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Chùm quỹ đạo tuần hoàn của một tự đẳng cấu hyperbolic trên xuyến $\mathbb{T}^2$

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TÓM TẮT

Bài báo nghiên cứu chùm quỹ đạo tuần hoàn của tự đẳng cấu T_A trên xuyến $\mathbb{T}^2$ cảm sinh bởi ma trận $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Chúng tôi chứng minh T_A thỏa mãn tiêu đề A. Chùm quỹ đạo tuần hoàn của T_A được nghiên cứu thông qua khái niệm '*p-gần nhau*' giữa các dãy tuần hoàn của hệ động lực ký tự tương ứng. Chúng tôi cũng đưa ra số chùm tuần hoàn các dãy tuần hoàn có chu kỳ cho trước trong trường hợp *p-gần* nhau.

Từ khóa: *Chùm quỹ đạo tuần hoàn, tự đẳng cấu hyperbolic, động lực học ký tự, xuyến 2 chiều.*

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Clustering of periodic orbits of a hyperbolic automorphism on the torus $\mathbb{T}^2$

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ABSTRACT

This paper deals with clustering of periodic orbits of the hyperbolic toral automorphism induced by matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. We prove that T_A satisfies the Axiom A. The clustering of periodic orbits of T_A is investigated via the notion of 'p-closeness' of periodic sequences of the respective symbolic dynamical system. We also provide the number of clusters of periodic sequences with given periods in the case of 2-closeness.

Keywords: *Clustering of periodic orbits, hyperbolic toral automorphism, symbolic dynamics, 2-torus*

1. INTRODUCTION

Symbolic dynamics is a powerful tool to investigate general dynamical systems. A dynamical system having a Markov partition will be represented as a symbolic dynamical system, which is the shift map on a set of bi-infinite sequences of symbols.

The construction of Markov partition for Axiom A diffeomorphisms given by R. Bowen[1] has many applications. Working on symbolic dynamics has several advantages since the theory of symbolic dynamics is almost complete.[2] Furthermore, since subshift of finite types are related to adjacency matrices and digraphs, we have many choices of tools to work on symbolic dynamics.

Clustering of periodic orbits is a beautiful phenomenon. B. Gutkin and V.A. Osipov[3] show that periodic orbits of the baker's map form clusters and have hierarchical structures, using symbolic dynamics. The corresponding symbolic dynamics of the baker's map is trivial with no grammar rule, i.e., each symbol can be followed by any other symbols.

In this paper, we consider the automorphism on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. We prove that T_A is an Axiom A diffeomorhism[4] and introduce symbolic dynamics provided by L. Barreira.[5] The corresponding symbolic dynamics is a subshift of finite type. The adjacency matrix is a 5×5 matrix with entries 0 and 1. This means that the respective symbolic dynamics has forbidden sequences. A periodic orbit of

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T_A corresponds to a periodic sequence up to a shift cycle. This paper deals with the clustering of periodic orbit based on the notion of 'p-closeness' introduced by Gutkin and Osipov. This is an equivalence relation and groups periodic sequences into clusters. One cluster can involve one or many orbits. We give a necessary and sufficient condition for sequences to be in the same cluster.

This paper is organized as follows. In the next section, we present symbolic dynamics of T_A and prove that T_A satisfies Axiom A. Section 3 studies clusters of periodic orbits of T_A via the notion of p-closeness and provides the number of 2-clusters.

2. SYMBOLIC DYNAMICS OF T_A

Consider $T_A: \mathbb{T}^2 \to \mathbb{T}^2$ induced by matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, i.e.

$$T_A(\mathsf{x}) = A\mathsf{x}, \quad \text{for all} \quad \mathsf{x} \in \mathbb{T}^2.$$

The map T_A is a diffeomorhism on $\mathbb{T}^2$ and is called hyperbolic since matrix A has two eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2},$$

which are not in the unit circle. The corresponding eigenvectors are

$$u_1 = \left(\frac{1+\sqrt{5}}{2}, 1\right) \text{ and } u_2 = \left(\frac{1-\sqrt{5}}{2}, 1\right).$$

The unstable manifold (resp. stable manifold) at $\mathsf{x} = x + \mathbb{Z}^2 \in \mathbb{T}^2$ is the projection of the line in $\mathbb{R}^2$ passing through x and in direction u_1 (resp. u_2) on $\mathbb{T}^2$. Therefore, u_1 and u_2 are called unstable direction and stable direction, respectively.

