TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Một số bất đẳng thức kiểu Fejér cho hàm $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ - lồi mạnh

Nguyễn Ngọc Huề*

Khoa Khoa học Tự nhiên và Công nghệ, Đại học Tây Nguyên, Đắk Lắk, Việt Nam

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TÓM TẮT

Trong bài báo này, chúng tôi xem xét một lớp hàm lồi mạnh mở rộng liên quan đến một cặp tựa trung bình số học, được gọi là hàm $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-lồi mạnh từ đó thiết lập một số bất đẳng thức kiểu Fejér cho lớp hàm lồi mạnh này. Các bất đẳng thức mới này là sự mở rộng thực sự của các bất đẳng thức Hermite-Hadamard và bất đẳng thức Fejér được thiết lập gần đây đối với hàm lồi mạnh và một số dạng mở rộng của lớp hàm lồi mạnh. Hơn nữa, các bất đẳng thức mới còn đặc trưng cho lớp hàm $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-lồi mạnh.

Từ khóa: *Hàm $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-lồi mạnh, tựa trung bình số học, hàm lồi, bất đẳng thức Hermite-Hadamard, bất đẳng thức Fejér.*

*Tác giả liên hệ chính.
Email: nnhue@ttn.edu.vn*

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Some Fejér type inequalities for strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ - convex functions

Nguyen Ngoc Hue*

Department of Technology and Science, Tay Nguyen University, Dak Lak, Vietnam

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ABSTRACT

In this paper, we propose and study a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means and called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, and establish various Fejér type inequalities for such a function class. These inequalities not merely provide a natural and intrinsic characterization of the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, but actually offer a generalization and refinement of some Hermite-Hadamard and Fejér type inequalities obtained in earlier studies for different kinds of strong convexity.

Keywords: *Strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, quasi-arithmetic mean, convexity, Hermite-Hadamard inequality, Fejér inequality.*

1. INTRODUCTION

In the field of mathematical inequalities, the well-known Hermite-Hadamard inequality for convex functions was first discovered by Hermite[1] in 1883 and independently discovered 10 years later by Hadamard.[2] This inequality says that if $f : [a, b] \to \mathbb{R}$ is a convex function then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

A weighted generalization of Hermite-Hadamard inequality was developed by Fejér:[3] If $f : [a, b] \to \mathbb{R}$ is a convex function, $g : [a, b] \to [0, \infty)$ is an integrable function with $\int_a^b g(x)dx > 0$ and it is symmetric to $\frac{a+b}{2}$, *i.e.* $g(x) = g(a + b - x)$ for all $x \in [a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq \frac{f(a)+f(b)}{2}. \quad (2)$$

Since then, the inequalities (1) and (2) have been generalized, extended and improved in various ways and found interesting applications to convex analysis, optimization theory and nonlinear analysis. One of such ways is to establish new inequalities for various generalized convex functions (see e.g.,[4–7]). Among them, an important subclass of convex functions in the optimization theory is strongly convex functions. This class was developed by Polyak[8] in 1966 for dealing with some related issues arisen from optimization theory.

Let c be a positive number. A function $f : [a, b] \to \mathbb{R}$ is called strongly convex with modulus c if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$$

for all $x, y \in [a, b]$. One says that f is strongly mid-convex with modulus c if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - c\frac{(x-y)^2}{4} \quad (3)$$

for all $x, y \in [a, b]$.

In 2010, Merentes and Nikodem[9] established a generalized version of Hermite-Hadamard inequality for strongly convex functions as follows: Let $f : [a, b] \to \mathbb{R}$ be a strongly convex function with modulus c. Then, the following inequality holds

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2. \quad (4)$$

In 2012, Azocar[10] et al. proposed a Fejér type inequality for strongly convex functions: Let $f : [a, b] \to \mathbb{R}$ be a strongly convex function with modulus c and $g : [a, b] \to [0, \infty)$ be an integrable function

**Corresponding author.*
Email: nnhue@ttn.edu.vn

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with $\int_a^b g(x)dx = 1$ and symmetric to $\frac{a+b}{2}$. Then,

$$f\left(\frac{a+b}{2}\right) + c\left[\int_a^b x^2 g(x)dx - \left(\frac{a+b}{2}\right)^2\right] \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} - c\left[\frac{a^2+b^2}{2} - \int_a^b x^2 g(x)dx\right]. \quad (5)$$

In recent years, some generalizations of inequality (5) have been established for strongly log-convex functions and strongly harmonic convex functions.[11,12] Motivated by the achievements, we continue the research direction. Our contributions in this paper are that we first deeply investigate the class of generalized strongly convex functions regarding to a pair of quasi-arithmetic means and then derive some new Fejér-type inequalities. The derived inequalities are not only characterizations for generalized strongly convex continuous functions, but they also generalize inequalities that were recently derived in the papers.[11,12]

The rest of this paper is organized as follows. In Section 2, we will introduce $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions with positive modulus and refer related particular cases. The main results of this paper will be presented in Section 3. Finally, the paper closes with the conclusions in Section 4.

2. Strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ - convex functions

Let I and J be the intervals of real numbers. Let $\phi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ be continuous and strictly monotone functions. Using a pair of quasi-arithmetic means $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, with $\mathcal{M}_\phi(a, b; \alpha) = \phi^{-1}(\alpha\phi(a) + (1-\alpha)\phi(b))$ Aumann[13] proposed the concept of $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions that is stated as follows.

Definition 1. [13] A function $f : I \to J$ is said to be $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex if

$$f(\mathcal{M}_\phi(a, b; \alpha)) \leq \mathcal{M}_\psi(f(a), f(b); \alpha) \quad (6)$$

for all $a, b \in I$ and $\alpha \in [0, 1]$.

In the case that f fulfills the inequality (6) with $\phi(x) = x$, f is called $\mathcal{M}_\psi$-convex. If f satisfies the inequality (6) with $\phi(x) = x$ and $\psi(x) = x$, then the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity of f reduces to the usual convexity in the literature of convex analysis.

For a pair of quasi-arithmetic means $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, we define a class of generalized strongly convex functions as follows.

Definition 2. Let c be a positive number. A function $f : I \to J$ is called strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex with modulus c if

$$f(\mathcal{M}_\phi(a, b; \alpha)) \leq \psi^{-1}\Big(\alpha\psi \circ f(a) + (1-\alpha)\psi \circ f(b) - c\,\alpha(1-\alpha)(\phi(a) - \phi(b))^2\Big) \quad (7)$$

for all $a, b \in I$ and $\alpha \in [0, 1]$. If the inequality (7) is reversed, we call that f is strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-concave with modulus c.

Note that if ψ is increasing then $f : I \to J$ is strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex with modulus c if and only if $\psi \circ f \circ \phi^{-1}$ is strongly convex function with modulus c on $\phi(I)$. If ψ is decreasing then $f : I \to J$ is strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex with modulus c iff $\psi \circ f \circ \phi^{-1}$ is strongly concave with modulus c on $\phi(I)$.

We say that a function f is strongly $\mathcal{M}_\psi$-convex with modulus c if f satisfies the inequality (7) for $\phi(x) = x$. For particular forms of ϕ and ψ, we obtain the following concepts:

- *strongly convex functions if we take $\phi(x) = x$ and $\psi(x) = x$:*
$$f(\alpha a + (1-\alpha)b) \leq \alpha f(a) + (1-\alpha)f(b) - c\alpha(1-\alpha)(a-b)^2$$
for all $a, b \in I$ and $\alpha \in [0, 1]$.
- *Strongly log-convex functions[14] if we take $\phi(x) = x$ and $\psi(x) = \ln x$:*
$$\ln f(\alpha a + (1-\alpha)b) \leq \alpha \ln f(a) + (1-\alpha)\ln f(b) - c\alpha(1-\alpha)(a-b)^2$$
for all $a, b \in I$ and $\alpha \in [0, 1]$.
- *Strongly exponentially convex functions[15] if we take $\phi(x) = x$ and $\psi(x) = e^x$:*
$$e^{f(\alpha a + (1-\alpha)b)} \leq \alpha e^{f(a)} + (1-\alpha)e^{f(b)} - c\alpha(1-\alpha)(a-b)^2$$
for all $a, b \in I$ and $\alpha \in [0, 1]$.
- *Strongly harmonic convex functions[11] if we take $\phi(x) = 1/x$ and $\psi(x) = x$:*
$$f\Big(\frac{ab}{\alpha a + (1-\alpha)b}\Big) \leq \alpha f(a) + (1-\alpha)f(b) - c\alpha(1-\alpha)\left(\frac{a-b}{ab}\right)^2$$
for all $a, b \in I$ and $\alpha \in [0, 1]$.
- *Strongly harmonic log-convex functions[11] if we take $\phi(x) = 1/x$ and $\psi(x) = \ln x$:*
$$f\Big(\frac{ab}{\alpha a + (1-\alpha)b}\Big) \leq f(a)^\alpha f(b)^{(1-\alpha)} - c\alpha(1-\alpha)\left(\frac{a-b}{ab}\right)^2$$
for all $a, b \in I$ and $\alpha \in [0, 1]$.

- *Strongly p-convex functions if* $\phi(x) = x^p$ *and* $\psi(x) = x$*:*

$$f([\alpha a^p + (1-\alpha)b^p]^{1/p}) \leq \alpha f(a) + (1-\alpha)f(b) - c\alpha(1-\alpha)(a^p - b^p)^2$$

for all $a, b \in I$ *and* $\alpha \in [0,1]$.

- *Strongly geometrically convex functions if we take* $\phi(x) = \ln x$ *and* $\psi(x) = \ln x$*:*

$$f(a^\alpha b^{(1-\alpha)}) \leq f(a)^\alpha f(b)^{(1-\alpha)} - c\alpha(1-\alpha)\ln^2\left(\frac{a}{b}\right)$$

for all $a, b \in I$ *and* $\alpha \in [0,1]$.

