories do not have Zeno property. To our best of knowledge, checking for non-Zeno property of time-delayed bimodal pieceiwse linear systems is impossible since there is no available paper about the non-Zenoness of time-delayed piecewise linear systems. Therefore, in this work, we develop Lyapunov-Krasovskii functionals in two aspects: requirements that V_1 is continuously differentiable and relaxation on the integral functional that it has more general form of piecewise quadratic one. It turns out that such requirements impose certain relations on the involved matrices in the literature of bimodal piecewise linear systems.

4.2. Stability analysis

By employing the proposed smooth piecewise quadratic LKFs, we now establish a novel stability criterion presented in term of linear matrix inequalities.

Theorem 4.1. *For system* (4)*, suppose that there exist the symmetric positive definite matrices $P, Q, R \in \mathbb{R}^{n\times n}$ and $h \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$ such that the following statements hold*

$$P + \gamma cc^T > 0, Q + hc^T + ch^T > 0, \tag{7a}$$

$$\begin{cases} 2\tau A_d^{2T} R A_d^2 - Q < 0, \\ 2\tau A_d^{2T} R A_d^2 - (hc^T + ch^T + Q) < 0, \end{cases} \tag{7b}$$

$$\begin{bmatrix} \Phi_1 & \tau P \\ \tau P & -\tau R \end{bmatrix} < 0, \tag{7c}$$

$$\begin{bmatrix} \Phi_2 & \tau(P+\gamma cc^T) \\ \tau(P+\gamma cc^T) & -\tau R \end{bmatrix} < 0, \tag{7d}$$

where $\Phi_1 = \mathrm{He}(P(A_1+A_d)) + Q + 2\tau A_1^T A_d^T R A_d A_1$ *and* $\Phi_2 = \mathrm{He}\{(P+\gamma cc^T)(A_2+A_d)\} + Q + \mathrm{He}(hc^T) + 2\tau A_2^T A_d^T R A_d A_2$*. Then, the system* (4) *is asymptotically stable.*

Proof. Let us consider the piecewise quadratic Lyapunov-Krasovskii functional

$$V : C^1([-\tau,0],\mathbb{R}^n) \to \mathbb{R}$$

defined by

$$V(\varphi) = V_1(\varphi(0)) + V_2(\varphi),$$

where

$$V_1(z) = \begin{cases} z^T P_1 z \text{ if } c^T z \leqslant 0, \\ z^T P_2 z \text{ if } c^T z \geqslant 0, \end{cases}$$

with $P_1 = P, P_2 = P + \gamma cc^T$ and

$$V_2(\varphi) = \int_{-\tau}^{0} F(\varphi(s))ds + \int_{-\tau}^{0}\int_{\eta}^{0} \dot{\varphi}^T(s) A_d^T R A_d \dot{\varphi}(s)dsd\eta.$$

where

$$F(z) = \begin{cases} z^T Q_1 z \text{ if } c^T z \leqslant 0, \\ z^T Q_2 z \text{ if } c^T z \geqslant 0, \end{cases}$$

with $Q_1 = Q, Q_2 = Q + hc^T + ch^T$. Due to (7a) and Lemma 2.1, V_2 is continuous and positive. Thus, one has

$$V(\varphi) = V_1(\varphi(0)) + \int_{-\tau}^{0} F(\varphi(s))ds$$

$$+\int_{-\tau}^{0}\int_{\eta}^{0}\dot{\varphi}^T(s)A_d^TRA_d\dot{\varphi}(s)dsd\eta$$
$$\geqslant V_1(\varphi(0)), \forall\varphi\in C^1([-\tau,0],\mathbb{R}^n).$$

This inequality shows that the functional V is positive definite on $C^1([-\tau,0],\mathbb{R}^n)$.

For any initial function $\varphi \in C([-\tau,0],\mathbb{R}^n)$, let $x(t;\varphi)$ be the corresponding trajectory of system (4) and define $x_t(s) = x(t+s;\varphi), s\in[-\tau,0]$. Then, $x_t \in C^1([-\tau,0],\mathbb{R}^n)$ for each $t\geqslant\tau$ and by simple transformations we get

$$V(x_t) = V_1(x(t;\varphi)) + \int_{t-\tau}^{t}F(x(s;\varphi))ds$$
$$+\int_{-\tau}^{0}\int_{t+\eta}^{t}\dot{x}^T(s;\varphi)A_d^TRA_d\dot{x}(s;\varphi)dsd\eta$$

for all $t\geqslant\tau$. Using Newton-Leibnitz formula, one can verify that

$$\dot{x}(t;\varphi) = G(x(t,\varphi)) - A_d\int_{t-\tau}^{t}\dot{x}(s;\varphi)ds, \quad (8)$$

where

$$G(x) = \begin{cases}(A_1+A_d)x, c^Tx\leqslant 0\\(A_2+A_d)x, c^Tx\geqslant 0.\end{cases}$$