The automorhism T_A is also called Arnold's cat map (see Figure 1). This map was used by Arnold and Avez[6] in 1968 to indicate ergodicity of the dynamical system. Later, people use 'CAT' to short for 'Continuous Automophirms on Torus'. The Arnold's cat map is also used to illustrate chaos property in chaos theory.

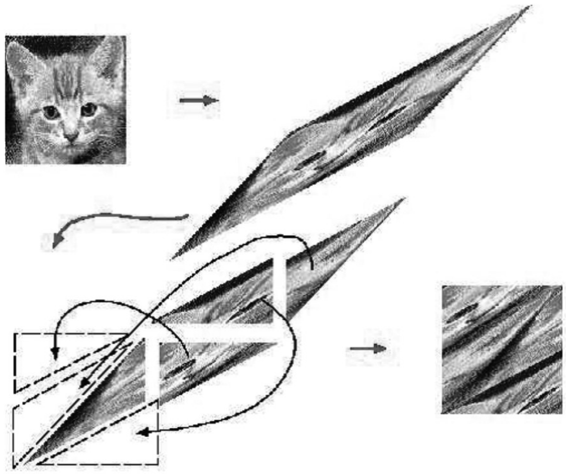

Figure 1. The Arnold's cat map

Next we recall some notions which will be used later. Let $f : M \to M$ is a diffeomorphism on a Riemannian manifold M.

Definition 2.1. A closed set $\Lambda \subset M$ is called *hyperbolic* if $f(\Lambda) = \Lambda$ and for $x \in \Lambda$, the tangent space T_xM has the splitting

$$T_xM = E^u(x) \oplus E^s(x)$$

such that

(i) $d_xf\big(E^s(x)\big) = E^s\big(f(x)\big), d_xf\big(E^u(x)\big) = E^u\big(f(x)\big)$;

(ii) there exist $c > 0$ and $\lambda \in (0,1)$ such that

$$\|d_xf^n(v)\| \le c\lambda^n\|v\|, \text{ khi } v \in E^s(x), n \ge 0$$

and

$$\|d_xf^{-n}(v)\| \le c\lambda^n\|v\|, \text{ khi } v \in E^u(x), n \ge 0;$$

If $\Lambda = M$ then we say f is hyperbolic.

Definition 2.2. A point $x \in M$ is called *non-wandering* if for any neighbourhood U of x, one has

$$U \bigcap \left(\bigcup_{n>0} f^n(U)\right) \ne \emptyset.$$

The set of non-wandering points of f is denoted by $\Omega(f)$.

Definition 2.3. A point $x \in M$ is called *periodic* if there exists $n > 0$ such that $f^n(x) = x$. The set of periodic points of f is denoted by $\mathcal{P}(f)$. Then x is called periodic with period n or n-periodic.

Proposition 2.1. $\overline{\mathcal{P}(f)} \subset \Omega(f)$.

Proof. Let $x \in \overline{\mathcal{P}(f)}$ and let U be a neighbourhood of x. Then there exists a point $y \in U \cap \mathcal{P}(f)$. Since y is a periodic point, there exists $n > 0$ such that $f^n(y) = y$. Then $y \in f^n(U)$ implies $y \in U \bigcap \left(\bigcup_{n>0} f^n(U) \right)$ or $U \bigcap \left(\bigcup_{n>0} f^n(U) \right) \neq \emptyset$. Hence $x \in \Omega(f)$.

□

Proposition 2.2. [7] *The set of periodic of T_A is* $\mathcal{P}(T_A) = \mathbb{Q}^2/\mathbb{Z}^2$.

This implies that the set of periodic points of T_A is dense in $\mathbb{T}^2$.