Lemma 3. [16]

1. *A function* $f : I \to J$ *is strongly convex with modulus* c *if and only if the function* $g(x) = f(x) - cx^2$ *is convex.*
2. *A function* $f : I \to J$ *is strongly midconvex with modulus* c *if and only if* $g(x) = f(x) - cx^2$ *is midconvex.*

Due to Lemma 3 and Jensen's inequality[17], one can verify that if f is continuous on I and strongly midconvex with modulus c then f is a strongly convex function with modulus c on I.

Lemma 4. *A function* f *is strongly* $(\mathcal{M}_\phi, \mathcal{M}_\psi)$*-convex with modulus* c *if and only if* $g(x) := \psi \circ f \circ \phi^{-1}(x) - cx^2$ *is strongly convex on* $\phi(I)$.

Proof. We have f is strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex with modulus c if and only if $\psi \circ f \circ \phi^{-1}$ is strongly convex with modulus c on $\phi(I)$. This achievement together with Lemma 3 yield $f : I \to J$ is strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex with modulus c if and only if $g(x) := \psi \circ f \circ \phi^{-1}(x) - cx^2$ is convex on $\phi(I)$. ☐

3. Fejér type inequalities for strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions

Throughout this paper, we assume that $f : I \to J$ is a strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex function with modulus c ($c > 0$); $a, b \in I$, $a < b$; $\alpha \in (0,1)$; $w_1, w_2 : [0,1] \to [0,\infty)$ are integrable functions and satisfy the condition $\int_0^s w_1(t)dt > 0$ for all $s \in (0,1]$. and $\int_s^1 w_2(t)dt > 0$ for all $s \in [0,1)$. Denote

$$\mathcal{L}(t) = \mathcal{M}_\phi(a, \mathcal{M}_\phi(a,b;\alpha); t)$$

and

$$\mathcal{R}(t) = \mathcal{M}_\phi(b, \mathcal{M}_\phi(a,b;\alpha); t)$$

for $t \in [0,1]$.

Theorem 5. *Let* $\mathcal{F}, \mathcal{G} : [0,1] \to \mathbb{R}$ *be two functions defined as*

$$\mathcal{F}(t) = \psi^{-1}\Big(\alpha(\psi \circ f \circ \mathcal{L}(t) - c[\phi \circ \mathcal{L}(t)]^2) + (1-\alpha)(\psi \circ f \circ \mathcal{R}(t) - c[\phi \circ \mathcal{R}(t)]^2)\Big)$$

and

$$\mathcal{G}(t) = t\mathcal{F}(1) + (1-t)\mathcal{F}(0).$$

Then,

(1) $\mathcal{F}$ *and* $\mathcal{G}$ *are* $\mathcal{M}_\psi$*-convex, increasing on* $[0,1]$ *and*

$$\begin{aligned} &\mathcal{F}(0) = \mathcal{G}(0) \\ &\mathcal{F}(t) \leq \mathcal{G}(t), \quad t \in [0,1], \\ &\mathcal{F}(1) = \mathcal{G}(1). \end{aligned} \tag{8}$$

(2) *For* $s \in (0,1]$*, let*

$$\mathcal{I}_1(s) = \psi^{-1}\left(\frac{\int_0^s \psi \circ \mathcal{F}(t) w_1(t)dt}{\int_0^s w_1(t)dt}\right)$$

and

$$\beta_1(s) = \frac{\int_0^s t w_1(t)dt}{\int_0^s w_1(t)dt}.$$

Then, $\mathcal{F} \circ \beta_1$, $\mathcal{I}_1$ *and* $\mathcal{G} \circ \beta_1$ *are increasing on* $(0,1]$ *and satisfy*

$$\lim_{s\to 0^+} \mathcal{F}\circ\beta_1(s) = \lim_{s\to 0^+} \mathcal{I}_1(s) = \lim_{s\to 0^+} \mathcal{G}\circ\beta_1(s) = \mathcal{G}(0),$$

$$\mathcal{F} \circ \beta_1(s) \leq \mathcal{I}_1(s) \leq \mathcal{G} \circ \beta_1(s), \; s \in (0,1]. \tag{9}$$

(3) *Similarly, for* $s \in [0,1)$*, we define*

$$\mathcal{I}_2(s) = \psi^{-1}\left(\frac{\int_s^1 \psi \circ \mathcal{F}(t) w_2(t)dt}{\int_s^1 w_2(t)dt}\right)$$

and

$$\beta_2(s) = \frac{\int_s^1 t w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then, $\mathcal{F} \circ \beta_2$, $\mathcal{I}_2$ *and* $\mathcal{G} \circ \beta_2$ *are increasing on* $[0,1)$ *and satisfy*

$$\mathcal{F} \circ \beta_2(s) \leq \mathcal{I}_2(s) \leq \mathcal{G} \circ \beta_2(s), \quad s \in [0,1), \tag{10}$$

$$\lim_{s\to 1^-} \mathcal{F}\circ\beta_2(s) = \lim_{s\to 1^-} \mathcal{I}_2(s) = \lim_{s\to 1^-} \mathcal{G}\circ\beta_2(s) = \mathcal{G}(1).$$

Moreover, if $w_1 = w_2$ *then* $\mathcal{I}_1(1) = \mathcal{I}_2(0)$.