Observe that by Lemma 2.1 and due to (7a), V_1 is continuously differentiable with respect to x. On the other hand, $x(t) := x(t;\varphi)$ is continuously differentiable with respect to t. Therefore, $V(x_t)$ is continuously differentiable as a function of variable t defined on $[\tau,\infty)$ and its derivative is computed as

$$\frac{d}{dt}V(x_t) = \frac{d}{dt}\Bigg\{V_1(x(t)) + \int_{t-\tau}^{t}F(x(s))ds$$
$$+\int_{-\tau}^{0}\int_{t+\eta}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)dsd\eta\Bigg\}. \quad (9)$$

For the first term on the right-hand side of (9), by taking the derivative and employing (8), one gets

$$\frac{d}{dt}V_1(x(t)) = \left\langle\frac{\partial V_1}{\partial x},\dot{x}(t)\right\rangle$$
$$= \dot{x}(t)^TP_ix(t) + x^T(t)P_i\dot{x}(t)$$
$$= 2x^T(t)P_i(A_i+A_d)x(t) - 2x^T(t)P_iA_d\int_{t-\tau}^{t}\dot{x}(s)ds.$$

On the other hand, due to Lemma 2.2, we can estimate the following term

$$-2x^T(t)P_iA_d\int_{t-\tau}^{t}\dot{x}(s)ds$$
$$=\int_{t-\tau}^{t}-2x^T(t)P_iA_d\dot{x}(s)ds$$
$$\leqslant \tau x^T(t)P_iR^{-1}P_ix(t) + \int_{t-\tau}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)ds.$$

Thus,

$$\frac{d}{dt}V_1(x(t)) \leqslant x^T(t)\{(A_i+A_d)^TP_i + P_i(A_i+A_d)$$
$$+\tau P_iR^{-1}P_i\}x(t) + \int_{t-\tau}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)ds. \quad (10)$$

For the second term on the right-hand side of (9), we have

$$\frac{d}{dt}\int_{t-\tau}^{t}F(x(s))ds = F(x(t)) - F(x(t-\tau))$$
$$= x^T(t)Q_ix(t) - x^T(t-\tau)Q_jx(t-\tau) \quad (11)$$

for some $i,j\in\{1,2\}$. For the third term on the right-hand side of (9), we note that

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+\eta}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)dsd\eta$$
$$=\int_{-\tau}^{0}\left\{\frac{d}{dt}\int_{t+\eta}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)ds\right\}d\eta$$
$$=\int_{-\tau}^{0}\dot{x}^T(t)A_d^TRA_d\dot{x}(t)d\eta$$
$$-\int_{-\tau}^{0}\dot{x}^T(t+\eta)A_d^TRA_d\dot{x}(t+\eta)d\eta$$
$$=\tau\dot{x}^T(t)A_d^TRA_d\dot{x}(t) - \int_{t-\tau}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)ds$$

and

$$\dot{x}^T(t)A_d^TRA_d\dot{x}(t)$$
$$=(A_ix(t)+A_dx(t-\tau))^TA_d^TR$$
$$\times A_d(A_ix(t)+A_dx(t-\tau))$$
$$=x^T(t)A_i^TA_d^TRA_dA_ix(t)$$
$$+x^T(t-\tau)A_d^{2T}RA_d^2x(t-\tau)$$
$$+x^T(t)A_i^TA_d^TRA_dA_dx(t-\tau)$$
$$+x^T(t-\tau)A_d^TA_d^TRA_dA_ix(t)$$
$$\leqslant 2x^T(t)A_i^TA_d^TRA_dA_ix(t) + 2x^T(t-\tau)$$
$$\times A_d^{2T}RA_d^2x(t-\tau)$$

where the last inequality is due to Lemma 2.2. Therefore, one obtains

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+\eta}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)dsd\eta$$
$$\leqslant 2\tau x^T(t)A_i^TA_d^TRA_dA_ix(t) + 2\tau x^T(t-\tau)A_d^{2T}$$
$$\times RA_d^2x(t-\tau) - \int_{t-\tau}^{t}\dot{x}^T(s)A_d^TRA_d\dot{x}(s)ds. \quad (12)$$