Definition 2.4. The diffeomorphism f is said to satisfy the *Axiom A* if $\Omega(f)$ is a hyperbolic set and

$$\Omega(f) = \overline{\mathcal{P}(f)}.$$

Theorem 2.1. *The diffeomorphism T_A satisfies the Axiom A.*

Proof. According propositions 2.1 and 2.2,

$$\Omega(T_A) = \overline{\mathcal{P}(T_A)} = \mathbb{T}^2.$$

It remains to show that $\mathbb{T}^2$ is a hyperbolic set. For $x \in \mathbb{T}^2$, let $E^u(x) = < u_1 >$ and $E^s(x) = < u_2 >$, which are eigenspaces. Then

$$E^u(x) \oplus E^s(x) = \mathbb{R}^2 = T_x\mathbb{T}^2.$$

Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$, $L_A(x) = Ax$, $x \in \mathbb{R}^2$. Since T_A is a linear map, $d_xT_A = L_A$. Then

$$d_xT_A\big(E^u(x)\big) = E^u\big(T_A(x)\big)$$

and

$$d_xT_A\big(E^s(x)\big) = E^s\big(T_A(x)\big).$$

We obtain that T_A satisfies (i).

Next, since again $d_xT_A = L_A$, $d_x{T_A}^n = d_xT_{A^n} = L_{A^n}$ for all $n \in \mathbb{Z}$. Then , for $n \in \mathbb{N}$, we have

$$d_xT_A^n(v) = A^n v = \lambda_2^n v \quad \text{for all} \quad v \in E^s(x)$$

and

$$d_xT_A^{-n}(v) = A^{-n}v = \lambda_1^{-n}v = \lambda_2^n v$$

for all $v \in E^u(x)$. This yields that for $n \geq 0$ one has

$$\|d_xT_A^n(v)\| = \lambda_2^n\|v\| \text{ for } v \in E^s(x)$$

and

$$\|d_xT_A^{-n}(v)\| = \lambda_2^n\|v\| \text{ for } v \in E^u(x).$$

This means that (ii) holds. □

Remark 2.1. (a) Since $\Omega(f) = \mathbb{T}^2$, $\mathbb{T}^2$ is a hyperbolic set of T_A and hence T_A is hyperbolic by Definition 2.1.

(b) The theorem is still true for any automorhism on $\mathbb{T}^2$ induced by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$ SL$(2, \mathbb{Z})$ with eigenvalues not in the unit circle. Then we call them hyperbolic automorphisms.
◇

Since T_A satisfies the Axiom A, it admits a Markov partition[1]. A Markov partition of T_A is constructed by Katok and Hasselblatt[8] including five rectangles R_0, R_1, R_2, R_3, R_4 (see Figure 2).

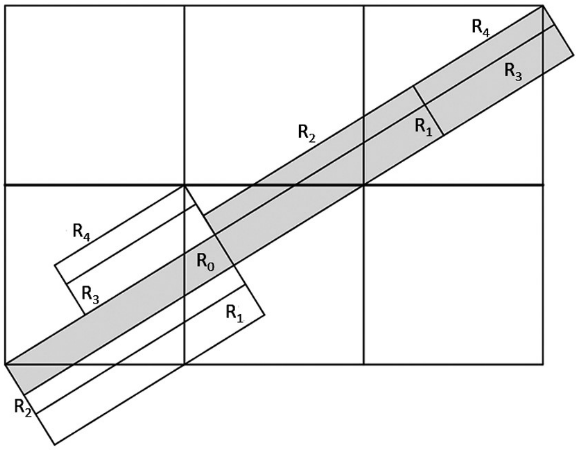

Figure 2. A Markov partion of T_A.

Rectangles R_i are the projections of rectangles in $\mathbb{R}^2$ with edges in directions u_2 and u_1; note that u_1 and u_2 are orthogonal. According to the property of Markov partition, $\cup_{i=0}^{4} R_i = \mathbb{T}^2$ and the rectangles intersect only on the boundaries. Furthermore, if $\mathrm{int}T_A(R_i) \cap R_j \neq 0$ then $T_A(R_i)$ intersects R_j along the stable direction, and if $\mathrm{int}{T_A}^{-1}(R_i) \cap R_j \neq 0$ then ${T_A}^{-1}(R_i)$ intersects R_j along the unstable direction; see Figure 3.