In order to prove Theorem 5, we need to introduce the following auxiliary result.

Lemma 6. [5] *Let* $P : [0,1] \to \mathbb{R}$ *be increasing and continuous.*

(1) *For* $s \in (0,1]$*, define*

$$P_1(s) = \frac{\int_0^s P(t) w_1(t)dt}{\int_0^s w_1(t)dt}.$$

Then, P_1 *is increasing on* $(0,1]$ *and*

$$\lim_{s\to 0^+} P_1(s) = P(0) \leq P_1(s) \leq P(s), \; s \in (0,1].$$

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(2) *Similarly, for* $s \in [0,1)$, *define*

$$P_2(s) = \frac{\int_s^1 P(t)w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then, P_2 *is increasing on* $[0,1)$ *and*

$$P(s) \le P_2(s) \le P(1) = \lim_{s\to 1^-} P_2(s),\ s \in [0,1).$$

We are now in a position to provide the proof of Theorem 5.

Proof of Theorem 5. Since ψ is strictly monotone, we consider two possible cases of ψ: strictly increasing and strictly decreasing.

First, suppose that ψ is strictly increasing on J. Since ψ is continuous on J, the function ψ^{-1} is continuous and strictly increasing on $\psi(J)$.

1. In order to prove that $\mathcal{F}$ is $\mathcal{M}_\psi$-convex on $[0,1]$, it suffices to show that $\psi \circ \mathcal{F}$ is convex on $[0,1]$. Indeed,

$$\begin{aligned}&\psi \circ \mathcal{F}(t)\\ &= \alpha\left(\psi \circ f \circ \phi^{-1}(A(t)) - c(A(t))^2\right)\\ &\quad + (1-\alpha)\left(\psi \circ f \circ \phi^{-1}(B(t)) - c(B(t))^2\right),\end{aligned}$$

where

$$A(t) = t\phi(a) + (1-t)(\alpha\phi(a) + (1-\alpha)\phi(b)) \tag{11}$$

and

$$B(t) = t\phi(b) + (1-t)(\alpha\phi(a) + (1-\alpha)\phi(b)). \tag{12}$$

By Lemma 4, $\psi \circ f \circ \phi^{-1}(x) - cx^2$ is convex on $\phi([a,b])$. Moreover, since $A(t)$ and $B(t)$ are linear on $[0,1]$, the function $\psi\circ\mathcal{F}$ is convex on $[0,1]$. The $\mathcal{M}_\psi$-convexity of $\mathcal{G}$ on $[0,1]$ immediately follows from the definition of $\mathcal{G}$.

By simple computations, one can verify that

$$\begin{aligned}&\mathcal{F}(0) = \mathcal{G}(0)\\ &= \psi^{-1}\Big(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha))\\ &\qquad - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2\Big)\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\mathcal{F}(1) = \mathcal{G}(1)\\ &= \psi^{-1}\Big(\alpha\psi \circ f(a) + (1-\alpha)\psi \circ f(b)\\ &\qquad - c(\alpha\phi^2(a) + (1-\alpha)\phi^2(b))\Big).\end{aligned} \tag{14}$$

Now, due to the convexity of $\psi \circ f \circ \phi^{-1}(x) - cx^2$, one gets

$$\begin{aligned}&\psi \circ f \circ \phi^{-1}(A(t)) - c(A(t))^2\\ &\le t(\psi \circ f(a) - c(\phi(a))^2)\\ &\quad + (1-t)(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha))\\ &\quad - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2)\end{aligned}$$

and

$$\begin{aligned}&\psi \circ f \circ \phi^{-1}(B(t)) - c(B(t))^2\\ &\le t(\psi \circ f(b) - c(\phi(b))^2)\\ &\quad + (1-t)(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha))\\ &\quad - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2).\end{aligned}$$

Thus,

$$\begin{aligned}\psi \circ \mathcal{F}(t) &\le t\psi^{-1}\Big(\alpha\psi \circ f(a) + (1-\alpha)\psi \circ f(b)\\ &\qquad - c(\alpha\phi^2(a) + (1-\alpha)\phi^2(b))\Big)\\ &\quad + (1-t)\psi^{-1}\Big(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha))\\ &\qquad - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2\Big)\\ &= \psi \circ \mathcal{G}(t).\end{aligned}$$

Since ψ^{-1} is increasing on $\psi(J)$,

$$\mathcal{F}(t) \le \mathcal{G}(t), \quad t \in [0,1],$$

the claims (8) hold.