A combination of (10), (11) and (12) yields

$$\frac{d}{dt}V(x_t) \leqslant \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \Pi_{11}^i & 0 \\ 0 & \Pi_{22}^j \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

where $\Pi_{22}^j = -Q_j + 2\tau A_d^{2T} R A_d^2$ and $\Pi_{11}^i = \mathrm{He}(P_i(A_i + A_d)) + \tau P_i R^{-1} P_i + Q_i + 2\tau A_i^T A_d^T R A_d A_i$. By Schur complement (Lemma 2.4) and the assumptions (7b)-(7d), we further obtain

$$\frac{d}{dt}V(x_t) \leqslant -\omega(x(t)), \forall t \geqslant \tau$$

where

$$\omega(x) = x^T \max\{-\lambda(\Pi_{11}^1), -\lambda(\Pi_{11}^2)\}x.$$

This fact together with Proposition 3.2 yields the asymptotic stability of system (4). ☐

4.3. Linear state feedback stabilization

An interesting application of stability conditions derived in Theorem 4.1 is that they can be employed to design a linear state feedback controller

$$u(t) = Kx(t) \tag{13}$$

that makes the following closed-loop system is asymptotically stable

$$\dot{x}(t) = \begin{cases} (A_1 + BK)x(t) + A_d x(t-\tau), \\ \qquad \text{if } c^T x(t) < 0 \\ (A_2 + BK)x(t) + A_d x(t-\tau), \\ \qquad \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{14a}$$

$$x(s) = \varphi(s), s \in [-\tau, 0]. \tag{14b}$$

Theorem 4.2. *Consider the system* (4). *Suppose that there exist symmetric positive definite matrices* $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathbb{R}^{n\times n}$, *a matrix* $U \in \mathbb{R}^{m\times n}$ *and scalars* $\gamma > 0, \mu > 0$ *such that*

$$\begin{bmatrix} -\tilde{\gamma}\tilde{R} & \tilde{R}^2 \\ \tilde{R}^2 & -\tilde{\gamma}\tilde{R} \end{bmatrix} < 0 \tag{15}$$

and the following statements hold

$$\begin{bmatrix} -\tilde{Q} & \tilde{P}c \\ c^T\tilde{P} & -\mu \end{bmatrix} < 0, \tag{16a}$$

$$\begin{bmatrix} -\tilde{Q} & \sqrt{2\tau}\tilde{P}A_d^{2T} & \tilde{P}c \\ \sqrt{2\tau}A_d^2\tilde{P} & -\tau\tilde{R} & 0 \\ c^T\tilde{P} & 0 & -\mu \end{bmatrix} < 0, \tag{16b}$$

$$\begin{bmatrix} \Psi_1 & \sqrt{2\tau}\Gamma_1^T A_d^T \\ \sqrt{2\tau}A_d\Gamma_1 & -\tau\tilde{R} \end{bmatrix} < 0, \tag{16c}$$

$$\begin{bmatrix} \Psi_2 & \sqrt{2\tau}\Gamma_2^T A_d^T & \tau\tilde{P}cc^T & \Delta \\ \sqrt{2\tau}A_d\Gamma_2 & -\tau\tilde{R} & 0 & 0 \\ cc^T\tilde{P} & 0 & -\tau\tilde{R} & 0 \\ \Delta^T & 0 & 0 & \Sigma \end{bmatrix} < 0, \tag{16d}$$

where $\Psi_i = \mathrm{He}\{(A_i + A_d)\tilde{P} + BU\} + \tilde{Q} + \tau\tilde{R}, \Gamma_i = A_i\tilde{P} + BU$, $\Sigma = \mathrm{diag}(-\tilde{\gamma}I, -\tilde{\gamma}I, -\tau\tilde{\gamma}, -\tau\tilde{\gamma})$, *and*

$$\Delta = \begin{bmatrix} \tilde{P}(A_2 + A_d)^T + U^T B^T & \tilde{P}cc^T & \tau\tilde{R}c & \tau\tilde{P}c \end{bmatrix}.$$

Then, there exists a linear state feedback controller $u(t) = Kx(t)$ *such that the closed-loop system* (14) *is asymptotically stable.*

Proof. Define $P := \tilde{P}^{-1} > 0$, $Q := \tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1} > 0$, $R := \tilde{R}^{-1} > 0$, $K := U\tilde{P}^{-1}$ and $h := -c/(2\mu)$. We prove that the matrices P, Q, R, h together with the scalar $\tilde{\gamma}^{-1}$ fulfill the conditions of Theorem 4.1, in the framework of closed-loop system (14).