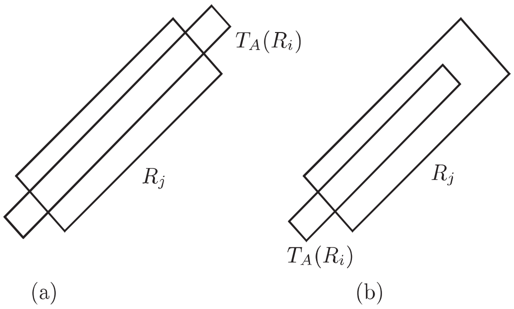

Figure 3. (a) possible, (b) impossible

The adjacency matrix $\mathcal{A} = (a_{i,j})_{i,j=0}^4$ is defined by

$$a_{i,j} = \begin{cases} 1 \text{ if } \mathrm{int}\, T_A(R_i) \cap \mathrm{int}\, R_j \neq \emptyset, \\ 0 \text{ if } \mathrm{int}\, T_A(R_i) \cap \mathrm{int}\, R_j = \emptyset \end{cases} \quad (1)$$

and explicitly

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The definition of a_{ij} in (1) can be explained as follows. If an orbit of T_A passes through $\mathrm{int}R_i$ and then passes through $\mathrm{int}R_j$, then $a_{i,j} = 1$, otherwise $a_{i,j} = 0$.

Denote

$$\mathscr{A} = \{0, 1, 2, 3, 4\}$$

and

$$\mathscr{A}^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathscr{A} \text{ for all } i \in \mathbb{Z}\}.$$

Definition 2.5. The map $\sigma : \mathscr{A}^{\mathbb{Z}} \longrightarrow \mathscr{A}^{\mathbb{Z}}$ defined by

$$(\sigma x)_i = x_{i+1} \text{ for all } i \in \mathbb{Z}$$

is called the *shift map.*

The distance in $\mathscr{A}^{\mathbb{Z}}$ is given by

$$d(x, y) = \sum_{n=-\infty}^{\infty} 2^{-n}|x_n - y_n|,$$

$x = (x_n),\ y = (y_n) \in \mathscr{A}^{\mathbb{Z}}$. Then $(\mathscr{A}^{\mathbb{Z}}, d)$ is a compact metric space and σ is a diffeomorphism on $\mathscr{A}^{\mathbb{Z}}$.

Define

$$\Lambda_{\mathcal{A}} = \{(x_n) \in \mathscr{A}^{\mathbb{Z}} : a_{x_i, x_{i+1}} = 1, \forall i \in \mathbb{Z}\}.$$

The symbols can follow 0,1,2 are 0,1,3; and the symbols can follow 3,4 are 2 and 4. Then $\Lambda_{\mathcal{A}}$ is a closed set of $\mathscr{A}^{\mathbb{Z}}$ and invariant under σ, i.e. $\sigma(\Lambda_{\mathcal{A}}) = \Lambda_{\mathcal{A}}$. The map $\sigma_{|\Lambda_{\mathcal{A}}} : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{A}}$ is called *subshift of finite type* induced by $\mathcal{A}$.

Definition 2.6. Sequence $x = (x_n)_{n=-\infty}^{\infty} \in \mathscr{A}^{\mathbb{Z}}$ is called periodic of period n if

$$\sigma^n(x) = x,$$

i.e.

$$x_{i+n} = x_i, \forall i \in \mathbb{Z}.$$

Then we write $x = [x_0 x_1 \ldots x_{n-1}]$. The set of all periodic sequences of period n in $\Lambda_{\mathcal{A}}$ is denoted by P_n.

For $n \geq 1$ denote by

$$X_n = \{x_0 \ldots x_{n-1} : x_0, \ldots, x_{n-1} \in \mathscr{A}, \\ a_{x_i, x_{i+1}} = 1, i \in \{0, \ldots, n-2\}\}$$

the set of all subsequences of n symbols in $\Lambda_{\mathcal{A}}$. For $n = 2$, we have $X_2 = \{00, 01, 03, 10, 11, 13, 20, 21, 23, 32, 34, 42, 44\}$.