Next, we prove that $\mathcal{F}$ is increasing. Suppose that $0 < t < r \le 1$. Due to the strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity of f and $\alpha A(t) + (1-\alpha)B(t) = \alpha\phi(a) + (1-\alpha)\phi(b)$, one has

$$\begin{aligned}&\psi \circ \mathcal{F}(0)\\ &= \psi \circ f \circ \phi^{-1}(\alpha\phi(a) + (1-\alpha)\phi(b))\\ &\quad - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2\\ &= \psi \circ f \circ \phi^{-1}\left(\alpha A(t)) + (1-\alpha)B(t)\right))\\ &\quad - c\left(\alpha A(t)) + (1-\alpha)B(t))\right)^2\\ &\le \alpha\left(\psi \circ f \circ \phi^{-1}(A(t)) - c(A(t))^2\right)\\ &\quad + (1-\alpha)\left(\psi \circ f \circ \phi^{-1}(B(t)) - c(B(t))^2\right)\\ &= \psi \circ \mathcal{F}(t).\end{aligned}$$

On the other hand, since $\psi \circ \mathcal{F}$ is convex,

$$\frac{\psi \circ \mathcal{F}(r) - \psi \circ \mathcal{F}(t)}{r-t} \ge \frac{\psi \circ \mathcal{F}(t) - \psi \circ \mathcal{F}(0)}{t-0}.$$

Thus,

$$\frac{\psi \circ \mathcal{F}(r) - \psi \circ \mathcal{F}(t)}{r-t} \ge \frac{\psi \circ \mathcal{F}(t) - \psi \circ \mathcal{F}(0)}{t-0} \ge 0,$$

i.e. $\psi \circ \mathcal{F}$ is increasing on $[0,1]$. Since ψ^{-1} is increasing on $\psi(J)$, $\mathcal{F}$ is increasing on $[0,1]$. Since

$$\psi \circ \mathcal{G}(t) = t[\mathcal{F}(1) - \mathcal{F}(0)] + \mathcal{F}(0)$$

and

$$\mathcal{F}(1) - \mathcal{F}(0) \ge 0,$$

one gets $\psi \circ \mathcal{G}$ is increasing on $[0,1]$ and then $\mathcal{G}$ is increasing on $[0,1]$.

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2. Applying Lemma 6 where $P = \psi \circ \mathcal{F}$, the function $\psi \circ \mathcal{I}_1$ is increasing on $(0, 1]$ and
$$\lim_{s\to 0^+} \psi \circ \mathcal{I}_1(s) = \psi \circ \mathcal{F}(0).$$
Since ψ^{-1} is strictly increasing and it is continuous on $\psi(J)$, the function $\mathcal{I}_1$ is increasing on $(0, 1]$ and
$$\lim_{s\to 0^+} \mathcal{I}_1(s) = \psi \circ \mathcal{F}(0).$$
Also, due to Lemma 6, the function β_1 is increasing on $(0, 1]$ and
$$\lim_{s\to 0^+} \beta_1(s) = 0 \leq \beta_1(s) \leq s, \quad s \in (0, 1].$$
Therfore, $\mathcal{F} \circ \beta_1$ and $\mathcal{G} \circ \beta_1$ are well-defined, increasing on $(0, 1]$ and
$$\lim_{s\to 0^+} \mathcal{F} \circ \beta_1(s) = \lim_{s\to 0^+} \mathcal{G} \circ \beta_1(s) = \mathcal{G}(0).$$
Next, we prove the inequalities in (9). Let us fix $s \in (0, 1]$. Applying Jensen's inequality[7] to the convex function $\psi\circ\mathcal{F}$ on the interval $[0, s]$, we obtain
$$\psi \circ \mathcal{F}\left(\frac{\int_0^s tw_1(t)dt}{\int_0^s w_1(t)dt}\right) \leq \frac{\int_0^s \psi \circ \mathcal{F}(t)w_1(t)dt}{\int_0^s w_1(t)dt}.$$
It follows
$$\mathcal{F} \circ \beta_1(s) \leq \mathcal{I}_1(s).$$
Since $\mathcal{F}(t) \leq \mathcal{G}(t),\ t \in [0, 1]$, we have
$$\frac{\int_0^s \psi \circ \mathcal{F}(t)w_1(t)dt}{\int_0^s w_1(t)dt} \leq \frac{\int_0^s \psi \circ \mathcal{G}(t)w_1(t)dt}{\int_0^s w_1(t)dt} = \psi \circ \mathcal{G} \circ \beta_1(s).$$
Since ψ^{-1} is increasing, one gets
$$\mathcal{I}_1(s) \leq \mathcal{G} \circ \beta_1(s).$$
3. Applying Lemma 6 with $P = \psi \circ \mathcal{F}$, we have that the function $\psi \circ \mathcal{I}_2$ is increasing on $[0, 1)$ and
$$\lim_{s\to 1^-} \psi \circ \mathcal{I}_2(s) = \psi \circ \mathcal{F}(1).$$
Since ψ^{-1} is strictly increasing and continuous on $\psi(J)$, $\mathcal{I}_2$ is increasing on $[0, 1)$ and
$$\lim_{s\to 1^-} \mathcal{I}_2(s) = \mathcal{F}(1).$$
Due to Lemma 6, β_2 is increasing $[0, 1)$ and
$$\lim_{s\to 1^-} \beta_2(s) = 1 \geq \beta_2(s) \geq s, \quad s \in [0, 1).$$
Thus, $\mathcal{F}\circ\beta_2$ and $\mathcal{G}\circ\beta_2$ is well-defined, increasing on $[0, 1)$ and
$$\lim_{s\to 1^-} \mathcal{F} \circ \beta_2(s) = \lim_{s\to 1^-} \mathcal{G} \circ \beta_2(s) = \mathcal{G}(1).$$
Now, we prove the inequalities (10). Fixing $s \in [0, 1)$ and applying Jensen's inequality[7] to convex function $\psi \circ \mathcal{F}$ on $[s, 1]$, we obtain
$$\psi \circ \mathcal{F}\left(\frac{\int_s^1 tw_2(t)dt}{\int_s^1 w_2(t)dt}\right) \leq \frac{\int_s^1 \psi \circ \mathcal{F}(t)w_2(t)dt}{\int_s^1 w_2(t)dt}$$
and hence
$$\mathcal{F} \circ \beta_2(s) \leq \mathcal{I}_2(s).$$
Since $\mathcal{F}(t) \leq \mathcal{G}(t),\ t \in [0, 1]$, it implies that
$$\frac{\int_s^1 \psi \circ \mathcal{F}(t)w_2(t)dt}{\int_s^1 w_2(t)dt} \leq \frac{\int_s^1 \psi \circ \mathcal{G}(t)w_2(t)dt}{\int_s^1 w_2(t)dt} = \psi \circ \mathcal{G} \circ \beta_2(s).$$
Since ψ^{-1} is increasing, one has
$$\mathcal{I}_2(s) \leq \mathcal{G} \circ \beta_2(s).$$
Moreover, if $w_1 = w_2$, due to the definitions of $\mathcal{I}_1$ and $\mathcal{I}_2$, one gets $\mathcal{I}_1(1) = \mathcal{I}_2(0)$.