1) First, it is obvious that $P + \tilde{\gamma}^{-1}cc^T > 0$ since $P > 0$ and $\tilde{\gamma} > 0$. Next, one has

$$\begin{aligned} Q + hc^T + ch^T &= Q - \mu^{-1}cc^T \\ &= \tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1} - \mu^{-1}cc^T \\ &= \tilde{P}^{-1}(\tilde{Q} - \mu^{-1}\tilde{P}cc^T\tilde{P})\tilde{P}^{-1} > 0 \end{aligned}$$

due to (16a) and Schur complement.

2) To verify the claim (7b), note that it follows from (16b)

$$-\tilde{Q} - \begin{bmatrix} \sqrt{2\tau}\tilde{P}A_d^{2T} & \tilde{P}c \end{bmatrix} \times \begin{bmatrix} -\tau\tilde{R} & 0 \\ 0 & -\mu \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{2\tau}\tilde{P}A_d^{2T} & \tilde{P}c \end{bmatrix}^T < 0$$

or equivalently

$$-\tilde{Q} + 2\tau\tilde{P}A_d^{2T}\tilde{R}^{-1}A_d^2\tilde{P} + \mu^{-1}\tilde{P}cc^T\tilde{P} < 0.$$

By pre-multiplying and post-multiplying by $\tilde{P}^{-1}$ in the above inequality, one gets

$$\begin{aligned} &2\tau\tilde{P}A_d^{2T}\tilde{R}^{-1}A_d^2\tilde{P} - Q - hc^T - ch^T \\ &\quad = 2\tau\tilde{P}A_d^{2T}\tilde{R}^{-1}A_d^2\tilde{P} - Q + \mu^{-1}cc^T < 0. \end{aligned}$$

Note that $\mu^{-1}cc^T > 0$, the above inequality also implies that $2\tau\tilde{P}A_d^{2T}\tilde{R}^{-1}A_d^2\tilde{P} - Q < 0$. The claim (7b) is verified.

3) Next, we verify the claim (7c). Due to (16c), we have

$$\begin{aligned} &\mathrm{He}\{(A_1 + A_d)\tilde{P} + BU\} + \tilde{Q} + \tau\tilde{R} \\ &+ 2\tau(A_1\tilde{P} + BU)^T A_d^T\tilde{R}^{-1}A_d(A_1\tilde{P} + BU) < 0 \end{aligned}$$

Substituting $U = K\tilde{P}$ and then pre-multiplying and post-multiplying by $\tilde{P}^{-1}$ in the obtained inequality, one gets

$$P(A_1 + BK + A_d) + (A_1 + BK + A_d)^T P + Q + 2\tau(A_1+BK)^T A_d^T RA_d(A_1+BK) + \tau PR^{-1}P < 0.$$

This is equivalent to (7c) in the context of closed-loop system. The claim (7c) is verified.

4) Finally, we verify the claim (7d). Note that the (16d) implies

$$\Psi_2 - \begin{bmatrix}\sqrt{2}\tau\Gamma_2^T A_d^T & \tau\tilde{P}cc^T & \Delta\end{bmatrix} \begin{bmatrix}-\tau\tilde{R} & 0 & 0\\ 0 & -\tau\tilde{R} & 0\\ 0 & 0 & \Sigma\end{bmatrix}^{-1} \times \begin{bmatrix}\sqrt{2}\tau\Gamma_2^T A_d^T & \tau\tilde{P}cc^T & \Delta\end{bmatrix}^T < 0$$

or equivalently

$$\begin{aligned}&\mathrm{He}\{(A_1 + A_d)\tilde{P} + BU\} + \tilde{Q} + \tau\tilde{R}\\ &+ 2\tau\Gamma_2^T A_d^T \tilde{R}^{-1} A_d \Gamma_2 + \tau\tilde{P}cc^T\tilde{R}^{-1}\tilde{P}cc^T\\ &+ (\tilde{P}(A_2 + A_d)^T + U^T B^T)\tilde{\gamma}^{-1}((A_2 + A_d)\tilde{P} + BU)\\ &+ \tilde{P}cc^T\tilde{\gamma}^{-1}cc^T\tilde{P} + \tau\tilde{R}c\tilde{\gamma}^{-1}c\tilde{R} + \tau\tilde{P}c\tilde{\gamma}^{-1}c\tilde{P} < 0.\end{aligned} \tag{17}$$