Next we find the number of P_n and X_n via $\mathcal{A}^n$. One has

$$\mathcal{A} = P\text{diag}(\lambda_1, \lambda_2, 0, 0, 0)P^{-1}$$

with

$$P = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & -1 & -1 & 0 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\text{tr}(\mathcal{A}^n) = \lambda_1^n + \lambda_2^n$$

and

$$\mathcal{A}^n = \begin{pmatrix} a_n & a_n & b_n & a_n & b_n \\ a_n & a_n & b_n & a_n & b_n \\ a_n & a_n & b_n & a_n & b_n \\ b_n & b_n & c_n & b_n & c_n \\ b_n & b_n & c_n & b_n & c_n \end{pmatrix},$$

where

$$a_n = \frac{5+\sqrt{5}}{10}\lambda_1^n + \frac{5-\sqrt{5}}{10}\lambda_2^n,$$

$$b_n = -\frac{5+3\sqrt{5}}{10}\lambda_1^n + \frac{-5+3\sqrt{5}}{10}\lambda_2^n,$$

$$c_n = \frac{5+2\sqrt{5}}{5}\lambda_1^n + \frac{5-2\sqrt{5}}{5}\lambda_2^n.$$

Proposition 2.3. (a) The number of subsequences of length n in $\Lambda_{\mathcal{A}}$ is

$$\text{card}(X_n) = \frac{25-11\sqrt{5}}{10}\lambda_1^{n-1} + \frac{25+11\sqrt{5}}{10}\lambda_2^{n-1}.$$

(b) The number of P_n is

$$\text{card}(P_n) = \text{card}\{x \in \Lambda_{\mathcal{A}} : \sigma^n(x) = x\} = \lambda_1^n + \lambda_2^n.$$

Proof. We use Proposition 2.2.12[2].

(a) The number of subsequences of length n in $\Lambda_{\mathcal{A}}$ is the sum of all entries in matrix $\mathcal{A}^{n-1}$, namely

$$9a_{n-1} + 12b_{n-1} + 4c_{n-1} \\ = \frac{25-11\sqrt{5}}{10}\lambda_1^{n-1} + \frac{25+11\sqrt{5}}{10}\lambda_2^{n-1}.$$

(b) The number of n-periodic sequences in $\Lambda_{\mathcal{A}}$ is equal to $\text{tr}(\mathcal{A}^n)$, that is $\lambda_1{}^n + \lambda_2{}^n$. □

For $x = (x_n)_{n\in\mathbb{Z}} \in \Lambda_{\mathcal{A}}$. Using the property of Markov partition, we can show that

$$\bigcap_{n\in\mathbb{Z}} T_A^{-n} R_{x_n}$$

is a single point in $\mathbb{T}^2$ [5]. We define

$$h : \Lambda_{\mathcal{A}} \to \mathbb{T}^2, \ h(x) = \bigcap_{n\in\mathbb{Z}} T_A^{-n} R_{x_n}. \quad (2)$$

Then h is a continuous surjection and satisfies

$$h \circ \sigma = T_A \circ h. \quad (3)$$

Then

$$h \circ \sigma^n = T_A^n \circ h, \text{ for all } n \geq 1. \quad (4)$$

Note that h is not injective, but finite to one. This does not influence the study of periodic orbits. The number of n-periodic points of T_A is given by the following result.

Proposition 2.4. [7] The number of n-periodic points of T_A is

$$\text{card}\{x \in \mathbb{T}^2 : T_A^n(x) = x\} = \lambda_1^n + \lambda_2^n - 2.$$

Remark 2.2. (a) For $x = (x_i) \in P_n, h(x) = x \in \mathbb{T}^2$ satisfies $T_A^n(x) = x$ and

$$T_A^k(x) \in R_{x_k}, k = 0, \ldots, n-1.$$

(b) If follows from (4) that if $x \in \Lambda_A$ is an n-periodic point of σ then $h(x)$ is an n-periodic point of T_A. Therefore,

$$h : P_n \backslash \{[0], [1], [4]\} \to \{x \in \mathbb{T}^2 : {T_A}^n(x) = x\} \backslash \{0 + \mathbb{Z}^2\}$$

is a bijection. Hence, instead of studying periodic points of T_A, we consider periodic sequences of Λ_A. ◇

3. CLUSTERING OF PERIODIC ORBITS

To define clustering of periodic orbits, we need some following notions.