The proof is similar for the case that ψ is decreasing. □

Note that Theorem 5 is not only a consequence of strong $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity, but it is also a characterization of strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex continuous functions with modulus c.

Corollary 7. *Let $f : I \to J$ be a continuous function. The following statements are equivalent:*

(1) *f is a strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex function with modulus c.*

(2) *$\mathcal{F}$ is increasing on $[0, 1]$ for all $a, b \in I, a < b$ and $\alpha = 1/2$.*

(3) *$\mathcal{I}_1$ is increasing on $(0, 1]$ for all $a, b \in I, a < b, \alpha = 1/2$ and $w_1 = 1$.*

(4) *For all $a, b \in I$ and $a < b$, we have*
$$\psi \circ f(\mathcal{M}_\phi(a, b; \alpha)) - c(\alpha\phi(a) + (1 - \alpha)\phi(b))^2 \leq \frac{1}{\phi(b) - \phi(a)} \int_a^b \left(\psi \circ f(x) - c\phi^2(x)\right) d\phi(x).$$

(5) *$\mathcal{I}_2$ is increasing on $[0, 1)$ for all $a, b \in I, a < b, \alpha = 1/2$ and $w_2 = 1$.*

(6) *For all $a, b \in I$ and $a < b$, we have*
$$\frac{1}{\phi(b) - \phi(a)} \int_a^b \left(\psi \circ f(x) - c\phi^2(x)\right) d\phi(x) \leq \alpha\psi \circ f(a) + (1 - \alpha)\psi \circ f(b) - c(\alpha\phi^2(a) + (1 - \alpha)\phi^2(b)).$$

(7) *$\mathcal{G}$ is increasing on $[0, 1]$ for all $a, b \in I, a < b$ and $\alpha = 1/2$.*

In order to prove Corollary 7, we need the following lemma.

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Lemma 8 ([9] Theorem 6)**.** *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ is a continuous function. Then, the following statements are equivalent.*

(1) *f is strongly convex with modulus c.*

(2) *For all $x, y \in I, x < y$, we have*

$$f\left(\frac{x+y}{2}\right) + \frac{c}{12}(y-x)^2 \le \frac{1}{y-x}\int_x^y f(t)dt.$$

(3) *For all $x, y \in I, x < y$, we have*

$$\frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} - \frac{c}{6}(y-x)^2.$$

Proof of Corollary 7. Due to Theorem 5, the implications (1) ⇒ (2) ⇒ (3) ⇒ (4), (1) ⇒ (5) ⇒ (6) và (1) ⇒ (7) hold. For the rest of proof, we prove the implications (4) ⇒ (1), (6) ⇒ (1) and (7) ⇒ (1). Without loss of generality, we assume that ψ is increasing. We need to prove that $\psi \circ f \circ \phi^{-1}$ is strongly convex on $\phi(I)$ provided that one of conditions (4), (6) và (7) holds. Since ϕ is continuous and strictly monotone on I, ϕ^{-1} is continuous and strictly monotone on $\phi(I)$. Now, the continuity of ψ, f and ϕ^{-1} imply that $\psi \circ f \circ \phi^{-1}$ is continuous on $\phi(I)$. Clearly, (7) implies that $\psi \circ f \circ \phi^{-1}$ is strongly midconvex on $\phi(I)$. Thus, $\psi \circ f \circ \phi^{-1}$ is strongly convex on $\phi(I)$. Finally, due to Corollary 8, if one of conditions (4) and (6) holds for the continuous function $\psi \circ f \circ \phi^{-1}$ on $\phi(I)$ then $\psi \circ f \circ \phi^{-1}$ is strongly convex on $\phi(I)$. □