Note that in the context of closed-loop system, the LMI

$$\begin{bmatrix}\Phi_2 & \tau(P + \tilde{\gamma}^{-1}cc^T)\\ \tau(P + \tilde{\gamma}^{-1}cc^T) & -\tau R\end{bmatrix} < 0, \tag{18}$$

is equivalent to

$$\begin{aligned}&\mathrm{He}\{(P+\tilde{\gamma}^{-1}cc^T)(A_2+BK+A_d)\} + Q - \mu^{-1}cc^T\\ &+ 2\tau(A_2 + BK)^T A_d^T RA_d(A_2 + BK)\\ &+ \tau(P + \tilde{\gamma}^{-1}cc^T)R^{-1}(P + \tilde{\gamma}^{-1}cc^T) < 0.\end{aligned}$$

By pre- and post-multiplying by P^{-1}, the above inequality is equivalent to

$$\begin{aligned}&(A_2 + BK + A_d)P^{-1} + P^{-1}(A_2 + BK + A_d)^T\\ &+ \tilde{\gamma}^{-1}P^{-1}cc^T(A_2 + BK + A_d)P^{-1}\\ &+ \tilde{\gamma}^{-1}P^{-1}(A_2 + BK + A_d)^T cc^T P^{-1} + P^{-1}QP^{-1}\\ &+ 2\tau P^{-1}(A_2^T + K^T B^T)A_d^T RA_d(A_2 + BK)P^{-1}\\ &- \mu^{-1}P^{-1}cc^T P^{-1} + \tau R^{-1} + \tau\tilde{\gamma}^{-1}P^{-1}cc^T R^{-1}\\ &+ \tau\tilde{\gamma}^{-1}R^{-1}cc^T P^{-1}\\ &+ \tau P^{-1}cc^T\tilde{\gamma}^{-2}R^{-1}cc^T P^{-1} < 0.\end{aligned} \tag{19}$$

On the other hand, applying Lemma 2.3, we have

$$\begin{aligned}&P^{-1}cc^T(A_2 + BK + A_d)P^{-1}\\ &+ P^{-1}(A_2 + BK + A_d)^T cc^T P^{-1}\\ &\leqslant P^{-1}cc^T cc^T P^{-1}\\ &+ P^{-1}(A_2 + BK + A_d)^T(A_2 + BK + A_d)P^{-1},\end{aligned}$$

and

$$P^{-1}cc^T R^{-1} + R^{-1}cc^T P^{-1} \leqslant P^{-1}cc^T P^{-1} + R^{-1}cc^T R^{-1}.$$

Moreover, the LMI (15) yields $\tilde{\gamma}^{-2}R^{-1} \leqslant R$. Therefore, the inequality (19) holds if the following one fulfills

$$\begin{aligned}&(A_2 + BK + A_d)P^{-1} + P^{-1}(A_2 + BK + A_d)^T\\ &+ \tilde{\gamma}^{-1}P^{-1}cc^T cc^T P^{-1} + P^{-1}QP^{-1} - \mu^{-1}P^{-1}cc^T P^{-1}\\ &+ \tilde{\gamma}^{-1}P^{-1}(A_2 + BK + A_d)^T(A_2 + BK + A_d)P^{-1}\\ &+ 2\tau P^{-1}(A_2^T + K^T B^T)A_d^T RA_d(A_2 + BK)P^{-1}\\ &+ \tau R^{-1} + \tau\tilde{\gamma}^{-1}P^{-1}cc^T P^{-1} + \tau\tilde{\gamma}^{-1}R^{-1}cc^T R^{-1}\\ &+ \tau P^{-1}cc^T Rcc^T P^{-1} < 0.\end{aligned}$$

Note that the later inequality is followed from (17) where $P = \tilde{P}^{-1} > 0$, $R = \tilde{R}^{-1} > 0$, $Q = \tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1} > 0$. The proof of claim (7d) is done. □

Without time-delays, we get the corollary.