For $x \in \mathbb{T}^2$ is an n-periodic point of T_A. The orbit of T_A through x is defined by

$$\mathcal{O}(x) = \{T_A^i(x), i = 0, 1, \ldots, n-1\}.$$

Definition 3.1. Let x and y be n-periodic points of T_A and $p \in \mathbb{N}^*$. We say that $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are *p-close* if there exists a permutation $\alpha : \{0, 1 \ldots, n-1\} \to \{0, 1 \ldots, n-1\}$ such that

$$d(T_A^k(x), T_A^{\alpha(k)}(y)) < 2^{-p}, \forall k = 0, \ldots, n-1.$$

Roughly speaking, for any point in the orbit of x we can find a point in the orbit of y such that the distance between them is less than 2^{-p}. This means that the two orbits enter the same parts of $\mathbb{T}^2$ but with different orders. We say that these two orbits are in the same p-cluster.

Next we define an equivalence relation $\sim$ in P_n as follows. We say $x \sim x'$ if there is $k \in \{0, \ldots, n-1\}$ such that $\sigma^k(x) = x'$, i.e. x and x' are different up to a shift map. Denote $\mathcal{P}_n = P_n/\sim$. To simplify, we also write $x = [x_0 x_1 \ldots x_{n-1}] \in \mathcal{P}_n$.

Definition 3.2. Let $1 \leq p \leq n$. Two periodic sequences $x = [x_0 \ldots x_{n-1}], y = [y_0 \ldots y_{n-1}] \in \mathcal{P}_n$ are called p-*close* if any subsequence of p consecutive symbols $a_1 a_2 \ldots a_p \in X_p$ appears the same number of times in both x and y.

We write $x \overset{p}{\sim} y$ if x and y are p-close. It is obvious that $\overset{p}{\sim}$ is an equivalence relation.

Proposition 3.1. [3] If $x \overset{p+1}{\sim} y$ then $x \overset{p}{\sim} y$.

Since $\overset{p}{\sim}$ is an equivalence relation on $\mathcal{P}_n$, the set $\mathcal{P}_n$ is decomposed into disjoint equivalence classes $\mathcal{C}_1^{(p)}, \ldots, \mathcal{C}_{\mathcal{N}_p}^{(p)}$. Each equivalence class consists of p-close sequences and is called a p-cluster.

Example 3.1. We consider $n = 7$.

(a) For p = 1, five sequences [0000132], [0000321], [0001032], [0003201], [0010032] belong to the same cluster since the number of times $0, 1, 2, 3, 4$ appear in these sequences are 4, 1, 1, 1, 0, respectively. This cluster is separated into three 2-clusters: [0000132] and [0000321] are independent clusters, and three sequences [0001032], [0003201], [0010032] belong to one cluster since 00 appears twice in the three sequences, 01, 10, 03, 32, 20 all appear once in the three sequences, while 34, 43, 44, 21, 23 do not appear. When $p = 3$, this third cluster is divided into three 3-clusters, each cluster contains only one sequence; see Figure 4 (b).

(b) For p = 1, six sequences [0011342], [0013421], [0101342], [0103421], [0110342], [0034211] are in the same cluster. For $p = 2$, they form six single 2-clusters; see Figure 4 (a).

Remark 3.1. (a) From Proposition 3.1, the periodic sequences in $\mathcal{P}_n$ can be represented as a line chart as in Figure 2[3]. For $p = 3$, only two clusters have more than one element, including the cluster [0001011], [0100011] and cluster [0010111], [0011101].