As a result of Theorem 5, one can derive new Fejér type inequalities for strongly $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions with modulus c by different choices of w_1 and w_2. For examples, we take

$$w_j(t) = (1-\alpha)g_j \circ \mathcal{L}(t) + \alpha g_j \circ \mathcal{R}(t), \quad t \in [0,1],$$

where $g_j : [a,b] \to [0,\infty)$, for $j = 1, 2$, is chosen such that

$$\frac{1-\alpha}{\alpha} g_1 \circ \mathcal{L}(t) = \frac{\alpha}{1-\alpha} g_1 \circ \mathcal{R}(t), \quad t \in [0,s] \quad (15)$$

and

$$\frac{1-\alpha}{\alpha} g_2 \circ \mathcal{L}(t) = \frac{\alpha}{1-\alpha} g_2 \circ \mathcal{R}(t), \quad t \in [s,1]. \quad (16)$$

Note that for $\alpha = 1/2$ and $\phi(x) = x$, the assumptions (15) and (16) hold if g_1 and g_2 are symmetric about $(a+b)/2$.

Due to (15) and $\mathcal{L}(0) = \mathcal{R}(0)$, one can verify that

$$\int_0^s w_1(t)dt = (1-\alpha)\int_0^s g_1 \circ \mathcal{L}(t)dt + \alpha \int_0^s g_1 \circ \mathcal{R}(t)dt$$
$$= \frac{1}{\phi(b)-\phi(a)} \int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x)$$

and

$$\int_0^s \psi \circ \mathcal{F}(t) w_1(t)dt$$
$$= (1-\alpha)\int_0^s \left(\psi \circ f \circ \mathcal{L}(t) - c[\phi \circ \mathcal{L}(t)]^2\right) g_1 \circ \mathcal{L}(t)dt$$
$$+ \alpha \int_0^s \left(\psi \circ f \circ \mathcal{R}(t) - c[\phi \circ \mathcal{R}(t)]^2\right) g_1 \circ \mathcal{R}(t)dt$$
$$= \frac{1}{\phi(b)-\phi(a)} \int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \left(\psi \circ f(x) - c\phi^2(x)\right) g_1(x)d\phi(x)$$

and hence

$$\mathcal{I}_1(s) = \psi^{-1}\left(\frac{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \left(\psi \circ f(x) - c\phi^2(x)\right) g_1(x)d\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x)}\right).$$

Similarly, since (16), $\mathcal{L}(1) = a$ and $\mathcal{R}(1) = b$,

$$\mathcal{I}_2(s)$$
$$= \psi^{-1}\left(\frac{\int_a^{\mathcal{L}(s)} \left(\psi \circ f(x) - c\phi^2(x)\right) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right.$$
$$\left. + \frac{\int_{\mathcal{R}(s)}^b \left(\psi \circ f(x) - c\phi^2(x)\right) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right).$$

Together with Theorem 5, we obtain the following result.

Corollary 9. *Let $g_1, g_2 : [a,b] \to [0,\infty)$ be integrable functions, where $\int_0^s g_1 \circ \mathcal{L}(t)dt > 0$ for all $s \in (0,1]$ and $\int_s^1 g_2 \circ \mathcal{R}(t)dt > 0$ for all $s \in [0,1)$ and satisfy (15), (16). Then,*

(i) For all $s \in (0,1]$, we have

$$\psi^{-1}\Big(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha)) - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2\Big)$$
$$\le \mathcal{F}\left(\frac{\int_0^s t g_1 \circ \mathcal{L}(t)dt}{\int_0^s g_1 \circ \mathcal{L}(t)dt}\right)$$
$$\le \psi^{-1}\left(\frac{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} \left(\psi \circ f(x) - c\phi^2(x)\right) g_1(x)d\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{R}(s)} g_1(x)d\phi(x)}\right)$$
$$\le \mathcal{G}\left(\frac{\int_0^s t g_1 \circ \mathcal{L}(t)dt}{\int_0^s g_1 \circ \mathcal{L}(t)dt}\right)$$
$$\le \psi^{-1}\Big(\alpha\psi \circ f(a) + (1-\alpha)\psi \circ f(b)$$
$$- c(\alpha\phi^2(a) + (1-\alpha)\phi^2(b))\Big). \quad (17)$$