Corollary 4.3. *Consider system* (4) *with* $\tau = 0$. *Suppose that there exist a positive definite symmetric matrix* $\tilde{P} \in \mathbb{R}^{n\times n}$, *a matrix* $U \in \mathbb{R}^{m\times n}$ *and scalar* $\tilde{\gamma} > 0$ *such that*

$$\mathrm{He}((A_1 + A_d)\tilde{P} + BU) < 0,$$

and

$$\begin{bmatrix}\Psi & \Delta & \tilde{P}cc^T\\ \Delta^T & -\tilde{\gamma}I & 0\\ cc^T\tilde{P} & 0 & -\tilde{\gamma}I\end{bmatrix} < 0,$$

where $\Psi = \mathrm{He}((A_2 + A_d)\tilde{P} + BU)$, *and* $\Delta = \tilde{P}(A_2 + A_d)^T + U^T B^T$. *Then, there exists a linear state feedback controller* $u(t) = Kx(t)$ *such that the closed-loop system* (14) *is asymptotically stable.*

Remark 4.4. Our developed results can be applied to discontinuous time-delayed bimodal piecewise linear systems with inputs. In fact, for such systems, we may employ a state feedback controller as

$$u(t) = Kx(t) + \begin{cases}K_1 x(t) \text{ nếu } c^T x(t) \leqslant 0\\ K_2 x(t) \text{ nếu } c^T x(t) \geqslant 0,\end{cases}$$

where the gains K_1, K_2 are first designed in such a way that the closed-loop system is continuous, *i.e.*, satisfying

$$(A_1 - A_2) + B(K_1 - K_2) = hc^T$$

https://doi.org/10.52111/qnjs.2025.19102

for some $h \in \mathbb{R}^n$ and the gain K is designed to stabilize the system

$$\dot{x}(t) = \begin{cases} (\tilde{A}_1 + BK)x(t) + A_d x(t-\tau) \\ \qquad\qquad \text{if } c^T x(t) \leqslant 0 \\ (\tilde{A}_2 + BK)x(t) + A_d x(t-\tau) \\ \qquad\qquad \text{if } c^T x(t) \geqslant 0 \end{cases} \quad (20)$$

with $\tilde{A}_i := A_i + BK_i,\ i = 1, 2.$

5. NUMERICAL EXAMPLES

In this section, we present two numerical examples to illustrate the effectiveness of the proposed stabilization conditions for both cases: with and without time delays. The first example is considered as a bimodal piecewise linear system with time delays. In the second example, we collect a bimodal piecewise linear system without time delays.

Example 5.1. Consider the planar time-delayed bimodal piecewise linear system

$$\dot{x}(t) = \begin{cases} A_1 x(t) + A_d x(t-\tau) + bu(t), \\ \qquad\qquad c^T x(t) \leqslant 0 \\ A_2 x(t) + A_d x(t-\tau) + bu(t), \\ \qquad\qquad c^T x(t) \geqslant 0 \end{cases} \quad (21)$$

where $c^T = \begin{bmatrix} -1 & 2 \end{bmatrix}, b^T = \begin{bmatrix} 0.3 & 0 \end{bmatrix}$ and

$$A_1 = \begin{bmatrix} -2 & -4 \\ 2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}.$$

For $u(t) \equiv 0$ and $\tau = 0$, the system (21) is not asymptotically stable as shown in Fig. 1,

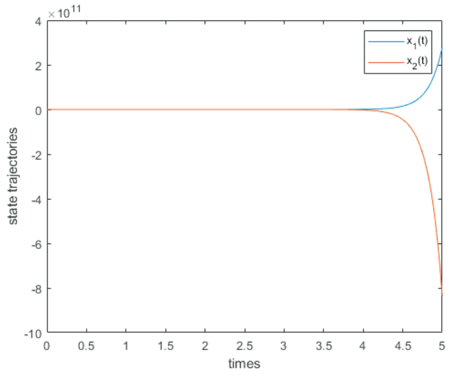

Figure 1. Trajectories of system (21) for $\tau = 0, u(t) = 0$ and starting at $x^0 = (-3, 3)^T$.

For $\tau = 0.025$, we would stabilize the system by using linear state feedback controller. To do so, we find the matrices $\tilde{P}, \tilde{Q}, \tilde{R}$, the numbers $\tilde{\gamma}, \mu$ satisfying (15) and LMIs (16a), (16b), (16c), (16d) of Theorem 4.2. Note that condition (15) is not an LMI. However, we can take $\tilde{R} = 5I_2$. Then, it is an LMI in $\tilde{\gamma}$. Solving the LMIs, we get $K = U\tilde{P}^{-1} = \begin{bmatrix} -1513.4 & 5681.6 \end{bmatrix}$.