◇

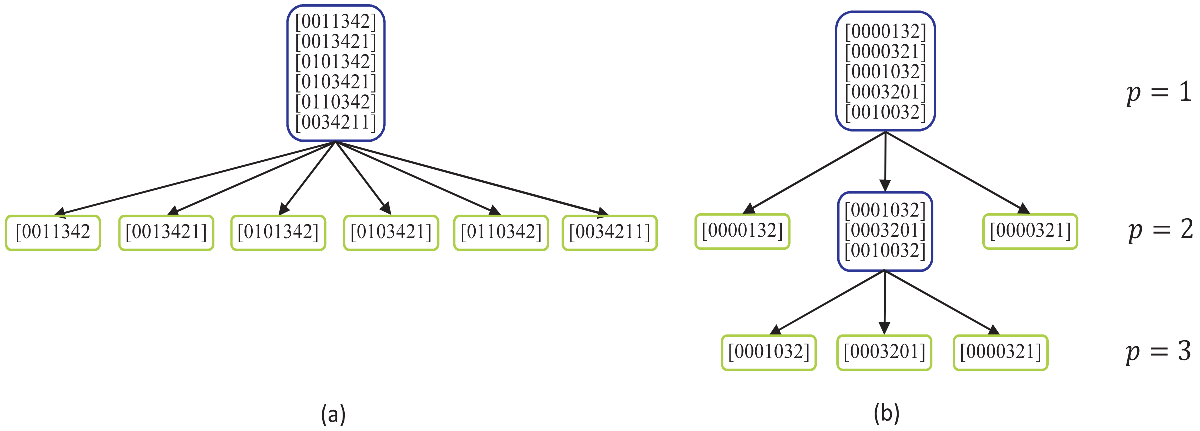

Figure 4. Some clusters with $n = 7$

According to Remark 2.2, each n-periodic sequence in $x \in \Lambda_A$ corresponds to an n-periodic point of T_A by the map h defined in (2). Hence, each element x in $\mathcal{P}_n$ corresponds to the n-periodic orbit $\mathcal{O}(h(x))$ of T_A. The next result allows us to consider clusters of periodic sequences in $\mathcal{P}_n$ instead of working on clusters of periodic orbits of T_A.

Proposition 3.2. [3]*If x, $y \in \mathcal{P}_n$ are 2p-close in $\mathscr{A}^{\mathbb{Z}}$ then two orbits $\mathcal{O}(h(x))$, $\mathcal{O}(h(y))$ of T_A are p-close in $\mathbb{T}^2$.*

The problem of counting clusters of periodic orbits is equivalent to the one of counting classes of closed paths in the de Bruijn graph G_p.

Definition 3.3. The *de Bruijn graph* G_p is defined by:

- each vertex corresponds to a sequence $x_0x_1 \ldots x_{p-2} \in X_{p-1}$;
- each directed edge connecting vertex $x_0x_1 \ldots x_{p-2}$ to vertex $x_1x_2 \ldots x_{p-1}$ corresponds $x_0x_1 \ldots x_{p-1} \in X_p$.

We see that the the set of vertices and the set of edges of de Bruijn graph G_p are X_{p-1} and X_p, respectively. The edges of G_p are the vertices of G_{p+1}. For instance, graph G_p has vertices $0, 1, 2, 3, 4$ and edges 00, 01, 03, 10, 11, 13, 20, 21, 23, 32, 34, 42, 44; see Figure 5 (a), while edges of G_2 are vertices of G_3, which has 34 edges; see Figure 5 (b).

Each closed path in G_p visiting n edges is represented by a sequence $x = x_0x_1 \ldots x_{n-1}$ in X_n. This finite sequence induces periodic orbit $[x_0x_1 \ldots x_{n-1}] \in P_n$. In this way, the ith edge of G_p corresponds to the code $x_ix_{i+1} \ldots x_{i+p-1}$ of x. Denote by g_x the corresponding closed path in G_p represented by x.

Next we calculate the number of 2-cluster, i.e. the number of equivalence classes of $\overset{2}{\sim}$ in $\mathcal{P}_n$.

Theorem 3.1. The number of 2-clusters in $\mathcal{P}_n$ is the number of vectors $N = (n_a)_{a \in X_2}$ satisfying