(ii) For all $s \in [0,1)$, we have

$$\psi^{-1}\Big(\psi \circ f(\mathcal{M}_\phi(a,b;\alpha)) - c(\alpha\phi(a) + (1-\alpha)\phi(b))^2\Big)$$

$$\mathcal{F}\left(\frac{\int_s^1 tg_2 \circ \mathcal{L}(t)dt}{\int_s^1 g_2 \circ \mathcal{L}(t)dt}\right)$$

$$= \psi^{-1}\left(\frac{\int_a^{\mathcal{L}(s)} \left(\psi \circ f(x) - c\phi^2(x)\right) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right.$$

$$\left. + \frac{\int_{\mathcal{R}(s)}^b \left(\psi \circ f(x) - c\phi^2(x)\right) g_2(x)d\phi(x)}{\int_a^{\mathcal{L}(s)} g_2(x)d\phi(x) + \int_{\mathcal{R}(s)}^b g_2(x)d\phi(x)}\right)$$

$$\leq \mathcal{G}\left(\frac{\int_s^1 tg_2 \circ \mathcal{L}(t)dt}{\int_s^1 g_2 \circ \mathcal{L}(t)dt}\right)$$

$$\leq \psi^{-1}\Big(\alpha\psi \circ f(a) + (1-\alpha)\psi \circ f(b)$$

$$- c(\alpha\phi^2(a) + (1-\alpha)\phi^2(b))\Big). \tag{18}$$

Remark 10. Corollary 9 actually generalizes some Hermite-Hadamard type inequalities that was recently established for strongly convex functions and generalized ones. We can list as below.

1. For $\alpha = 1/2$ and $\psi(x) = \phi(x) = x$, due to (17) we obtain an inequality that is sharper than Fejér (5) type inequality establised by Azocar et al.[10] Theorem 5 for strongly convex functions.

2. Let $\alpha = 1/2$, $g_1 = 1$ and $\psi(x) = \phi(x) = x$. Then, the inequality (17) implies

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2$$

$$\leq \frac{f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right)}{2} + \frac{13c}{192}(b-a)^2$$

$$\leq \frac{2}{b-a}\int_{\frac{a+3b}{4}}^{\frac{3a+b}{4}} f(x)dx + \frac{c}{16}(b-a)^2$$

$$\leq \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} + \frac{c}{48}(b-a)^2$$

$$\leq \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{c}{24}(b-a)^2$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2,$$

where $f : [a,b] \to \mathbb{R}$ is strongly convex with modulus c. This is a refinement of inequality (4).

3. Let $\alpha = 1/2$, $g_1 = 1$ and $\phi(x) = 1/x, \psi(x) = x$. Then, (17) reduces to the following inequality.[11]

$$f\left(\frac{2ab}{a+b}\right) + \frac{c}{12}\left(\frac{b-a}{ab}\right)^2$$

$$\leq \frac{f\left(\frac{8ab}{5a+3b}\right) + f\left(\frac{8ab}{3a+5b}\right)}{2} + \frac{13c}{192}\left(\frac{b-a}{ab}\right)^2$$

$$\leq \frac{(3a+b)(a+3b)}{8(b-a)}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{f(x)}{x^2}dx + \frac{c}{16}\left(\frac{b-a}{ab}\right)^2$$

$$\leq \frac{f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right)}{2} + \frac{c}{48}\left(\frac{b-a}{ab}\right)^2$$

$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

$$\leq \frac{1}{2}\left(f\left(\frac{2ab}{a+b}\right) + \frac{f(a)+f(b)}{2}\right) - \frac{c}{24}\left(\frac{b-a}{ab}\right)^2$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{c}{6}\left(\frac{b-a}{ab}\right)^2.$$

The above inequality is a refinement of the inequality in[11] that was established by Noor et al. for strongly harmonic convex functions with modulus c.

4. Due to (17), we obtain a refinement of Hermite-Hadamard inequality for strongly log-convex functions[12] if $\alpha = \frac{1}{2}$, $g_1 = 1$ and $\phi(x) = x, \psi(x) = \ln x$:

$$\exp\left(\ln f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2\right)$$

$$\leq \exp\left(\frac{\ln f\left(\frac{5a+3b}{8}\right) + \ln f\left(\frac{3a+5b}{8}\right)}{2} + \frac{13c}{192}(b-a)^2\right)$$

$$\leq \exp\left(\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \ln f(x)dx + \frac{c}{16}(b-a)^2\right)$$

$$\leq \exp\left(\frac{\ln f\left(\frac{3a+b}{4}\right) + \ln f\left(\frac{a+3b}{4}\right)}{2} + \frac{c}{48}(b-a)^2\right)$$

$$\leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)dx\right)$$

$$\leq 1/2\exp\left(\ln f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2\right)$$

$$+ 1/2\exp\left(\frac{\ln f(a) + \ln f(b)}{2} - \frac{c}{6}(b-a)^2\right)$$

$$\leq \exp\left(\frac{\ln f(a) + \ln f(b)}{2} - \frac{c}{6}(b-a)^2\right).$$

4. Conclusions

In the paper, we studied a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means, called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, and derived various Fejér type inequalities. These not merely provide a natural and intrinsic characterization of the $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, but actually offer generalizations and refinements of some Hermite-Hadamard and Fejér type inequalities obtained in earlier studies for different kinds of strong convexity.

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