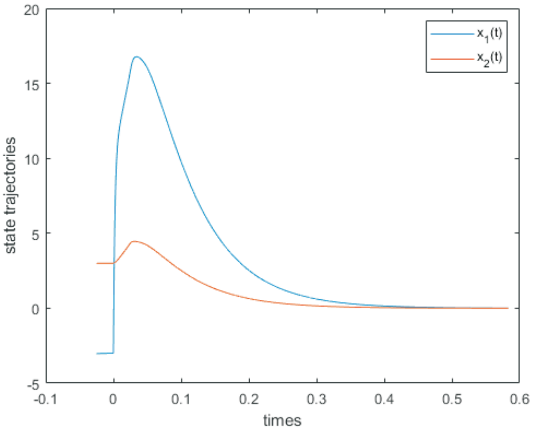

Figure 2. Trajectories of the closed-loop system between system (21) and controller (13).

Figure 2 shows the trajectories of the closed-loop system that is composed from the system (21) and the state feedback controller $u(t) = Kx(t) = \begin{bmatrix} -1513.4 & 5681.6 \end{bmatrix} x(t)$ for initial function

$$\varphi(t) = \begin{bmatrix} -3 + \sin t \\ 2 + \cos t \end{bmatrix}, \ t \in [-0.025, 0],$$

This trajectory asymptotically converges to the origin.

In the rest of this paper, we validate our method to stabilize a practical bimodal piecewise linear system without time-delays that appeared in the work,[14] Example 21, and compare our achievements with the available methods in[14].

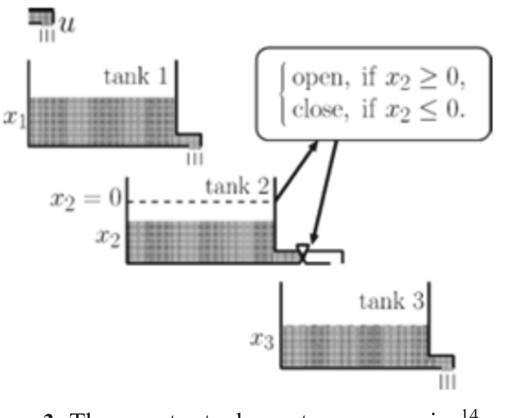

Figure 3. Three water tanks system, source in.[14]

Example 5.2 [(14)]**.** Consider a three water tanks system as illustrated in Fig. 3 that is taken from the paper.[14] Let x_i be the water level of tank i, $(i = 1, 2, 3)$, and u be the volume of water discharged into tank 1. The valve at tank 2 is open if $x_2 \geqslant 0$ and closed if $x_2 < 0$. For simplicity, all coefficients are normalized to 1. Then, dynamic equation of the system in the neighborhood of origin is

$$\dot{x}(t) = \begin{cases} A_1 x(t) + bu(t) \text{ if } c^T x(t) \leqslant 0 \\ A_2 x(t) + bu(t) \text{ if } c^T x(t) \geqslant 0 \end{cases} \tag{22}$$

where $c^T = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, b^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Without input, *i.e.*, $u(t) = 0$, the origin is not asymptotically stable.[14] Based on the theory developed in,[14] a state feedback controller has been derived to stabilize the system using piecewise linear functions as follows

$$u(t) = \begin{cases} K_1 x(t) \text{ if } c^T x(t) \leqslant 0 \\ K_2 x(t) \text{ if } c^T x(t) \geqslant 0 \end{cases}$$

with $K_1 = \begin{bmatrix} -1 & -2 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & -2 & 0 \end{bmatrix}$. In fact, such controller transforms a continuous bimodal system into a discontinuous closed-loop one, but it is still well-posed in the sense of Carathéodory solutions.[15] It worth to mention that one does not get such lucky in general.

Taking our proposed approach, we solve the involved LMIs of Corollary 4.3 and get the matrices $U = \begin{bmatrix} -0.2548 & -0.4668 & -0.0143 \end{bmatrix}$ and

$$\tilde{P} = \begin{bmatrix} 0.2690 & -0.2156 & -0.0517 \\ -0.2156 & 0.3749 & 0.0513 \\ -0.0517 & 0.0513 & 0.6138 \end{bmatrix}.$$

Then, the gain K of linear state feedback controller is

$$K = U\tilde{P}^{-1} = \begin{bmatrix} -3.6140 & -3.3159 & -0.0504 \end{bmatrix}.$$

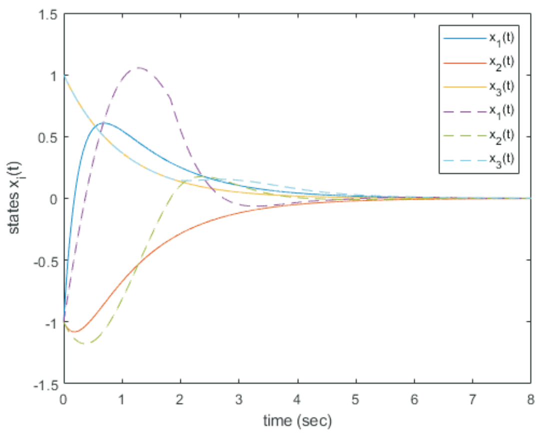

Figure 4. Trajectories of the closed-loop three tanks system with two kinds of controllers.