$$\sum_{a \in X_2} n_a = n \tag{5}$$

and

$$S_2N = R_2^TN, \tag{6}$$

where S_2 and N_2 are given by

$$R_2^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

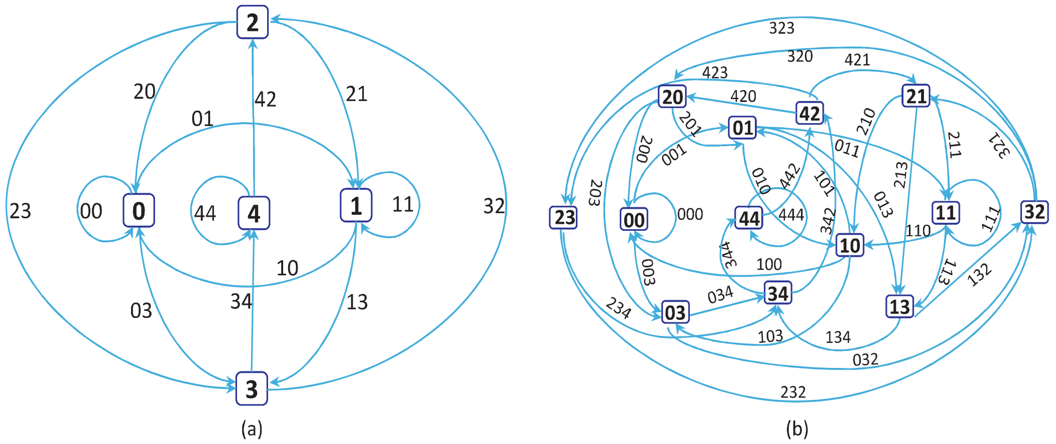

Figure 5. Edges of G_2 (a) are vertices of G_3 (b)

and

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. For $p \geq 2$. Denote by $N = (n_a)_{a\in X_p}$ the vector in which n_a is the number of times the closed path g_x according to $x \in \mathcal{P}_n$ passes through the edge $a \in X_p$. Then, two sequences x, $y \in X_n$ are p-close (i.e. $x \overset{p}{\sim} y$) if and only if two respective paths g_x, g_y in G_p visit each edge of G_p the same number of times, i.e. $N(x) = N(y)$. Therefore, each equivalence class of p-close sequences is uniquely determined by vector $N = (n_a)_{a\in X_p}$. Each equivalence class in $\mathcal{P}_n/\overset{p}{\sim}$ is identified with a vector $N = (n_a)_{a\in X_p}$, where n_a is the number of times the corresponding path visits the edge a of graph G_p. For $p = 2$, since vector N corresponds to a periodic orbit of period n, its coordinates must satisfy:

(i) the length of closed path is equal to n, so (5) holds;

(ii) the number of times a periodic orbit visits a vertex of G_2 is equal to the number of times this orbit exits that vertex. This is illustrated by equation (6). The theorem is proved. □

Example 3.2. The number of 2-clusters in P_7 is represented by vector $N = (n_{00}, n_{01}, n_{03}, n_{10}, n_{11}, n_{13}, n_{20}, n_{21}, n_{23}, n_{42}, n_{44})$ satisfying the following system

$$\begin{cases} n_{00} + n_{01} + n_{03} + n_{10} + n_{11} + n_{13} + n_{32} & \\ + n_{34} + n_{20} + n_{21} + n_{23} + n_{42} + n_{44} & = 7 \\ n_{01} + n_{03} - n_{10} - n_{20} & = 0 \\ -n_{01} + n_{10} + n_{13} - n_{21} & = 0 \\ n_{03} + n_{13} + n_{23} - n_{32} - n_{34} & = 0 \\ n_{20} + n_{21} + n_{23} - n_{32} - n_{42} & = 0 \\ n_{34} - n_{42} & = 0. \end{cases}$$

By solving this system to find non-negative integer solutions, one has 94 solutions, corresponding to 94 2-clusters. There are 76 clusters including single element, 14 clusters including two elements and four clusters including three elements. The 2-clusters with three elements are

$$\{[0001011], [0010011], [0001101]\},$$

$$\{[0010111], [0011011], [0011101]\},$$

$$\{[0001032], [0003201], [0010032]\},$$

$$\{[0111321], [0113211], [0132111]\}.$$

4. CONCLUSION AND OUTLOOK

The paper studies clustering of periodic orbits of the automorphism T_A via symbolic dynamics. We only consider the case $p = 2$, although the de Brujin graph is defined for $p \geq 2$. Furthermore, Theorem 3.1 gives us the information of the number of clusters. The number of elements in the same cluster has not been given. The matrices S_p and R_p for general p as well as algorithms to list all p-clusters will be investigated in the near future.

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