Figure 4 shows the trajectories of closed-loop systems for the initial state $x(0) = \text{col}(-1, -1, 1)$ that is composed from system (22) and state feedback controllers: the linear state feedback controller $u(t) = Kx(t)$ (solid lines) and piecewise linear state feedback controller (dash lines)

$$u(t) = \begin{cases} K_1 x(t) \text{ if } c^T x(t) \leqslant 0 \\ K_2 x(t) \text{ if } c^T x(t) \geqslant 0. \end{cases}$$

Our controller yields a better stabilization.

6. CONCLUSIONS

In this paper, we studied the stability and stabilization of time-delayed bimodal piecewise linear systems via smooth Lyapunov-Krasovskii functionals. The main contributions of the paper are including: (1) new stability criteria based on the proposed smooth Lyapunov-Krasovskii functional were derived to guarantee asymptotic stability in the zero inputs and (2) an interesting condition was established to design a linear state feedback controller to stabilize the system which is less conservative than before in the literature. Finally, some numerical examples illustrate the effectiveness of proposed methods.

Acknowledgment

This research is conducted within the framework of science and technology projects at institutional level of Quy Nhon University under the project code T2024.831.02.

REFERENCES

1. L. Q. Thuan, M. K. Camlibel. Controllability and stabilizability of a class of continuous piecewise affine dynamical systems, *SIAM Journal on Control and Optimization*, **2014**, *52*(3), 1914-1934.
2. M. Hovd, S. Olaru. Relaxing PWQ Lyapunov stability criteria for PWA systems, *Automatica*, **2013**, *49*(2), 667-670.
3. Z. Sun. Stability of piecewise linear systems revisited, *Annual Reviews in Control*, **2010**, *34*(2), 221-231.
4. Y. Eren, J. Shen, M. K. Camlibel. Quadratic stability and stabilization of bimodal piecewise linear systems, *Automatica*, **2014**, *50*, 1444-1450.
5. R. Iervolino, F. Vasca, L. Iannelli. Cone-copositive piecewise quadratic Lyapunov functions for conewise linear systems, *IEEE Transactions on Automatic Control*, **2015**, *60*(11), 3077-3082.
6. R. Iervolino, D. Tangredi, F. Vasca. Lyapunov stability for piecewise affine systems via cone-copositivity, *Automatica*, **2017**, *81*, 22-29.
7. V. Kulkarni, M. Jun, J. Hespanha. *Piecewise quadratic Lyapunov functions for piecewise affine time-delay systems*, in Proceeding of the 2004 American Control Conference, Boston, Massacusetts, 2004.
8. X. Song, S. Zhou, W.X. Zheng. *Delay-dependent stabilization for continuous-time piecewise time-delay systems*, in Proceeding of the 6^{th} World Congress on Intelligent Control and Automation, Dalian, China, 2006.
9. J. Shen. Robust non-Zenoness of piecewise affine systems with applications to linear complementarity systems, *SIAM Journal on Optimization*, **2014**, *24*(4), 2023-2056.
10. L. Q. Thuan. Non-Zenoness of piecewise affine dynamical systems and affine complementarity systems with inputs, *Control Theory Technology*, **2014**, *12*, 35–47.
11. L. Q. Thuan, M. K. Camlibel. Continuous piecewise affine dynamical systems do not exhibit Zeno behavior, *IEEE Transactions on Automatic Control*, **2011**, *56*(8), 1932–1936.
12. S. Boyd, L. Vandenberghe. *Convex optimization*, Cambridge University Press, 2004.
13. V. L. Kharitonov. *Time-delay systems: Lyapunov functionals and matrices*, Springer, New York, 2013.
14. Y. Iwatani, S. Hara. Stability tests and stabilization for piecewise linear systems based on poles and zeros of subsystems, *Automatica*, **2006**, *42*, 1685–1695.
15. J. Imura, A. V. D. Schaft. Characterization of well-posedness of piecewise-linear systems, *IEEE Transactions on Automatic Control*, **2000**, *45*(9),1600-1619.
16. J. K. Hale. *Theory of functional differential equations*, Springer New York, NY, 2012.

https://doi.org/10.52111/qnjs.2025.19102